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Symmetric identities of higher-order degenerate $q-$Euler polynomials

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Abstract

In this paper, we study the higher-order degenerate q-Euler polynomials and give some identities of symmetry on these polynomials derived from symmetric properties for certain multivariate fermionic p-adic q-integrals on $\mathbb{Z}_p$.

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1. Introduction

Let p be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized so that $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-analogue of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let $f(x)$ be a continuous functional $\mathbb{Z}_p$. Then, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \tag{1.1}$$
$$= \frac{[2]_q}{2} \lim_{N\to\infty} \sum_{x=0}^{P^N-1} f(x)\, q^x (-1)^x, \quad \text{(see [12, 14])},$$

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where $[x]_{-q}=\frac{1-(-q)^x}{1+q}$.

Note that

$$\lim_{q\to 1}\int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) \tag{1.2}$$

is the ordinary fermonic p-adic integral on $\mathbb{Z}_p$.

From (1.1), we can easily derive the following equation:

$$q^n\int_{\mathbb{Z}_p} f(x+n)\,d\mu_{-q}(x)+(-1)^{n-1}\int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x)=[2]_q\sum_{x=0}^{n-1} f(x)(-1)^{n-1-x}, \tag{1.3}$$

and

$$q\int_{\mathbb{Z}_p} f(x+1)\,d\mu_{-q}(x)+\int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x)=[2]_q f(0),\quad (\text{see [14]}). \tag{1.4}$$

As is well known, the higher-order Euler polynomials are defined by the generating function

$$\left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}. \tag{1.5}$$

When $x=0$, $E_n^{(r)}=E_n^{(r)}(0)$ are called the higher-order Euler numbers (see [1]–[23]).

From (1.2), we note that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)=\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}.$$

Carlitz considered q-Bernoulli numbers defined by the recurrence relation

$$\beta_{0,q}=1,\quad q(q\beta_q+1)^n-\beta_{n,q}=\begin{cases}1, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases} \tag{1.6}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$ (see [4]).

In [12, 14], Kim defined Carlitz's type q-Euler numbers given by

$$\mathcal{E}_{0,q}=1,\quad q(q\mathcal{E}_q+1)^n-\mathcal{E}_{n,q}=[2]_q\delta_{0,n}, \tag{1.7}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Recently, the higher-order q-Euler polynomials are defined by the multivariate fermionic p-adic q-integral on $\mathbb{Z}_p$

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{[x_1+\cdots+x_r+x]_q t}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)=\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(r)}(x)\frac{t^n}{n!},\quad (\text{see [14]}). \tag{1.8}$$

When $x=0$, $\mathcal{E}_{n,q}^{(r)}=\mathcal{E}_{n,q}^{(r)}(0)$ are called the higher-order q-Euler numbers. In particular, $r=1$, then $\mathcal{E}_{n,q}^{(1)}(x)=\mathcal{E}_{n,q}(x)$.

From (1.8), we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r+x]_q^n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)=\mathcal{E}_{n,q}^{(r)}(x) \tag{1.9}$$

$$= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \left(\frac{[2]_q}{1+q^{l+1}} \right)^r q^{lx}$$
$$= [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r} [m_1+\cdots+m_r+x]_q^n ,$$

where $r \in \mathbb{N}$ and $n \geq 0$.

By (1.9), we get the generating function of the higher-order q-Euler polynomials as follows:

$$[2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r]_q t} = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [14, 15]}). \tag{1.10}$$

Carlitz introduced the higher-order degenerate Euler polynomials given by the generating function

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} \right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)}(x \mid \lambda) \frac{t^n}{n!}. \tag{1.11}$$

When $x = 0$, $\mathcal{E}_n^{(r)}(x) = \mathcal{E}_n^{(r)}(0 \mid \lambda)$ are called the higher-order degenerate Euler numbers (see [5]). In particular, $r = 1$, $\mathcal{E}_n^{(1)}(x \mid \lambda) = \mathcal{E}_n(x \mid \lambda)$ are called degenerate Euler polynomials.

Note that $\lim_{\lambda \to 0} \mathcal{E}_n^{(r)}(x \mid \lambda) = E_n^{(r)}(x)$, $(n \geq 0)$.

In this paper, we study the higher-order degenerate q-Euler polynomials and give some identities of symmetry on these polynomials derived from symmetric properties for certain multivariate fermionic p-adic q-integrals on $\mathbb{Z}_p$.

2. Symmetric identities of higher-order degenerate *q*-Euler polynomials

Let $\lambda, t \in \mathbb{C}_p$ be such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$. From (1.2) and (1.3), we note that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x_1+\cdots+x_r+x}{\lambda}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} \right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)}(x \mid \lambda) \frac{t^n}{n!}. \tag{2.1}$$

In view of (1.8), we define the higher-order degenerate q-Euler polynomials by the generating function as

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}[x_1+\cdots+x_r+x]_q} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q,\lambda}^{(r)}(x) \frac{t^n}{n!}. \tag{2.2}$$

Thus, by (2.2), we get

$$\lim_{\lambda \to 0} \mathcal{E}_{n,q,\lambda}^{(r)}(x) = \mathcal{E}_{n,q}^{(r)}(x), \quad (n \geq 0).$$

From (2.2), we can derive

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left([x_1+\cdots+x_r+x]_q \right)_{n,\lambda} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \mathcal{E}_{n,q,\lambda}^{(r)}(x), \quad (n \geq 0), \tag{2.3}$$

where

$$\left([x]_q \right)_{n,\lambda} = [x]_q \left([x]_q - \lambda \right) \left([x]_q - 2\lambda \right) \cdots \left([x]_q - (n-1)\lambda \right), \quad (n \geq 1)$$

and $\left([x]_q\right)_{0,\lambda} = 1$.

By (2.3), we get

$$\begin{aligned} \mathcal{E}_{n,q,\lambda}^{(r)}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left([x_1 + \cdots + x_r]_q\right)_{n,\lambda} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\ &= \sum_{l=0}^{n} S_1(n,l)\, \lambda^{n-l} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_r + x]_q^l \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\ &= \sum_{l=0}^{n} S_1(n,l)\, \lambda^{n-l} \mathcal{E}_{l,q}^{(r)}(x), \end{aligned} \tag{2.4}$$

where $S_1(n,l)$ is the Stirling number of the first kind.

From (1.9) and (2.4), we have

$$\mathcal{E}_{n,q,\lambda}^{(r)}(x) = [2]_q^r \sum_{l=0}^{n} \sum_{m_1,\ldots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r} [m_1 + \cdots + m_r + x]_q^l \, S_1(n,l)\, \lambda^{n-l}. \tag{2.5}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$\begin{aligned} \mathcal{E}_{n,q,\lambda}^{(r)}(x) &= \sum_{l=0}^{n} S_1(n,l)\, \lambda^{n-l} \mathcal{E}_{l,q}^{(r)}(x) \\ &= [2]_q^r \sum_{l=0}^{n} \sum_{m_1,\ldots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r} [m_1 + \cdots + m_r + x]_q^l \, S_1(n,l)\, \lambda^{n-l}. \end{aligned}$$

Now, we observe that

$$\begin{aligned} &[2]_q^r \sum_{l=0}^{n} \sum_{m_1,\ldots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r} [m_1 + \cdots + m_r + x]_q^l \, S_1(n,l)\, \lambda^{n-l} \\ &= [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r} \left([m_1 + \cdots + m_r + x]_q\right)_{n,\lambda}. \end{aligned} \tag{2.6}$$

Thus, by (2.6), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_{n,q,\lambda}^{(r)}(x) \frac{t^n}{n!} &= [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r} \sum_{n=0}^{\infty} \frac{\left([m_1 + \cdots + m_r + x]_q\right)_{n,\lambda}}{n!} t^n \\ &= [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r} (1 + \lambda t)^{\frac{[m_1+\cdots+m_r+x]_q}{\lambda}}. \end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. *For $r \in \mathbb{N}$, we have*

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q,\lambda}^{(r)}(x) \frac{t^n}{n!} = [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r} (1 + \lambda t)^{\frac{[m_1+\cdots+m_r+x]_q}{\lambda}}.$$

By replacing t by $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$ in (2.2), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{[x_1+\cdots+x_r+x]_q t} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) &= \sum_{m=0}^{\infty} \mathcal{E}_{m,q,\lambda}^{(r)}(x)\,\lambda^{-m}\frac{\left(e^{\lambda t}-1\right)^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}\mathcal{E}_{m,q,\lambda}^{(r)}(x)\,S_2(n,m)\right)\frac{t^n}{n!},\end{aligned} \tag{2.8}$$

where $S_2(n,m)$ is the Stirling number of the second kind.

Therefore, by (1.8) and (2.8), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,q}^{(r)}(x) = \sum_{m=0}^{n}\lambda^{n-m}\mathcal{E}_{m,q,\lambda}^{(r)}(x)\,S_2(n,m).$$

Let $w_1, w_2 \in \mathbb{N}$ be such that $w_1 \equiv 1, w_2 \equiv 1 \pmod 2$. Then, by (2.2), we get

$$\begin{aligned}&\frac{1}{[w_1]_{-q}^r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{\left[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l\right]_q}{\lambda}} d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r) \\ &= \frac{1}{[w_1]_{-q}^r}\lim_{N\to\infty}\sum_{y_1,\ldots,y_r=0}^{p^N-1}\frac{1}{[p^N]_{-q^{w_1}}^r}(1+\lambda t)^{\frac{\left[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l\right]_q}{\lambda}} \\ &\quad\times(-q^{w_1})^{y_1+\cdots+y_r} \\ &= \frac{1}{[w_1]_{-q}^r}\lim_{N\to\infty}\frac{1}{[w_2p^N]_{-q^{w_1}}^r}\sum_{y_1,\ldots,y_r=0}^{w_2p^N-1}(1+\lambda t)^{\frac{\left[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l\right]_q}{\lambda}} \\ &\quad\times(-q)^{w_1y_1+\cdots+w_1y_r} \\ &= \frac{[2]_q^r}{2^r}\lim_{N\to\infty}\sum_{i_1,i_2,\ldots,i_r=0}^{w_2-1}\sum_{y_1,\ldots,y_r=0}^{p^N-1}(1+\lambda t)^{\frac{\left[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r (i_l+w_2y_l)\right]_q}{\lambda}} \\ &\quad\times(-1)^{y_1+\cdots+y_r}\,q^{w_1(i_1+w_2y_1)+w_1(i_2+w_2y_2)+\cdots+w_1(i_r+w_2y_r)}\times(-1)^{i_1+\cdots+i_r} \\ &= \frac{[2]_q^r}{2^r}\sum_{i_1,\ldots,i_r=0}^{w_2-1}(-1)^{\sum_{l=1}^r i_l}\,q^{w_1\sum_{l=1}^r i_l} \\ &\quad\times\lim_{N\to\infty}\sum_{y_1,\ldots,y_r=0}^{p^N-1}(1+\lambda t)^{\frac{\left[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r (i_l+w_2y_l)\right]_q}{\lambda}} \\ &\quad\times(-1)^{y_1+\cdots+y_r}\,q^{w_1w_2y_1+w_1w_2y_2+\cdots+w_1w_2y_r}.\end{aligned} \tag{2.9}$$

From (2.9), we note that

$$\begin{aligned}&\frac{1}{[w_1]_{-q}^r}\sum_{j_1,\ldots,j_r=0}^{w_1-1}q^{w_2\sum_{l=1}^r j_l}(-1)^{\sum_{l=1}^r j_l} \\ &\quad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{\left[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l\right]_q}{\lambda}} d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r) \\ &= \frac{[2]_q^r}{2^r}\lim_{N\to\infty}\sum_{i_1,\ldots,i_r=0}^{w_2-1}\sum_{j_1,\ldots,j_r=0}^{w_1-1}\sum_{y_1,\ldots,y_r=0}^{p^N-1}(-1)^{\sum_{l=1}^r (j_l+i_l+y_l)}\end{aligned} \tag{2.10}$$

$$\times q^{w_1\sum_{l=1}^r i_l+w_2\sum_{l=1}^r j_l+w_1w_2\sum_{l=1}^r y_l}$$
$$\times (1+\lambda t)^{\frac{1}{\lambda}\left[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r y_l\right]_q}.$$

On the other hand,

$$\begin{aligned}
&\frac{1}{[w_2]_{-q}^r}\sum_{j_1,\ldots,j_r=0}^{w_2-1} q^{w_1\sum_{l=1}^r j_l}(-1)^{\sum_{l=1}^r j_l} \\
&\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[w_1w_2x+w_1\sum_{l=1}^r j_l+w_2\sum_{l=1}^r y_l]_q}{\lambda}}\,d\mu_{-q^{w_2}}(y_1)\cdots d\mu_{-q^{w_2}}(y_r) \\
=&\frac{[2]_q^r}{2^r}\lim_{N\to\infty}\sum_{i_1,\ldots,i_r=0}^{w_1-1}\sum_{j_1,\ldots,j_r=0}^{w_2-1}\sum_{y_1,\ldots,y_r=0}^{p^N-1}(-1)^{\sum_{l=1}^r(i_l+j_l+y_l)} \\
&\times q^{w_1\sum_{l=1}^r j_l+w_2\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r y_l} \\
&\times(1+\lambda t)^{\frac{1}{\lambda}\left[w_1w_2x+w_1\sum_{l=1}^r j_l+w_2\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r y_l\right]_q}.
\end{aligned}\tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.4. *Let $w_1, w_2 \in \mathbb{N}$ such that $w_1 \equiv 1 \pmod 2$ and $w_2 \equiv 1 \pmod 2$. Then, we have*

$$\begin{aligned}
&\frac{1}{[w_1]_{-q}^r}\sum_{j_1,\ldots,j_r=0}^{w_1-1} q^{w_2\sum_{l=1}^r j_l}(-1)^{\sum_{l=1}^r j_l} \\
&\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l]_q}{\lambda}}\,d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r) \\
=&\frac{1}{[w_2]_{-q}^r}\sum_{j_1,\ldots,j_r=0}^{w_2-1} q^{w_1\sum_{l=1}^r j_l}(-1)^{\sum_{l=1}^r j_l} \\
&\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[w_1w_2x+w_1\sum_{l=1}^r j_l+w_2\sum_{l=1}^r y_l]_q}{\lambda}}\,d\mu_{-q^{w_2}}(y_1)\cdots d\mu_{-q^{w_2}}(y_r).
\end{aligned}$$

We observe that

$$\left[w_1w_2x+\sum_{l=1}^r j_lw_2+\sum_{l=1}^r y_lw_1\right]_q=[w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l\right]_{q^{w_1}}. \tag{2.12}$$

From (2.12), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}\left[w_1w_2x+\sum_{l=1}^r j_lw_2+\sum_{l=1}^r y_lw_1\right]_q}\,d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r) \\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[w_1]_q}{\lambda}\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l\right]_{q^{w_1}}}\,d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r) \\
=&\sum_{n=0}^\infty \mathcal{E}^{(r)}_{n,q^{w_1},\frac{\lambda}{[w_1]_q}}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_r)\right)[w_1]_q^n\frac{t^n}{n!}.
\end{aligned}\tag{2.13}$$

Therefore, by Theorem 2.4, (2.12) and (2.13), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have*

$$\frac{[w_1]_q^n}{[w_1]_{-q}^r}\sum_{j_1,\ldots,j_r=0}^{w_1-1}(-1)^{j_1+\cdots+j_r}q^{w_2(j_1+\cdots+j_r)}\mathcal{E}^{(r)}_{n,q^{w_1},\frac{\lambda}{[w_1]_q}}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_r)\right)$$

$$= \frac{[w_2]_q^n}{[w_2]_{-q}^r} \sum_{j_1,\ldots,j_r=0}^{w_2-1} (-1)^{j_1+\cdots+j_r} q^{w_1(j_1+\cdots+j_r)} \mathcal{E}^{(r)}_{n,q^{w_2},\frac{\lambda}{[w_2]_q}} \left(w_1 x + \frac{w_1}{w_2}(j_1+\cdots+j_r) \right).$$

From (2.3), we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{1}{[w_1]_q}\right)^n \left([w_1]_q \left[w_2 x + \frac{w_2}{w_1}\sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}} \right)_{n,\lambda} \\ &\quad \times d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r) \\ &= \sum_{l=0}^n S_1(n,l) \left(\frac{\lambda}{[w_1]_q}\right)^{n-l} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_2 x + \frac{w_2}{w_1}\sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}}^l \\ &\quad \times d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r) \\ &= \sum_{l=0}^n S_1(n,l) \left(\frac{\lambda}{[w_1]_q}\right)^{n-l} \sum_{i=0}^l \binom{l}{i} \left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j_1+\cdots+j_r]_{q^{w_2}}^i q^{w_2(l-i)\sum_{k=1}^r j_k} \\ &\quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [w_2 x + y_1 + \cdots + y_r]_{q^{w_1}}^{l-i} d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r) \\ &= \sum_{l=0}^n \sum_{i=0}^l S_1(n,l)\lambda^{n-l} [w_1]_q^{l-n-i} [w_2]_q^i [j_1+\cdots+j_r]_{q^{w_2}}^i \\ &\quad \times q^{w_2(l-i)\sum_{k=1}^r j_k} \binom{l}{i} \mathcal{E}^{(r)}_{l-i,q^{w_1}}(w_2 x). \end{aligned} \tag{2.14}$$

Thus, by (2.14), we get

$$\begin{aligned} &\frac{[w_1]_q^n}{[w_1]_{-q}^r} \sum_{j_1,\ldots,j_r=0}^{w_1-1} q^{w_2\sum_{l=1}^r j_l} (-1)^{\sum_{l=1}^r j_l} \left(\frac{1}{[w_1]_q}\right)^n \\ &\quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left([w_1]_q \left[w_2 x + \frac{w_2}{w_1}\sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}} \right)_{n,\lambda} d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r) \\ &= \frac{1}{[w_1]_{-q}^r} \sum_{l=0}^n \sum_{i=0}^l \binom{l}{i} S_1(n,l)\lambda^{n-l} [w_1]_q^{l-i} [w_2]_q^i \mathcal{E}^{(r)}_{l-i,q^{w_1}}(w_2 x) \tilde{T}^{(r)}_{l+1,i}(w_1 \mid q^{w_2}), \end{aligned} \tag{2.15}$$

where

$$\tilde{T}^{(r)}_{n,i}(w \mid q) = \sum_{j_1,\ldots,j_r=0}^{w-1} (-1)^{j_1+\cdots+j_r} [j_1+\cdots+j_r]_q^i q^{(n-i)\sum_{l=1}^r j_l}. \tag{2.16}$$

On the other hand,

$$\begin{aligned} &\frac{[w_2]_q^n}{[w_2]_{-q}^r} \sum_{j_1,\ldots,j_r=0}^{w_2-1} q^{w_1\sum_{l=1}^r j_l} (-1)^{\sum_{l=1}^r j_l} \left(\frac{\lambda}{[w_2]_q}\right)^n \\ &\quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{[w_2]_q}{\lambda} \left[w_1 x + \frac{w_1}{w_2}\sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_2}} \right)_{n,\lambda} d\mu_{-q^{w_2}}(y_1)\cdots d\mu_{-q^{w_2}}(y_r) \\ &= \frac{1}{[w_2]_{-q}^r} \sum_{l=0}^n \sum_{i=0}^l \binom{l}{i} S_1(n,l)\lambda^{n-l} [w_2]_q^{l-i} [w_1]_q^i \mathcal{E}^{(r)}_{l-i,q^{w_2}}(w_1 x) \tilde{T}^{(r)}_{l+1,i}(w_2 \mid q^{w_1}). \end{aligned} \tag{2.17}$$

Therefore, by (2.15) and (2.17), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have*

$$\begin{aligned} &\frac{1}{[w_1]_{-q}^r} \sum_{l=0}^{n} \sum_{i=0}^{l} \binom{l}{i} S_1(n,l) \lambda^{n-l} [w_1]_q^{l-i} [w_2]_q^i \mathcal{E}_{l-i,q^{w_1}}^{(r)}(w_2 x) \tilde{T}_{l+1,i}^{(r)}(w_1 \mid q^{w_2}) \\ &= \frac{1}{[w_2]_{-q}^r} \sum_{l=0}^{n} \sum_{i=0}^{l} \binom{l}{i} S_1(n,l) \lambda^{n-l} [w_2]_q^{l-i} [w_1]_q^i \mathcal{E}_{l-i,q^{w_2}}^{(r)}(w_1 x) \tilde{T}_{l+1,i}^{(r)}(w_2 \mid q^{w_1}). \end{aligned}$$

Remark 2.7. If we take $\lambda \to 0$, then we get

$$\begin{aligned} &\frac{1}{[w_1]_{-q}^r} \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^i \mathcal{E}_{n-i,q^{w_1}}^{(r)}(w_2 x) \tilde{T}_{n+1,i}^{(r)}(w_1 \mid q^{w_2}) \\ &= \frac{1}{[w_2]_{-q}^r} \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^i \mathcal{E}_{n-i,q^{w_2}}^{(r)}(w_1 x) \tilde{T}_{n+1,i}^{(r)}(w_2 \mid q^{w_1}). \end{aligned}$$

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