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A fixed point technique for some iterative algorithm with applications to generalized right fractional calculus

George A. Anastassiou[a], Ioannis K. Argyros[b,*]

[a]Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA.
[b]Department of Mathematical Sciences, Cameron University, Lawton, Ok 73505, USA.

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Abstract

We present a fixed point technique for some iterative algorithms on a generalized Banach space setting to approximate a locally unique zero of an operator. Earlier studies such as [I. K. Argyros, Approx. Theory Appl., **9** (1993), 1–9], [I. K. Argyros, Southwest J. Pure Appl. Math., **1** (1995), 30–36], [I. K. Argyros, Springer-Verlag Publ., New York, (2008)], [P. W. Meyer, Numer. Funct. Anal. Optim., **9** (1987), 249–259] require that the operator involved is Fréchet-differentiable. In the present study we assume that the operator is only continuous. This way we extend the applicability of these methods to include right fractional calculus as well as problems from other areas. Some applications include fractional calculus involving right generalized fractional integral and the right Hadamard fractional integral. Fractional calculus is very important for its applications in many applied sciences.

Keywords: Generalized Banach space, fixed point iterative algorithm, semilocal convergence, fixed point right generalized fractional integral.
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1. Introduction

We present a semilocal convergence analysis for some fixed point iterative algorithms on a generalized Banach space setting to approximate a zero of an operator. The semilocal convergence is, based on the

*Corresponding author
Email addresses: ganastss@memphis.edu (George A. Anastassiou), iargyros@cameron.edu (Ioannis K. Argyros)

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information around an initial point, to give conditions ensuring the convergence of the iterative algorithm. A generalized norm is defined to be an operator from a linear space into a partially order Banach space (to be precised in section 2). Earlier studies such as [3, 4, 5, 7] for Newton's method have shown that a more precise convergence analysis is obtained when compared to the real norm theory. However, the main assumption is that the operator involved is Fréchet-differentiable. This hypothesis limits the applicability of Newton's method. In the present study using a fixed point technique (see iterative algorithm 3.1), we show convergence by only assuming the continuity of the operator. This way we expand the applicability of these iterative algorithms.

The rest of the paper is organized as follows: section 2 contains the basic concepts on generalized Banach spaces and auxiliary results on inequalities and fixed points. In section 3 we present the semilocal convergence analysis. Finally, in the concluding sections 4-5, we present special cases and applications in generalized right fractional calculus.

2. Generalized Banach spaces

We present some standard concepts that are needed in what follows to make the paper as self contained as possible. More details on generalized Banach spaces can be found in [3, 4, 5, 7], and the references there in.

Definition 2.1. A generalized Banach space is a triplet $(x, E, /\cdot/)$ such that

(i) X is a linear space over $\mathbb{R}(\mathbb{C})$.

(ii) $E = (E, K, \|\cdot\|)$ is a partially ordered Banach space, i.e.

(ii$_1$) $(E, \|\cdot\|)$ is a real Banach space,

(ii$_2$) E is partially ordered by a closed convex cone K,

(ii$_3$) The norm $\|\cdot\|$ is monotone on K.

(iii) The operator $/\cdot/ : X \to K$ satisfies

$/x/ = 0 \Leftrightarrow x = 0$, $/\theta x/ = |\theta| /x/$,

$/x + y/ \leq /x/ + /y/$ for each $x, y \in X$, $\theta \in \mathbb{R}(\mathbb{C})$.

(iv) X is a Banach space with respect to the induced norm $\|\cdot\|_i := \|\cdot\| \cdot /\cdot/$.

Remark 2.2. The operator $/\cdot/$ is called a generalized norm. In view of (iii) and (iii$_3$) $\|\cdot\|_i$, is a real norm. In the rest of this paper all topological concepts will be understood with respect to this norm.

Let $L\left(X^j, Y\right)$ stand for the space of j-linear symmetric and bounded operators from X^j to Y, where X and Y are Banach spaces. For X, Y partially ordered $L_+\left(X^j, Y\right)$ stands for the subset of monotone operators P such that

$$0 \leq a_i \leq b_i \Rightarrow P(a_1, \cdots, a_j) \leq P(b_1, \cdots, b_j).$$

Definition 2.3. The set of bounds for an operator $Q \in L(X, X)$ on a generalized Banach space $(X, E, /\cdot/)$ the set of bounds is defined to be:

$$B(Q) := \{P \in L_+(E, E),\ /Qx/ \leq P/x/ \text{ for each } x \in X\}.$$

Let $D \subset X$ and $T : D \to D$ be an operator. If $x_0 \in D$ the sequence $\{x_n\}$ given by

$$x_{n+1} := T(x_n) = T^{n+1}(x_0)$$

is well defined. We write in case of convergence

$$T^{\infty}(x_0) := \lim (T^n(x_0)) = \lim_{n\to\infty} x_n.$$

We need some auxiliary results on inequations.

Lemma 2.4. *Let $(E, K, \|\cdot\|)$ be a partially ordered Banach space, $\xi \in K$ and $M, N \in L_+(E, E)$.*

(i) *Suppose there exists $r \in K$ such that*

$$R(r) := (M + N) r + \xi \leq r \tag{2.1}$$

and

$$(M + N)^k r \to 0 \quad as \quad k \to \infty. \tag{2.2}$$

Then, $b := R^\infty(0)$ is well defined satisfies the equation $t = R(t)$ and is the smaller than any solution of the inequality $R(s) \leq s$.

(ii) *Suppose there exists $q \in K$ and $\theta \in (0, 1)$ such that $R(q) \leq \theta q$, then there exists $r \leq q$ satisfying (i).*

Proof. (i) Define sequence $\{b_n\}$ by $b_n = R^n(0)$. Then, we have by (2.1) that $b_1 = R(0) = \xi \leq r \Rightarrow b_1 \leq r$. Suppose that $b_k \leq r$ for each $k = 1, 2, \cdots, n$. Then, we have by (2.1) and the inductive hypothesis that $b_{n+1} = R^{n+1}(0) = R(R^n(0)) = R(b_n) = (M + N) b_n + \xi \leq (M + N) r + \xi \leq r \Rightarrow b_{n+1} \leq r$. Hence, sequence $\{b_n\}$ is bounded above by r. Set $P_n = b_{n+1} - b_n$. We shall show that

$$P_n \leq (M + N)^n r \quad \text{for each } n = 1, 2, \cdots. \tag{2.3}$$

We have by the definition of P_n and (2.2) that

$$\begin{aligned} P_1 &= R^2(0) - R(0) = R(R(0)) - R(0) = R(\xi) - R(0) \\ &= \int_0^1 R'(t\xi)\, \xi dt \leq \int_0^1 R'(\xi)\, \xi dt \leq \int_0^1 R'(r)\, r dt \leq (M + N) r, \end{aligned}$$

which shows (2.3) for $n = 1$. Suppose that (2.3) is true for $k = 1, 2, \cdots, n$. Then, we have in turn by (2.2) and the inductive hypothesis that

$$\begin{aligned} P_{k+1} &= R^{k+2}(0) - R^{k+1}(0) = R^{k+1}(R(0)) - R^{k+1}(0) \\ &= R^{k+1}(\xi) - R^{k+1}(0) = R\left(R^k(\xi)\right) - R\left(R^k(0)\right) \\ &= \int_0^1 R'\left(R^k(0) + t\left(R^k(\xi) - R^k(0)\right)\right)\left(R^k(\xi) - R^k(0)\right) dt \\ &\leq R'\left(R^k(\xi)\right)\left(R^k(\xi) - R^k(0)\right) = R'\left(R^k(\xi)\right)\left(R^{k+1}(0) - R^k(0)\right) \\ &\leq R'(r)\left(R^{k+1}(0) - R^k(0)\right) \leq (M + N)(M + N)^k r = (M + N)^{k+1} r, \end{aligned}$$

which completes the induction for (2.3). It follows that $\{b_n\}$ is a complete sequence in a Banach space and as such it converges to some b. Notice that $R(b) = R\left(\lim_{n\to\infty} R^n(0)\right) = \lim_{n\to\infty} R^{n+1}(0) = b \Rightarrow b$ solves the equation $R(t) = t$. We have that $b_n \leq r \Rightarrow b \leq r$, where r a solution of $R(r) \leq r$. Hence, b is smaller than any solution of $R(s) \leq s$.

(ii) Define sequences $\{v_n\}$, $\{w_n\}$ by $v_0 = 0$, $v_{n+1} = R(v_n)$, $w_0 = q$, $w_{n+1} = R(w_n)$. Then, we have that

$$\begin{gathered} 0 \leq v_n \leq v_{n+1} \leq w_{n+1} \leq w_n \leq q, \\ w_n - v_n \leq \theta^n (q - v_n) \end{gathered} \tag{2.4}$$

and sequence $\{v_n\}$ is bounded above by q. Hence, it converges to some r with $r \leq q$. We also get by (2.4) that $w_n - v_n \to 0$ as $n \to \infty \Rightarrow w_n \to r$ as $n \to \infty$. $\square$

We also need the auxiliary result for computing solutions of fixed point problems.

Lemma 2.5. *Let $(X,(E,K,\|\cdot\|),/\cdot/)$ be a generalized Banach space, and $P \in B(Q)$ be a bound for $Q \in L(X,X)$. Suppose there exists $y \in X$ and $q \in K$ such that*

$$Pq + /y/ \leq q \text{ and } P^k q \to 0 \text{ as } k \to \infty.$$

Then, $z = T^{\infty}(0)$, $T(x) := Qx + y$ is well defined and satisfies: $z = Qz + y$ and $/z/ \leq P/z/ + /y/ \leq q$. Moreover, z is the unique solution in the subspace $\{x \in X | \exists\ \theta \in \mathbb{R} : \{x\} \leq \theta q\}$.

The proof can be found in [7, Lemma 3.2].

3. Semilocal convergence

Let $(X,(E,K,\|\cdot\|),/\cdot/)$ and Y be generalized Banach spaces, $D \subset X$ an open subset, $G : D \to Y$ a continuous operator and $A(\cdot) : D \to L(X,Y)$.

A zero of operator G is to be determined by an iterative algorithm starting at a point $x_0 \in D$. The results are presented for an operator $F = JG$, where $J \in L(Y,X)$. The iterates are determined through a fixed point problem:

$$x_{n+1} = x_n + y_n,\ A(x_n) y_n + F(x_n) = 0 \Leftrightarrow y_n = T(y_n) := (I - A(x_n)) y_n - F(x_n). \tag{3.1}$$

Let $U(x_0,r)$ stand for the ball defined by

$$U(x_0,r) := \{x \in X : /x - x_0/ \leq r\}$$

for some $r \in K$.

Next, we present the semilocal convergence analysis of iterative algorithm 3.1 using the preceding notation.

Theorem 3.1. *Let $F : D \subset X$, $A(\cdot) : D \to L(X,Y)$ and $x_0 \in D$ be as defined previously. Suppose:*

(H_1) *There exists an operator $M \in B(I - A(x))$ for each $x \in D$.*

(H_2) *There exists an operator $N \in L_+(E,E)$ satisfying for each $x,y \in D$*

$$/F(y) - F(x) - A(x)(y-x)/ \leq N/y-x/.$$

(H_3) *There exists a solution $r \in K$ of*

$$R_0(t) := (M+N)t + /F(x_0)/ \leq t.$$

(H_4) *$U(x_0,r) \subseteq D$.*

(H_5) *$(M+N)^k r \to 0$ as $k \to \infty$.*

Then, the following hold:

(C_1) *The sequence $\{x_n\}$ defined by*

$$x_{n+1} = x_n + T_n^{\infty}(0),\ \ T_n(y) := (I - A(x_n)) y - F(x_n)$$

is well defined, remains in $U(x_0,r)$ for each $n = 0,1,2,\cdots$ and converges to the unique zero of operator F in $U(x_0,r)$.

(C_2) *An apriori bound is given by the null-sequence $\{r_n\}$ defined by $r_0 := r$ and for each $n = 1,2,\cdots$*

$$r_n = P_n^{\infty}(0),\quad P_n(t) = Mt + Nr_{n-1}.$$

(C_3) *An aposteriori bound is given by the sequence $\{s_n\}$ defined by*

$$s_n := R_n^{\infty}(0),\quad R_n(t) = (M+N)t + Na_{n-1},$$

$$b_n := /x_n - x_0/ \leq r - r_n \leq r,$$

where

$$a_{n-1} := /x_n - x_{n-1}/ \text{ for each } n = 1,2,\cdots.$$

Proof. Let us define for each $n \in \mathbb{N}$ the statement:
(I_n) $x_n \in X$ and $r_n \in K$ are well defined and satisfy

$$r_n + a_{n-1} \leq r_{n-1}.$$

We use induction to show (I_n). The statement (I_1) is true: By Lemma 2.4 and (H_3), (H_5) there exists $q \leq r$ such that:

$$Mq + /F(x_0)/ = q \text{ and } M^k q \leq M^k r \to 0 \text{ as } k \to \infty.$$

Hence, by Lemma 2.5 x_1 is well defined and we have $a_0 \leq q$. Then, we get the estimate

$$\begin{aligned} P_1(r-q) &= M(r-q) + Nr_0 \\ &\leq Mr - Mq + Nr = R_0(r) - q \\ &\leq R_0(r) - q = r - q. \end{aligned}$$

It follows with Lemma 2.4 that r_1 is well defined and

$$r_1 + a_0 \leq r - q + q = r = r_0.$$

Suppose that (I_j) is true for each $j = 1, 2, \cdots, n$. We need to show the existence of x_{n+1} and to obtain a bound q for a_n. To achieve this notice that:

$$Mr_n + N(r_{n-1} - r_n) = Mr_n + Nr_{n-1} - Nr_n = P_n(r_n) - Nr_n \leq r_n.$$

Then, it follows from Lemma 2.4 that there exists $q \leq r_n$ such that

$$q = Mq + N(r_{n-1} - r_n) \text{ and } (M+N)^k q \to 0, \text{ as } k \to \infty. \tag{3.2}$$

By (I_j) it follows that

$$b_n = /x_n - x_0/ \leq \sum_{j=0}^{n-1} a_j \leq \sum_{j=0}^{n-1} (r_j - r_{j+1}) = r - r_n \leq r.$$

Hence, $x_n \in U(x_0, r) \subset D$ and by (H_1) M is a bound for $I - A(x_n)$. We can write by (H_2) that

$$\begin{aligned} /F(x_n)/ &= /F(x_n) - F(x_{n-1}) - A(x_{n-1})(x_n - x_{n-1})/ \\ &\leq Na_{n-1} \leq N(r_{n-1} - r_n). \end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$Mq + /F(x_n)/ \leq q.$$

By Lemma 2.5, x_{n+1} is well defined and $a_n \leq q \leq r_n$. In view of the definition of r_{n+1} we have that

$$P_{n+1}(r_n - q) = P_n(r_n) - q = r_n - q,$$

so that by Lemma 2.4, r_{n+1} is well defined and

$$r_{n+1} + a_n \leq r_n - q + q = r_n,$$

which proves (I_{n+1}). The induction for (I_n) is complete. Let $m \geq n$, then we obtain in turn that

$$/x_{m+1} - x_n/ \leq \sum_{j=n}^{m} a_j \leq \sum_{j=n}^{m} (r_j - r_{j+1}) = r_n - r_{m+1} \leq r_n. \tag{3.4}$$

Moreover, we get inductively the estimate

$$r_{n+1} = P_{n+1}(r_{n+1}) \leq P_{n+1}(r_n) \leq (M+N)\, r_n \leq \cdots \leq (M+N)^{n+1}\, r.$$

It follows from (H_5) that $\{r_n\}$ is a null-sequence. Hence, $\{x_n\}$ is a complete sequence in a Banach space X by (3.4) and as such it converges to some $x^* \in X$. By letting $m \to \infty$ in (3.4) we deduce that $x^* \in U(x_n, r_n)$. Furthermore, (3.3) shows that x^* is a zero of F. Hence, (C_1) and (C_2) are proved. In view of the estimate

$$R_n(r_n) \leq P_n(r_n) \leq r_n$$

the apriori, bound of (C_3) is well defined by Lemma 2.4. That is s_n is smaller in general than r_n. The conditions of Theorem 3.1 are satisfied for x_n replacing x_0. A solution of the inequality of (C_2) is given by s_n (see (3.3)). It follows from (3.4) that the conditions of Theorem 3.1 are easily verified. Then, it follows from (C_1) that $x^* \in U(x_n, s_n)$ which proves (C_3). □

In general the aposterior, estimate is of interest. Then, condition (H_5) can be avoided as follows:

Proposition 3.2. *Suppose: condition (H_1) of Theorem 3.1 is true.*

(H_3') *There exists $s \in K$, $\theta \in (0,1)$ such that*

$$R_0(s) = (M+N)\, s + /F(x_0)/ \leq \theta s.$$

(H_4') $U(x_0, s) \subset D$.

Then, there exists $r \leq s$ satisfying the conditions of Theorem 3.1. Moreover, the zero x^* of F is unique in $U(x_0, s)$.

Remark 3.3.

(i) Notice that by Lemma 2.4 $R_n^{\infty}(0)$ is the smallest solution of $R_n(s) \leq s$. Hence any solution of this inequality yields on upper estimate for $R_n^{\infty}(0)$. Similar inequalities appear in (H_2) and (H_2').

(ii) The weak assumptions of Theorem 3.1 do not imply the existence of $A(x_n)^{-1}$. In practice the computation of $T_n^{\infty}(0)$ as a solution of a linear equation is no problem and the computation of the expensive or impossible to compute in general $A(x_n)^{-1}$ is not needed.

(iii) We can used the following result for the computation of the aposteriori estimates. The proof can be found in [7, Lemma 4.2] by simply exchanging the definitions of R.

Lemma 3.4. *Suppose that the conditions of* Theorem 3.1 *are satisfied. If $s \in K$ is a solution of $R_n(s) \leq s$, then $q := s - a_n \in K$ and solves $R_{n+1}(q) \leq q$. This solution might be improved by $R_{n+1}^k(q) \leq q$ for each $k = 1, 2, \cdots$.*

4. Special cases and applications

Application 4.1. The results obtained in earlier studies such as [3, 4, 5, 7] require that operator F (i.e. G) is Fréchet-differentiable. This assumption limits the applicability of the earlier results. In the present study we only require that F is a continuous operator. Hence, we have extended the applicability of the iterative algorithms include to classes of operators that are only continuous. If $A(x) = F'(x)$ iterative algorithm 3.1 reduces to Newton's method considered in [7].

Example 4.2. The j-dimensional space $\mathbb{R}^j$ is a classical example of a generalized Banach space. The generalized norm is defined by componentwise absolute values. Then, as ordered Banach space we set $E = \mathbb{R}^j$ with componentwise ordering with e.g. the maximum norm. A bound for a linear operator (a matrix) is given by the corresponding matrix with absolute values. Similarly, we can define the "N" operators. Let $E = \mathbb{R}$. That is we consider the case of a real normed space with norm denoted by $\|\cdot\|$. Let us see how the conditions of Theorem 3.1 look like.

Theorem 4.3.

(H_1) $\|I - A(x)\| \leq M$ *for some* $M \geq 0$.
(H_2) $\|F(y) - F(x) - A(x)(y - x)\| \leq N \|y - x\|$ *for some* $N \geq 0$.
(H_3) $M + N < 1$,

$$r = \frac{\|F(x_0)\|}{1 - (M + N)}. \tag{4.1}$$

(H_4) $U(x_0, r) \subseteq D$.
(H_5) $(M + N)^k r \to 0$ *as* $k \to \infty$, *where* r *is given by* (4.1).
Then, the conclusions of Theorem 3.1 hold.

5. Applications to generalized right fractional calculus

Background

We use Theorem 4.3 in this section.

We use here the following right generalized fractional integral.

Definition 5.1 ([6], p. 99)**.** The right generalized fractional integral of a function f with respect to given function g is defined as follows:

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here $g \in AC([a, b])$ (absolutely continuous functions) and is strictly increasing, $f \in L_\infty([a, b])$. We set

$$\left(I^\alpha_{b-;g} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) f(t)\, dt, \quad x \leq b, \tag{5.1}$$

clearly $\left(I^\alpha_{b-;g} f\right)(b) = 0$.

When g is the identity function id, we get that $I^\alpha_{b-;id} = I^\alpha_{b-}$, the ordinary right Riemann-Liouville fractional integral, where

$$\left(I^\alpha_{b-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t)\, dt, \quad x \leq b,$$

$\left(I^\alpha_{b-} f\right)(b) = 0$.

When $g(x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$, we get

Definition 5.2 ([6], p. 110)**.** Let $0 < a < b < \infty$, $\alpha > 0$. The right Hadamard fractional integral of order α is given by

$$\left(J^\alpha_{b-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{y}{x}\right)^{\alpha-1} \frac{f(y)}{y} dy, \quad x \leq b,$$

where $f \in L_\infty([a, b])$.

We mention:

Definition 5.3 ([1])**.** The right fractional exponential integral is defined as follows: Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a, b])$. We set

$$\left(I^\alpha_{b-;e^x} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(e^t - e^x\right)^{\alpha-1} e^t f(t)\, dt, \quad x \leq b.$$

Definition 5.4 ([1])**.** Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a, b])$, $A > 1$. We give the right fractional integral

$$\left(I^\alpha_{b-;A^x} f\right)(x) = \frac{\ln A}{\Gamma(\alpha)} \int_x^b \left(A^t - A^x\right)^{\alpha-1} A^t f(t)\, dt, \quad x \leq b.$$

We also give:

Definition 5.5 ([1]). Let $\alpha, \sigma > 0$, $0 \leq a < b < \infty$, $f \in L_\infty([a,b])$. We set

$$\left(K_{b-;x^\sigma}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t^\sigma - x^\sigma)^{\alpha-1} f(t) \sigma t^{\sigma-1} dt, \quad x \leq b.$$

We mention the following generalized right fractional derivatives.

Definition 5.6 ([1]). Let $\alpha > 0$ and $\lceil \alpha \rceil = m$ ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in AC^m([a,b])$ (space of functions f with $f^{(m-1)} \in AC([a,b])$). We define the right generalized fractional derivative of f of order α as follows

$$\left(D_{b-;g}^{\alpha} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) f^{(m)}(t) dt,$$

for any $x \in [a,b]$, where Γ is the gamma function.

We set

$$\begin{aligned} D_{b-;g}^{m} f(x) &= (-1)^m f^{(m)}(x), \\ D_{b-;g}^{0} f(x) &= f(x), \ \forall x \in [a,b]. \end{aligned}$$

When $g = id$, then $D_{b-}^{\alpha} f = D_{b-;id}^{\alpha} f$ is the right Caputo fractional derivative.

So we have the specific generalized right fractional derivatives.

Definition 5.7 ([1]).

$$D_{b-;\ln x}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \left(\ln \frac{y}{x}\right)^{m-\alpha-1} \frac{f^{(m)}(y)}{y} dy, \quad 0 < a \leq x \leq b,$$

$$D_{b-;e^x}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \left(e^t - e^x\right)^{m-\alpha-1} e^t f^{(m)}(t) dt, \quad a \leq x \leq b,$$

and

$$D_{b-;A^x}^{\alpha} f(x) = \frac{(-1)^m \ln A}{\Gamma(m-\alpha)} \int_x^b \left(A^t - A^x\right)^{m-\alpha-1} A^t f^{(m)}(t) dt, \quad a \leq x \leq b,$$

$$\left(D_{b-;x^\sigma}^{\alpha} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (t^\sigma - x^\sigma)^{m-\alpha-1} \sigma t^{\sigma-1} f^{(m)}(t) dt, \quad 0 \leq a \leq x \leq b.$$

We make:

Remark 5.8 ([1]). Here $g \in AC([a,b])$ (absolutely continuous functions), g is increasing over $[a,b]$, $\alpha > 0$. Then

$$\int_x^b (g(t) - g(x))^{\alpha-1} g'(t) dt = \frac{(g(b) - g(x))^\alpha}{\alpha}, \quad \forall\, x \in [a,b].$$

Finally we will use:

Theorem 5.9 ([1]). *Let $\alpha > 0$, $\mathbb{N} \ni m = \lceil \alpha \rceil$, and $f \in C^m([a,b])$. Then $\left(D_{b-;g}^{\alpha} f\right)(x)$ is continuous in $x \in [a,b]$, $-\infty < a < b < \infty$.*

Results

I) We notice the following ($a \leq x \leq b$):

$$\begin{aligned}\left|\left(I_{b-;g}^{\alpha} f\right)(x)\right| &\leq \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (g(t)-g(x))^{\alpha-1} g'(t)|f(t)|\,dt \\ &\leq \frac{\|f\|_{\infty}}{\Gamma(\alpha)} \int_{x}^{b} (g(t)-g(x))^{\alpha-1} g'(t)\,dt = \frac{\|f\|_{\infty}}{\Gamma(\alpha)} \frac{(g(b)-g(x))^{\alpha}}{\alpha} \\ &= \frac{\|f\|_{\infty}}{\Gamma(\alpha+1)} (g(b)-g(x))^{\alpha} \leq \frac{\|f\|_{\infty}}{\Gamma(\alpha+1)} (g(b)-g(a))^{\alpha}.\end{aligned}$$

In particular it holds

$$\left(I_{b-;g}^{\alpha} f\right)(b) = 0,$$

and

$$\left\|I_{b-;g}^{\alpha} f\right\|_{\infty,[a,b]} \leq \frac{(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)} \|f\|_{\infty}, \tag{5.2}$$

proving that $I_{b-;g}^{\alpha}$ is a bounded linear operator. We use:

Theorem 5.10 ([2])**.** *Let $r > 0$, $a < b$, $F \in L_{\infty}([a,b])$, $g \in AC([a,b])$ and g is strictly increasing. Consider*

$$B(s) := \int_{s}^{b} (g(t)-g(s))^{r-1} g'(t) F(t)\,dt, \quad \textit{for all } s \in [a,b].$$

Then $B \in C([a,b])$.

By Theorem 5.10, the function $\left(I_{b-;g}^{\alpha} f\right)$ is a continuous function over $[a,b]$. Consider $a < b^{*} < b$. Therefore $\left(I_{b-;g}^{\alpha} f\right)$ is also continuous over $[a,b^{*}]$. Thus, there exist $x_1, x_2 \in [a,b^{*}]$ such that

$$\begin{aligned}\left(I_{b-;g}^{\alpha} f\right)(x_1) &= \min \left(I_{b-;g}^{\alpha} f\right)(x), \\ \left(I_{b-;g}^{\alpha} f\right)(x_2) &= \max \left(I_{b-;g}^{\alpha} f\right)(x), \ \text{ where } x \in [a,b^{*}].\end{aligned}$$

We assume that

$$\left(I_{b-;g}^{\alpha} f\right)(x_1) > 0.$$

Hence

$$\left\|I_{b-;g}^{\alpha} f\right\|_{\infty,[a,b^{*}]} = \left(I_{b-;g}^{\alpha} f\right)(x_2) > 0.$$

Here it is

$$J(x) = mx, \quad m \neq 0.$$

Therefore the equation

$$Jf(x) = 0, \quad x \in [a,b^{*}], \tag{5.3}$$

has the same solutions as the equation

$$F(x) := \frac{Jf(x)}{2\left(I_{b-;g}^{\alpha} f\right)(x_2)} = 0, \quad x \in [a,b^{*}].$$

Notice that

$$I_{b-;g}^{\alpha}\left(\frac{f}{2\left(I_{b-;g}^{\alpha} f\right)(x_2)}\right)(x) = \frac{\left(I_{b-;g}^{\alpha} f\right)(x)}{2\left(I_{b-;g}^{\alpha} f\right)(x_2)} \leq \frac{1}{2} < 1, \quad x \in [a,b^{*}].$$

Call

$$A(x) := \frac{\left(I_{b-;g}^{\alpha} f\right)(x)}{2\left(I_{b-;g}^{\alpha} f\right)(x_2)}, \quad \forall\, x \in [a, b^*].$$

We notice that

$$0 < \frac{\left(I_{b-;g}^{\alpha} f\right)(x_1)}{2\left(I_{b-;g}^{\alpha} f\right)(x_2)} \le A(x) \le \frac{1}{2}, \;\; \forall\, x \in [a, b^*].$$

We observe

$$|1 - A(x)| = 1 - A(x) \le 1 - \frac{\left(I_{b-;g}^{\alpha} f\right)(x_1)}{2\left(I_{b-;g}^{\alpha} f\right)(x_2)} =: \gamma_0, \quad \forall\, x \in [a, b^*].$$

Clearly $\gamma_0 \in (0,1)$.

I.e.

$$|1 - A(x)| \le \gamma_0, \quad \forall\, x \in [a, b^*], \, \gamma_0 \in (0,1).$$

Next we assume that $F(x)$ is a contraction, i.e.

$$|F(x) - F(y)| \le \lambda |x - y|; \quad \forall\, x, y \in [a, b^*],$$

and $0 < \lambda < \frac{1}{2}$. Equivalently we have

$$|Jf(x) - Jf(y)| \le 2\lambda \left(I_{b-;g}^{\alpha} f\right)(x_2) |x - y|, \quad \text{all } x, y \in [a, b^*].$$

We observe that

$$\begin{aligned} |F(y) - F(x) - A(x)(y - x)| &\le |F(y) - F(x)| + |A(x)| |y - x| \le \lambda |y - x| + |A(x)| |y - x| \\ &= (\lambda + |A(x)|) |y - x| =: (\psi_1), \;\; \forall\, x, y \in [a, b^*]. \end{aligned}$$

By (5.2) we get

$$\left|\left(I_{b-;g}^{\alpha} f\right)(x)\right| \le \frac{\|f\|_\infty}{\Gamma(\alpha + 1)} (g(b) - g(a))^{\alpha}, \quad \forall\, x \in [a, b^*].$$

Hence

$$|A(x)| = \frac{\left|\left(I_{b-;g}^{\alpha} f\right)(x)\right|}{2\left(I_{b-;g}^{\alpha} f\right)(x_2)} \le \frac{\|f\|_\infty (g(b) - g(a))^{\alpha}}{2\Gamma(\alpha + 1)\left(I_{b-;g}^{\alpha} f\right)(x_2)} < \infty, \quad \forall\, x \in [a, b^*].$$

Therefore we get

$$(\psi_1) \le \left(\lambda + \frac{\|f\|_\infty (g(b) - g(a))^{a}}{2\Gamma(\alpha + 1)\left(I_{b-;g}^{\alpha} f\right)(x_2)}\right) |y - x|, \quad \forall\, x, y \in [a, b^*].$$

Call

$$0 < \gamma_1 := \lambda + \frac{\|f\|_\infty (g(b) - g(a))^{a}}{2\Gamma(\alpha + 1)\left(I_{b-;g}^{\alpha} f\right)(x_2)},$$

choosing $(g(b) - g(a))$ small enough we can make $\gamma_1 \in (0,1)$. We have proved that

$$|F(y) - F(x) - A(x)(y - x)| \le \gamma_1 |y - x|, \;\; \forall\, x, y \in [a, b^*], \, \gamma_1 \in (0,1).$$

Next we call and we need that

$$0<\gamma:=\gamma_0+\gamma_1=1-\frac{\left(I^{\alpha}_{b-;g}f\right)(x_1)}{2\left(I^{\alpha}_{b-;g}f\right)(x_2)}+\lambda+\frac{\|f\|_{\infty}\left(g(b)-g(a)\right)^{a}}{2\Gamma(\alpha+1)\left(I^{\alpha}_{b-;g}f\right)(x_2)}<1,$$

$$\lambda+\frac{\|f\|_{\infty}\left(g(b)-g(a)\right)^{a}}{2\Gamma(\alpha+1)\left(I^{\alpha}_{b-;g}f\right)(x_2)}<\frac{\left(I^{\alpha}_{b-;g}f\right)(x_1)}{2\left(I^{\alpha}_{b-;g}f\right)(x_2)},$$

equivalently,

$$2\lambda\left(I^{\alpha}_{b-;g}f\right)(x_2)+\frac{\|f\|_{\infty}\left(g(b)-g(a)\right)^{a}}{\Gamma(\alpha+1)}<\left(I^{\alpha}_{b-;g}f\right)(x_1),$$

which is possible for small λ, and small $(g(b)-g(a))$. That is $\gamma\in(0,1)$. So our method solves (5.3).

II) Let $\alpha\notin\mathbb{N}$, $\alpha>0$ and $\lceil\alpha\rceil=m$, $a<b^*<b$, $G\in AC^m([a,b])$, with $0\neq G^{(m)}\in L_{\infty}([a,b])$. Here we consider the right generalized (Caputo type) fractional derivative:

$$\left(D^{\alpha}_{b-;g}G\right)(x)=\frac{(-1)^m}{\Gamma(m-\alpha)}\int_x^b\left(g(t)-g(x)\right)^{m-\alpha-1}g'(t)\,G^{(m)}(t)\,dt,$$

for any $x\in[a,b]$.

By Theorem 5.10 we get that $\left(D^{\alpha}_{b-;g}G\right)\in C([a,b])$, in particular $\left(D^{\alpha}_{b-;g}G\right)\in C([a,b^*])$. Here notice that $\left(D^{\alpha}_{b-;g}G\right)(b)=0$.

Therefore there exist $x_1,x_2\in[a^*,b]$ such that $D^{\alpha}_{b-;g}G(x_1)=\min D^{\alpha}_{b-;g}G(x)$, and $D^{\alpha}_{b-;g}G(x_2)=\max D^{\alpha}_{b-;g}G(x)$, for $x\in[a,b^*]$.
We assume that

$$D^{\alpha}_{b-;g}G(x_1)>0.$$

(i.e. $D^{\alpha}_{b-;g}G(x)>0$, $\forall\,x\in[a,b^*]$).
Furthermore

$$\left\|D^{\alpha}_{b-;g}G\right\|_{\infty,[a,b^*]}=D^{\alpha}_{b-;g}G(x_2).$$

Here it is

$$J(x)=mx,\ m\neq 0.$$

The equation

$$JG(x)=0,\quad x\in[a,b^*],\tag{5.4}$$

has the same set of solutions as the equation

$$F(x):=\frac{JG(x)}{2D^{\alpha}_{b-;g}G(x_2)}=0,\quad x\in[a,b^*].$$

Notice that

$$D^{\alpha}_{b-;g}\left(\frac{G(x)}{2D^{\alpha}_{b-;g}G(x_2)}\right)=\frac{D^{\alpha}_{b-;g}G(x)}{2D^{\alpha}_{b-;g}G(x_2)}\leq\frac{1}{2}<1,\quad\forall\,x\in[a,b^*].$$

We call

$$A(x):=\frac{D^{\alpha}_{b-;g}G(x)}{2D^{\alpha}_{b-;g}G(x_2)},\quad\forall\,x\in[a,b^*].$$

We notice that

$$0<\frac{D^{\alpha}_{b-;g}G(x_1)}{2D^{\alpha}_{b-;g}G(x_2)}\leq A(x)\leq\frac{1}{2}.$$

Hence it holds

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{D^{\alpha}_{b-;g} G(x_1)}{2D^{\alpha}_{b-;g} G(x_2)} =: \gamma_0, \quad \forall\, x \in [a, b^*].$$

Clearly $\gamma_0 \in (0,1)$. We have proved that

$$|1 - A(x)| \leq \gamma_0 \in (0,1), \quad \forall\, x \in [a, b^*].$$

Next we assume that $F(x)$ is a contraction over $[a, b^*]$, i.e.

$$|F(x) - F(y)| \leq \lambda |x - y|; \quad \forall\, x, y \in [a, b^*],$$

and $0 < \lambda < \frac{1}{2}$. Equivalently we have

$$|JG(x) - JG(y)| \leq 2\lambda \left(D^{\alpha}_{b-;g} G(x_2)\right) |x - y|, \quad \forall\, x, y \in [a, b^*].$$

We observe that

$$\begin{aligned} |F(y) - F(x) - A(x)(y - x)| &\leq |F(y) - F(x)| + |A(x)| |y - x| \\ &\leq \lambda |y - x| + |A(x)| |y - x| \\ &= (\lambda + |A(x)|) |y - x| =: (\xi_2), \quad \forall\, x, y \in [a, b^*]. \end{aligned}$$

We observe that

$$\begin{aligned} \left|D^{\alpha}_{b-;g} G(x)\right| &\leq \frac{1}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) \left|G^{(m)}(t)\right| dt \\ &\leq \frac{1}{\Gamma(m-\alpha)} \left(\int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t)\, dt\right) \left\|G^{(m)}\right\|_{\infty} \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{(g(b) - g(x))^{m-\alpha}}{(m-\alpha)} \left\|G^{(m)}\right\|_{\infty} \\ &= \frac{1}{\Gamma(m-\alpha+1)} (g(b) - g(x))^{m-\alpha} \left\|G^{(m)}\right\|_{\infty} \leq \frac{(g(b) - g(a))^{m-\alpha}}{\Gamma(m-\alpha+1)} \left\|G^{(m)}\right\|_{\infty}. \end{aligned}$$

That is

$$\left|D^{\alpha}_{b-;g} G(x)\right| \leq \frac{(g(b) - g(a))^{m-\alpha}}{\Gamma(m-\alpha+1)} \left\|G^{(m)}\right\|_{\infty} < \infty, \quad \forall\, x \in [a, b].$$

Hence, $\forall\, x \in [a, b^*]$ we get that

$$|A(x)| = \frac{\left|D^{\alpha}_{b-;g} G(x)\right|}{2D^{\alpha}_{b-;g} G(x_2)} \leq \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\|G^{(m)}\right\|_{\infty}}{D^{\alpha}_{b-;g} G(x_2)} < \infty.$$

Consequently we observe

$$(\xi_2) \leq \left(\lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\|G^{(m)}\right\|_{\infty}}{D^{\alpha}_{b-;g} G(x_2)}\right) |y - x|, \quad \forall\, x, y \in [a, b^*].$$

Call

$$0 < \gamma_1 := \lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\|G^{(m)}\right\|_{\infty}}{D^{\alpha}_{b-;g} G(x_2)},$$

choosing $(g(b) - g(a))$ small enough we can make $\gamma_1 \in (0,1)$. We proved that

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \quad \text{where } \gamma_1 \in (0,1), \ \forall\, x, y \in [a, b^*].$$

Next we call and need

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{D^{\alpha}_{b-;g} G(x_1)}{2 D^{\alpha}_{b-;g} G(x_2)} + \lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\| G^{(m)} \right\|_{\infty}}{D^{\alpha}_{b-;g} G(x_2)} < 1,$$

equivalently we find,

$$\lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\| G^{(m)} \right\|_{\infty}}{D^{\alpha}_{b-;g} G(x_2)} < \frac{D^{\alpha}_{b-;g} G(x_1)}{2 D^{\alpha}_{b-;g} G(x_2)},$$

equivalently,

$$2\lambda D^{\alpha}_{b-;g} G(x_2) + \frac{(g(b) - g(a))^{m-\alpha}}{\Gamma(m-\alpha+1)} \left\| G^{(m)} \right\|_{\infty} < D^{\alpha}_{b-;g} G(x_1),$$

which is possible for small λ, $(g(b) - g(a))$. That is $\gamma \in (0,1)$. Hence equation (5.4) can be solved with our presented iterative algorithms.

Conclusion: Our presented earlier semilocal fixed point iterative algorithms, see Theorem 4.3, can apply in the above two generalized fractional settings since the following inequalities have been fulfilled:

$$\|1 - A(x)\|_{\infty} \leq \gamma_0,$$

and

$$|F(y) - F(x) - A(x)(y-x)| \leq \gamma_1 |y - x|,$$

where $\gamma_0, \gamma_1 \in (0,1)$, furthermore it holds

$$\gamma = \gamma_0 + \gamma_1 \in (0,1),$$

for all $x, y \in [a, b^*]$, where $a < b^* < b$. The specific functions $A(x)$, $F(x)$ have been described above.

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