Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 506–517
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Asymptotic periodicity for a class of fractional integro-differential equations

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Communicated by C. Park

Abstract

In this paper, we are concerned with the existence and uniqueness of S-asymptotically ω-periodic solutions to a class of fractional integro-differential equations. Some sufficient conditions are established about the existence and uniqueness of S-asymptotically ω-periodic solutions to the fractional integro-differential equation by applying fixed point theorem combined with sectorial operator, where the nonlinear perturbation term f is a Lipschitz case and non-Lipschitz case.

Keywords: Fractional integro-differential equation, S-asymptotically ω-periodic, fixed point, mild solutions, sectorial operator.
2010 MSC: 34G20, 35B40, 42A75.

1. Introduction

In this article, we study the existence and uniqueness of S-asymptotically ω-periodic solutions of the following fractional integro-differential equation

$$\begin{aligned} D_t^\alpha(u(t)-g(t,u(t))) &= A(u(t)-g(t,u(t))+D_t^{\alpha-1}f(t,u(t),\tilde{K}u(t)),\\ \tilde{K}u(t) &= \int_0^t R(t-s)h(s,u(s))ds, \\ u(0) &= u_0, \quad t\geq 0, \end{aligned} \tag{1.1}$$

where $1<\alpha<2$ and $A: D(A)\subseteq \mathbb{X}\to\mathbb{X}$ is a linear densely defined operator of sectorial type on a complex Banach space $(\mathbb{X},\|\cdot\|)$, $\tilde{K}$ is a bounded linear operator, $\|R(t)\|\leq M_0e^{-\mu t}$ for $t\geq 0$ and $M_0,\ \mu$ are positive

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Received 2015-07-05

constants, $f : [0,\infty) \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, $h : [0,\infty) \times \mathbb{X} \to \mathbb{X}$ and $g : [0,\infty) \times \mathbb{X} \to \mathbb{X}$ are an S-asymptotically ω-periodic functions satisfying suitable conditions given later. The fractional derivative D_t^α is to be understood in Riemann-Liouville sense.

Recently emerged the notion of S-asymptotically ω-periodic functions have many applications in several problems like functional differential equations, integro-differential equations, fractional differential equations and fractional integro-differential equations. The concept of S-asymptotically ω-periodic function was first introduced in the literature by Henríquez and Pierri et al. in [29, 31]. In these paper the author discussed the concept of S-asymptotically ω-periodicity, studied the existence of S-asymptotically ω-periodic mild solutions of abstract differential equations, abstract neutral differential equations and in [31] the authors introduced a composition theory of such type of functions.

Due to their applications in various branches of science such as physics, mechanics, chemistry engineering, etc., fractional calculus have gained considerable attention. Significant development has been made in fractional differential equations and fractional integro-differential equations. For details, including some applications and recent results, see the monographs [22, 28, 32, 39, 40]. On the S-asymptotically ω-periodicity and fractional order equations related aspects, in [12] the authors studied the existence of S-asymptotically ω-periodic mild solutions for semilinear fractional integro-differential equations and extended these results to certain class of semilinear Volterra equations in [15]. In [4] the authors studied the existence and uniqueness of asymptotically ω-periodic and weighted S-asymptotically ω-periodic mild solutions for some classes of integro-differential equations. In[13] the authors studied S-asymptotically ω-periodic solutions of the semilinear fractional integro-differential equation in the phase space. In [17] the authors studied the existence of weighted S-asymptotically ω-periodic mild solutions for a class of abstract fractional integro-differential equation. In [14] the authors studied the existence of pseudo S-asymptotically ω-periodic mild solutions for a class of abstract fractional differential equation. In [41] the author discussed the existence of an S-asymptotically ω-periodic mild solution of semilinear fractional integro-differential equations in Banach space, where the nonlinear perturbation is S-asymptotically ω-periodic or S-asymptotically ω-periodic in the Stepanov sense. In [16] the authors proved existence and uniqueness of S-asymptotically ω-periodic mild solutions for a class of linear and semilinear fractional order differential equations by using a generalization of the semigroup theory of linear operators. In [46] the authors investigated the existence and uniqueness of asymptotically ω-periodic mild solutions to semilinear fractional integro-differential equations with Stepanov asymptotically ω-periodic coefficients.

Also in virtue of their numerous applications value in several fields of sciences and engineering, abstract integro-differential equations and fractional integro-differential equations relating to topics of interest have received much attention in recent years. Some properties of the solution for the equations relating to topics have been studied from different point of view. Abbas[1] studied existence and uniqueness of a pseudo almost automorphic solution of some nonlinear integro-differential equation in a Banach space. Mishra and Bahuguna [35] proved weighted pseudo almost automorphic solution of an integro-differential equation, with weighted Stepanov-like pseudo almost automorphic forcing term in a Banach space. Xia [42] investigated weighted pseudo almost automorphic solutions of hyperbolic semilinear integro-differential equations in intermediate Banach spaces. Mophou [36] discussed the existence and uniqueness of weighted pseudo almost automorphic mild solution to the semilinear fractional equation.

For further literature concerning topic we refer the reader to [2, 3, 6–10, 19–21, 23–27, 30, 33, 37, 38, 43–45]. The topic about the existence of S-asymptotically ω- periodic solutions for fractional integral-differential equation (1.1) is untreated in the literature, which is one of the key motivations of this study.

This work is organized as follows. In Section 2, we recall some preliminary facts which will be used throughout this paper. In Section 3, we establish some sufficient conditions for S-asymptotically ω-periodic solutions to the problem (1.1).

2. Preliminaries

In this section, we introduce some preliminary results needed in what follows.

Throughout this paper, we assume that $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are two Banach spaces. We let $C([0,\infty),\mathbb{X})$ (respectively, $C([0,\infty)\times\mathbb{Y},\mathbb{X})$) stand for the collection of all continuous functions from $[0,\infty)$ into $\mathbb{X}$ (respectively, the collection of all jointly continuous functions $f : [0,\infty)\times\mathbb{Y} \to \mathbb{X}$). $BC([0,\infty),\mathbb{X})$ (respectively, $BC([0,\infty)\times\mathbb{Y},\mathbb{X})$) denotes the class of all bounded continuous functions from $[0,\infty)$ into $\mathbb{X}$ (respectively, the class of all jointly bounded continuous functions from $[0,\infty)\times\mathbb{Y}$ into $\mathbb{X}$). Note that $BC([0,\infty),\mathbb{X})$ is a Banach space with the sup norm $\|\cdot\|_\infty$. Moreover, we denote by $B(\mathbb{X})$ the space of bounded linear operators form $\mathbb{X}$ into $\mathbb{X}$ endowed with the operator topology. In this work $C_b([0,\infty),\mathbb{X})$ denotes the Banach space consisting of all continuous and bounded functions from $[0,\infty)$ into $\mathbb{X}$ with the norm of the uniform convergence.

Definition 2.1 ([11]). A closed linear operator $(A, D(A))$ with dense domain $D(A)$ in the Banach space $\mathbb{X}$ is said to be sectorial of type $\tilde{\omega}$ and angle θ if there are constants $\tilde{\omega} \in \mathbb{R}$; $\theta \in (0, \frac{\pi}{2})$ and $M > 0$ such that its resolvent exists outside the sector

$$\tilde{\omega} + \Sigma_\theta := \{\lambda + \tilde{\omega} : \lambda \in C, \quad |arg(-\lambda)| < \theta\}, \tag{2.1}$$

$$\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda - \tilde{\omega}|}, \quad \lambda \notin \tilde{\omega} + \Sigma_\theta. \tag{2.2}$$

Definition 2.2 ([12]). Let $1 < \alpha < 2$ and also A be a closed and linear operator with a domain $D(A)$ defined on a Banach space $\mathbb{X}$. We say that A is the generator of a solution operator if there exist $\tilde{\omega} \in \mathbb{R}$ and a strongly continuous function $S_\alpha : [0,\infty) \to B(\mathbb{X})$ such that $\{\lambda^\alpha : Re\lambda > \tilde{\omega}\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad Re\lambda > \tilde{\omega}, \quad x \in \mathbb{X}.$$

From [11], if A is sectional of type $\tilde{\omega} \in \mathbb{R}$ with $0 \le \theta < \pi(1 - \alpha/2)$, then A is a generator of a solution operator given by

$$S_\alpha(t) = \frac{1}{2\pi i}\int_{\mathbb{G}} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} d\lambda, \quad t \ge 0,$$

with $\mathbb{G}$ a suitable path lying outside the sector $\tilde{\omega} + \Sigma_\theta$. Furthermore, the following lemma holds.

Lemma 2.3 ([11]). *Let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator in a complex Banach space $\mathbb{X}$, satisfying hypothesis* (2.1) *and* (2.2)*, for some $M > 0$; $\tilde{\omega} < 0$ and $0 \le \theta < \pi(1 - \alpha/2)$. Then there exists $C(\theta;\alpha) > 0$ depending solely on θ and α, such that*

$$\|S_\alpha(t)\|_{B(\mathbb{X})} \le \frac{C(\theta,\alpha)M}{1 + |\tilde{\omega}|t^\alpha}, \quad t \ge 0. \tag{2.3}$$

Definition 2.4 ([31]). A function $f \in BC([0,\infty),\mathbb{X})$ is called S-asymptotically ω-periodic if there exists ω such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. In this case we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic. The collection of all such functions will be denoted by $SAP_\omega(\mathbb{X})$.

Definition 2.5 ([31]). A continuous function $f : [0,\infty)\times\mathbb{X} \to \mathbb{X}$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for every bounded set $K^* \subset \mathbb{X}$, the set $\{f(t,x) : t \ge 0, x \in K^*\}$ is bounded and $\lim_{t\to\infty}(f(t,x) - f(t+\omega,x)) = 0$ uniformly in $x \in K^*$.

Definition 2.6 ([31]). A continuous function $f : [0,\infty)\times\mathbb{X} \to \mathbb{X}$ is said to be asymptotically uniformly continuous on bounded sets if for every $\epsilon > 0$ and every bounded set $K^* \subset \mathbb{X}$, there exist $L_{\epsilon,K^*} > 0$ and $\delta_{\epsilon,K^*} > 0$ such that $||f(t,x) - f(t,y)\| < \epsilon$ for all $t \ge L_{\epsilon,K^*}$ and all $x, y \in K^*$ with $\|x - y\| < \delta_{\epsilon,K^*}$.

Lemma 2.7 ([5])**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces, and denote by $B(\mathbb{X},\mathbb{Y})$, the space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$. Let $A \in B(\mathbb{X},\mathbb{Y})$. Then when $f \in SAP_\omega(\mathbb{X})$, we have $Af := [t \to Af(t)] \in SAP_\omega(\mathbb{Y})$.*

Definition 2.8 ([31])**.** Let $f : [0,\infty) \times \mathbb{X} \to \mathbb{X}$ be a function which uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let $u : [0,\infty) \to \mathbb{X}$ be S-asymptotically ω-periodic function. Then the Nemytskii operator $\phi(\cdot) := f(\cdot, u(\cdot))$ is S-asymptotically ω-periodic function.

Lemma 2.9. *Assume $f : [0,\infty) \times \mathbb{X} \to \mathbb{X}$ is uniformly S-asymptotically ω-periodic on bounded sets and satisfies the Lipschitz condition, that is, there exists a constant $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le L\|x-y\| \quad \forall t \ge 0, \forall x, y \in \mathbb{X}.$$

If $u \in SAP_\omega(\mathbb{X})$, then the function $t \to f(t, u(t))$ belongs to $SAP_\omega(\mathbb{X})$.

Proof. The proof is similar to the following Corollary 3.3 and simplified. □

Lemma 2.10 ([13])**.** *Assume that A is sectorial of type $\tilde{\omega} < 0$, if $f \in SAP_\omega(\mathbb{X})$ and $\Lambda f : [0,\infty) \to \mathbb{X}$ is expressed by*

$$\Lambda f(t) = \int_0^t S_\alpha(t-s) f(s) ds,$$

then $\Lambda f \in SAP_\omega(\mathbb{X})$.

The next, we list the useful compactness.

Let $h^* : [0,\infty) \to [1,\infty)$ be a continuous function such that $h^*(t) \to \infty$ as $t \to \infty$. We consider the space $C_{h^*}(\mathbb{X}) = (u \in C([0,\infty),\mathbb{X}) : \lim_{t\to\infty}(\frac{u(t)}{h^*(t)}) = 0)$ endowed with the norm $\|u\|_{h^*} = \sup_{t\ge0}(\frac{\|u(t)\|}{h^*(t)})$.

Lemma 2.11 ([18])**.** *A set $K' \subseteq C_{h^*}(\mathbb{X})$ is relatively compact in $C_{h^*}(\mathbb{X})$ if it verifies the following conditions:*

(c1) *For all $b > 0$, the set $K'_b = \left\{ u|_{[0,b]} : u \in K' \right\}$ is relatively compact in $C([0,b],\mathbb{X})$.*

(c2) $\lim_{t\to\infty}(\frac{\|u(t)\|}{h^*(t)}) = 0$ *uniformly for $u \in K'$.*

3. Main results

Definition 3.1. A continuous function $u : [0,\infty) \to \mathbb{X}$ satisfying the integral equation

$$u(t) = S_\alpha(t)(u_0 - g(0,u_0)) + g(t,u(t)) + \int_0^t S_\alpha(t-s) f(s, u(s), \tilde{K}u(s)) ds$$

is called the mild solution of the problem (1.1).

Now, we list the following basic hypotheses.

(H1) A is a sectorial operator of type $\tilde{\omega} < 0$ with $0 \le \theta < \pi(1-\alpha/2)$;

(H2) there exist constants $L_g, L_h > 0$, such that

$$\|g(t,x) - g(t,y)\| \le L_g\|x-y\|$$
$$\|h(t,x) - h(t,y)\| \le L_h\|x-y\|$$

for all $t \ge 0$ and each $x, y \in \mathbb{X}$;

(H3) there exist constants $L_{f_1}, L_{f_2} > 0$, such that

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leq L_{f_1}\|x_1 - x_2\| + L_{f_2}\|y_1 - y_2\|$$

for all $t \geq 0$ and each $x_1, y_1, x_2, y_2 \in \mathbb{X}$;

(H4) $f : [0,\infty) \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, g and $h : [0,\infty) \times \mathbb{X} \to \mathbb{X}$ are uniformly S-asymptotically ω-periodic on bounded sets;

(H5) $f : [0,\infty) \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, g and $h : [0,\infty) \times \mathbb{X} \to \mathbb{X}$ are asymptotically uniformly continuous on bounded sets function;

(H6) There exists a continuous nondecreasing function $W : [0,\infty) \to [0,\infty)$ such that $\|f(t,x,y)\| \leq W(\|x\| + \|y\|)$ for all $t \in [0,\infty)$and $x \in \mathbb{X}$.

We shall present and prove our main results.

Lemma 3.2. *Let $f : [0,\infty) \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ be a uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets function and, let $u, u_1 : [0,\infty) \to \mathbb{X}$ be an S-asymptotically ω-periodic function. Then the function $v(t) = f(t, u(t), u_1(t)) \in SAP_\omega(\mathbb{X})$.*

Proof. Since $R(u) = \{u(t) \mid t \geq 0\}, R(u_1) = \{u_1(t) \mid t \geq 0\}$ is bounded set, for each $\varepsilon > 0$ there exist constants $\delta > 0$ and $L^1_\varepsilon > 0$ such that

$$max\{\|f(t+\omega,z,z_1) - f(t,z,z_1)\|, \|f(t,x,x_1) - f(t,y,y_1)\|\} \leq \varepsilon$$

for every $t \geq L^1_\varepsilon$ and $x, y, z \in R(u),\ x_1, y_1, z_1 \in R(u_1)$ with $\|x - y\| + \|x_1 - y_1\| \leq 2\delta$. Likewise, there exists $L^2_\varepsilon > 0$ such that $\|u(t+\omega) - u(t)\| + \|u_1(t+\omega) - u_1(t)\| \leq 2\delta$, for every $t \geq L^2_\varepsilon$. Thus, for $t \geq \max\{L^1_\varepsilon, L^2_\varepsilon\}$, we obtain

$$\begin{aligned}\|f(t+\omega,u(t+\omega),u_1(t+\omega)) - f(t,u(t),u_1(t))\| &\leq \|f(t+\omega,u(t+\omega),u_1(t+\omega)) - f(t+\omega,u(t),u_1(t))\| \\ &\quad + \|f(t+\omega,u(t),u_1(t)) - f(t,u(t),u_1(t))\| \\ &\leq 2\varepsilon,\end{aligned}$$

which proves the assertion. □

Corollary 3.3. *Let $f : [0,\infty) \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ be a uniformly S-asymptotically ω-periodic on bounded sets and Lipschitzian(H3), let $u, u_1 : [0,\infty) \to \mathbb{X}$ be an S-asymptotically ω-periodic function. Then the function $v(t) = f(t, u(t), u_1(t)) \in SAP_\omega(\mathbb{X})$.*

Proof. $R(u) = \{u(t) \mid t \geq 0\}, R(u_1) = \{u_1(t) \mid t \geq 0\}$ is bounded set, which follows that $f(t, u(t), u_1(t)) \in C_b([0,\infty), \mathbb{X}, \mathbb{X})$. For $\varepsilon > 0$ there exist constant $L_\varepsilon > 0$ such that

$$\|f(t+\omega,z,z_1) - f(t,z,z_1)\| \leq \frac{\varepsilon}{3}, \quad \|u(t+\omega) - u(t)\| \leq \frac{\varepsilon}{3L_{f_1}},$$

$$\|u_1(t+\omega) - u_1(t)\| \leq \frac{\varepsilon}{3L_{f_2}}$$

for every $t \geq L_\varepsilon$ and $z \in R(u), z_1 \in R(u_1)$. Thus, for $t \geq L_\varepsilon$, we have

$$\begin{aligned}&\|f(t+\omega,u(t+\omega),u_1(t+\omega)) - f(t,u(t),u_1(t))\| \\ &\quad \leq \|f(t+\omega,u(t+\omega),u_1(t+\omega)) - f(t+\omega,u(t),u_1(t))\| \\ &\quad\quad + \|f(t+\omega,u(t),u_1(t)) - f(t,u(t),u_1(t))\| \\ &\quad \leq L_{f_1}\|u(t+\omega) - u(t)\| + L_{f_2}\|u_1(t+\omega) - u_1(t)\| \\ &\quad\quad + \|f(t+\omega,u(t),u_1(t)) - f(t,u(t),u_1(t))\| \\ &\quad \leq \varepsilon,\end{aligned}$$

which proves the assertion. □

A different Lipschitz condition is considered in the following results.

Theorem 3.4. *Assume that* (H1)-(H4) *hold. Then* (1.1) *has a unique S-asymptotically ω-periodic mild solution provided*

$$L_g + (L_{f_1} + L_{f_2} L_h \frac{M_0}{\mu}) C(\theta,\alpha) M \frac{|\tilde{\omega}|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} < 1.$$

Proof. We define the nonlinear operator Γ by the expression

$$(\Gamma u)(t) = S_\alpha(t)(u_0 - g(0,u_0)) + g(t,u(t)) + \int_0^t S_\alpha(t-s) f(s,u(s),\tilde{K}u(s))ds, t \geq 0. \tag{3.1}$$

For given $u \in SAP_\omega([0,\infty),\mathbb{X})$, it follows from Lemmas 2.7, 2.9 and Corollary 3.3 that the function $t \to f(t,u(t),\tilde{K}u(t))$, $t \to g(t,u(t))$ is in $SAP_\omega(\mathbb{X})$. Moreover, from Lemma 2.10, we deduce that $\Gamma u \in SAP_\omega([0,\infty),\mathbb{X})$, that is, Γ maps $SAP_\omega([0,\infty),\mathbb{X})$ into itself. Next, we show that the operator Γ has a unique fixed point in $SAP_\omega([0,\infty),\mathbb{X})$. Indeed, for each $t \in [0,\infty)$, $u, v \in SAP_\omega([0,\infty),\mathbb{X})$, we have

$$\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\| &= \|[g(t,u(t)) - g(t,v(t))] + \int_0^t S_\alpha(t-s)[f(s,u(s),\tilde{K}u(s)) - f(s,v(s),\tilde{K}v(s))]ds\| \\
&\leq \|g(t,u(t)) - g(t,v(t))\| \\
&\quad + \int_0^t \|S_\alpha(t-s)\|_{B(u)} \|f(s,u(s),\tilde{K}u(s)) - f(s,v(s),\tilde{K}v(s))\| ds \\
&\leq L_g \|u(t) - v(t)\| + \int_0^t \frac{C(\theta,\alpha)M}{1+|\tilde{\omega}|(t-s)^\alpha}[L_{f_1}\|u(s) - v(s)\| + L_{f_2}\|\tilde{K}u(s) - \tilde{K}v(s)\|]ds,
\end{aligned} \tag{3.2}$$

note

$$\begin{aligned}
\|\tilde{K}u(s) - \tilde{K}v(s)\| &\leq \int_0^t \|R(t-s)\| \|h(s,u(s))\| - h(s,v(s))ds \\
&\leq \int_0^t \|R(t-s)\| L_h \|u(s) - v(s)\| ds \\
&\leq \sup_{t\geq 0} \|u(t) - v(t)\| L_h \int_0^t \|R(t-s)\| ds \\
&\leq \sup_{t\geq 0} \|u(t) - v(t)\| L_h (\int_0^t \|R(s)\| ds) \\
&\leq \sup_{t\geq 0} \|u(t) - v(t)\| L_h (\int_0^t M_0 e^{-\mu s} ds) \\
&\leq \sup_{t\geq 0} \|u(t) - v(t)\| L_h (M_0 \frac{1-e^{-\mu t}}{\mu}).
\end{aligned}$$

Combining the above estimate, inequality (3.2) implies,

$$\begin{aligned}
\|(\Gamma u)(t) - (\Gamma v)(t)\| &\leq L_g \sup_{t\geq 0} \|u(t) - v(t)\| + (L_{f_1} + L_{f_2} L_h M_0 \frac{1-e^{-\mu t}}{\mu}) \sup_{t\geq 0} \|u(t) - v(t)\| \int_0^t \frac{C(\theta,\alpha)M}{1+|\tilde{\omega}|(s)^\alpha} ds \\
&\leq [L_g + (L_{f_1} + L_{f_2} L_h M_0 \frac{1-e^{-\mu t}}{\mu}) \frac{|\tilde{\omega}|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)} C(\theta,\alpha)M] \|u - v\|_\infty.
\end{aligned}$$

Hence,we have

$$\|\Gamma u - \Gamma v\|_\infty \leq [L_g + (L_{f_1} + L_{f_2} L_h M_0 \frac{1-e^{-\mu t}}{\mu}) \frac{|\tilde{\omega}|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)} C(\theta,\alpha)M] \|u - v\|_\infty,$$

which proves that Γ is a contraction we conclude that Γ has a unique fixed point in $SAP_\omega(\mathbb{X})$. The proof is complete. □

Next, we establish a local version of the previous result.

Theorem 3.5. *Assume that conditions* (H1),(H4) *hold and there are continuous and nondecreasing functions* $L_{f_1}, L_{f_2}, L_g, L_h : [0,\infty) \to [0,\infty)$, *such that for each positive number* $\tilde{R}$, *and* $x, y, x_1, y_1 \in \mathbb{X}$ *and* $\|x\|, \|x_1\|, \|y\|, \|y_1\| \leq \tilde{R}$, *one has*

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leq L_{f_1}(\tilde{R})\|x_1 - x_2\| + L_{f_2}(\tilde{R})\|y_1 - y_2\|,$$

$$\|g(t,x) - g(t,y)\| \leq L_g(\tilde{R})\|x - y\|,$$

$$\|h(t,x) - h(t,y)\| \leq L_h(\tilde{R})\|x - y\|$$

for all $t \geq 0$, *where* $L_{f_1}(0) = L_{f_2}(0) = L_g(0) = L_h(0) = 0$ *and* $f(t,0,0) = g(t,0) = h(t,0) = 0$ *for every* $t \geq 0$. *Then there exists* $\varepsilon > 0$ *such that for each* u_0 *satisfying* $\|u_0\| \leq \varepsilon$, *there is a unique S-asymptotically* ω*-periodic mild solution of* (1.1).

Proof. We choose $\tilde{R} > 0$ and $\lambda \in (0,1)$ small enough such that

$$\begin{aligned}\Theta &:= C(\theta,\alpha)M[1 + L_g(\lambda\tilde{R})]\lambda + L_g(\tilde{R}) \\ &\quad + \frac{C(\theta,\alpha)M|\tilde{\omega}|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)} L_{f_2}(\frac{M_0(1-e^{-\mu t})}{\mu}L_h(\tilde{R})\tilde{R})\frac{M_0(1-e^{-\mu t})}{\mu}L_h(\tilde{R}) < 1,\end{aligned}$$

where $C(\theta,\alpha)$ and M are the constants given in (2.3). We consider u_0 such that $\|u_0\| \leq \varepsilon$, with $\varepsilon = \lambda\tilde{R}$; we define the space $\Xi u_0 = \{u \in SAP_\omega([0,\infty),\mathbb{X}) : u(0) = u_0, \|u\|_\infty \leq \tilde{R}\}$ endowed with the metric $d(u,v) = \|u - v\|_\infty$. We also define the operator Γ on the space Ξu_0 by (3.1). Let u be in Ξu_0 in a similar way as that of proof of Theorem 3.4; we have that $\Gamma u \in SAP_\omega([0,\infty),\mathbb{X})$. Moreover, we obtain the estimate

$$\begin{aligned}\|\Gamma u(t)\| &\leq C(\theta,\alpha)M[1 + L_g(\lambda\tilde{R})]\lambda\tilde{R} + L_g(\tilde{R})\tilde{R} \\ &\quad + \frac{C(\theta,\alpha)M|\tilde{\omega}|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)} L_{f_2}(\frac{M_0(1-e^{-\mu t})}{\mu}L_h(\tilde{R})\tilde{R})\frac{M_0(1-e^{-\mu t})}{\mu}L_h(\tilde{R})\tilde{R} = \Theta\tilde{R} < \tilde{R}.\end{aligned}$$

Therefore, $\Gamma(\Xi u_0) \subset \Xi u_0$. On the other hand, for $u, v \in \Xi u_0$, we see that

$$\|\Gamma u - \Gamma v\|_\infty \leq [L_g(\tilde{R}) + \frac{C(\theta,\alpha)M|\tilde{\omega}|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)} L_{f_2}(\frac{M_0(1-e^{-\mu t})}{\mu}L_h(\tilde{R})\tilde{R})\frac{M_0(1-e^{-\mu t})}{\mu}L_h(\tilde{R})]\|u - v\|_\infty,$$

which shows that Γ is a contraction from Ξu_0 into Ξu_0. The assertion is now a consequence of the contraction mapping principle. □

In the following, we discuss the existence of S-asymptotically ω-periodic solutions to the problem (1.1) when f is not necessarily Lipschitz continuous.

Theorem 3.6. *Assume that the conditions* (H1) *and* (H4)–(H6) *hold. In addition, suppose the following properties hold:*

(i) *For each* $C \geq 0$

$$\lim_{t\to\infty}\frac{1}{h^*(t)}\int_0^t \frac{W(1 + \|\tilde{K}\|)Ch^*(s)}{1 + |\tilde{\omega}|(t-s)^\alpha}ds = 0,$$

where h^* *is the function given in* Lemma 2.11.
We set

$$\beta(C) := C(\theta,\alpha)M\int_0^t \frac{W((1 + \|\tilde{K}\|)Ch^*(s)}{1 + |\tilde{\omega}|(t-s)^\alpha})ds,$$

where $C(\theta,\alpha)$ *and* M *are constants given in* (2.3).

(ii) *There is a constant* $L_{g_1} > 0$ *such that* $\|g(t, h^*(t)x) - g(t, h^*(t)y)\| \leq L_{g_1}\|x - y\|$ *for all* $t \geq 0$ *and* $x, y \in \mathbb{X}$, h^* *is given in* Lemma 2.11.

(iii) *For each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for every* $u, v \in Ch^*(\mathbb{X})$; $\|u - v\|_{h^*} \leq \delta$ *implies that*

$$C(\theta,\alpha)M\int_0^t \frac{\|f(s,u(s),\tilde{K}u(s)) - f(s,v(s),\tilde{K}v(s))\|}{1+|\tilde{\omega}|(t-s)^\alpha}ds \leq \varepsilon,$$

for all $t \geq 0$.

(iv) $L_{g_1} + \liminf_{\xi\to\infty} \frac{\beta(\xi)}{\xi} < 1$.

(v) *For all* $a, b \geq 0$, $a < b$, *and* $r > 0$, *the set* $\{f(s, h^*(s)x, \tilde{K}(h^*(s)x)) : a \leq s \leq b,\ x \in C_{h^*}(\mathbb{X}),\ \|x\|_{h^*} \leq r\}$ *is relatively compact in* $\mathbb{X}$.

Then equation (1.1) *has an* S*-asymptotically* ω*-periodic mild solution.*

Proof. We consider the nonlinear operator $\Gamma = \Gamma_1 + \Gamma_2 : C_{h^*}(\mathbb{X}) \to C_{h^*}(\mathbb{X})$ given by

$$(\Gamma_1 u)(t) := S_\alpha(t)(u_0 - g(0,u_0)) + g(t,u(t)), \quad t \geq 0,$$
$$(\Gamma_2 u)(t) := \int_0^t S_\alpha(t-s)f(s,u(s),\tilde{K}s)ds, \quad t \geq 0.$$

We will show that the operator Γ_1 is contraction and Γ_2 is completely continuous. For the sake of convenience, we divide the proof into several steps.

(I) We show that Γ_1 is $C_{h^*}(\mathbb{X})$-valued.

For $u \in C_{h^*}(\mathbb{X})$, we have that

$$\begin{aligned}\frac{\|(\Gamma_1 u)(t)\|}{h^*(t)} &\leq \frac{1}{h^*(t)}(\|S_\alpha(t)\|(\|u_0\| + \|g(0,u_0)\| + \|g(t,u(t)) - g(t,0)\| + \|g(t,0)\|))\\ &\leq \frac{1}{h^*(t)}(\|S_\alpha(t)\|(\|u_0\| + \|g(0,u_0)\| + \|g(t,h^*(t)\frac{u(t)}{h^*(t)}) - g(t,0)\| + \|g(t,0)\|))\\ &\leq \frac{1}{h^*(t)}(\|S_\alpha(t)\|(\|u_0\| + \|g(0,u_0)\| + L_{g_1}\frac{\|u(t)\|}{h^*(t)} + \|g(\cdot,0)\|_\infty))\\ &\leq \frac{1}{h^*(t)}(C(\theta,\alpha)M(\|u_0\| + \|g(0,u_0)\| + L_{g_1}\|u\|_{h^*} + \|g(\cdot,0)\|_\infty)).\end{aligned}$$

Therefore, Γ_1 is $C_{h^*}(\mathbb{X})$-valued.

(II) Γ_1 is an L_{g_1}-contraction.

For $x, y \in C_{h^*}(\mathbb{X})$, we have that

$$\|\Gamma_1 x(t) - \Gamma_1 y(t)\| \leq \|g(t,x(t)) - g(t,y(t))\| \leq L_{g_1}\frac{\|x(s) - y(s)\|}{h^*(t)} \leq L_{g_1}\|x - y\|_{h^*}.$$

Considering that $h^*(t) \geq 1$, we get

$$\|\Gamma_1 x - \Gamma_1 y\|_{h^*} \leq L_{g_1}\|x - y\|_{h^*}.$$

By (iv), Γ_1 is an L_{g_1}-contraction.

Next we show that Γ_2 is completely continuous.

(III) For $u \in C_{h^*}(\mathbb{X})$, we have that

$$\begin{aligned}\|(\Gamma_2 u)(t)\| &\leq C(\theta,\alpha)M\int_0^t \frac{W(\|u(s)\| + \tilde{K}\|u(s)\|)}{1+|\tilde{\omega}|(t-s)^\alpha}ds\\ &\leq C(\theta,\alpha)M\int_0^t \frac{W(1+\tilde{K})\|u\|_{h^*}h^*(s)}{1+|\tilde{\omega}|(t-s)^\alpha}ds.\end{aligned}$$

It follows from condition (i) that Γ_2 is well defined.

(IV) We will show that the operator Γ_2 is continuous.
In fact,for any $\varepsilon > 0$, we choose $\delta > 0$ involved in condition (iii) . If $u, v \in C_{h^*}(\mathbb{X})$ and $\|u - v\|_{h^*} \le \delta$ then

$$(\Gamma_2 u)(t) - (\Gamma_2 v)(t) \le C(\theta,\alpha)M \int_0^t \frac{\|f(s,u(s),\tilde{K}u(s)) - f(s,v(s),\tilde{K}v(s))\|}{1 + |\tilde{\omega}|(t-s)^\alpha} ds \le \varepsilon,$$

which shows the assertion.

(V) Next, we show that Γ_2 is completely continuous.
Let $\tilde{V} = \Gamma_2(B_r^*(C_{h^*}(\mathbb{X})))$ and $\tilde{v} = \Gamma_2 u$ for $u \in B_r^*(C_{h^*}(\mathbb{X})))$. Initially, we can infer that $\tilde{V}_b(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in [0, b]$. In fact, using condition (V), we get that $\mathcal{N} := \{S_\alpha(s)f(\xi, h^*(\xi)x, \tilde{K}(h^*(\xi)x)) : 0 \le s \le t, 0 \le \xi \le t, \|x\|_{h^*} \le r\}$ is relatively compact. It is easy to see that $\tilde{V}_b(t) \subseteq t\overline{C(\mathcal{N})}$, which establishes our assertion.

$$\tilde{v}(t+s) - \tilde{v}(t) = \int_t^{t+s} S_\alpha(t+s-\xi)f(\xi,u(\xi),\tilde{K}u(\xi))d\xi + \int_0^t [S_\alpha(\xi+s) - S_\alpha(\xi)]f(t-\xi,u(t-\xi),\tilde{K}u(t-\xi))d\xi.$$

It follows that the set $\tilde{V}_b$ is equicontinuous.
For each $\varepsilon > 0$, we can take $\delta_1 > 0$ such that

$$\| \int_t^{t+s} S_\alpha(t+s-\xi)f(\xi,u(\xi),\tilde{K}u(\xi))d\xi\| \le C(\theta,\alpha)M \int_t^{t+s} \frac{W((1+\|\tilde{K}\|)Ch^*(s)}{1+|\tilde{\omega}|(t+s-\xi)^\alpha} d\xi \le \frac{\varepsilon}{2}$$

for $s \le \delta_1$. Furthermore, since $\{f(t-\xi, u(t-\xi), \tilde{K}u(t-\xi)) : 0 \le \xi \le t, u \in B_r^*(C_{h^*}(\mathbb{X}))\}$ is a relatively compact set and $S_\alpha(\cdot)$ is strongly continuous, we can choose $\delta_2 > 0$ such that $\|[S_\alpha(\xi+s) - S_\alpha(\xi)]f(t - \xi, u(t-\xi), \tilde{K}u(t-\xi))\| \le \frac{\varepsilon}{2t}$ for $s \le \delta_2$. Combining these estimates, we get $\|\tilde{v}(t+s) - \tilde{v}(t)\| \le \varepsilon$ for s small enough and independent of $u \in B_r^*(C_{h^*}(\mathbb{X}))$.
From the condition (i), we have,

$$\frac{\|\tilde{v}(t)\|}{h^*(t)} \le \frac{1}{h^*(t)}[C(\theta,\alpha)M \int_0^t \frac{W((1+\|\tilde{K}\|)Ch^*(s)}{1+|\tilde{\omega}|(t-s)^\alpha} ds] \to 0, \quad \text{as } t \to \infty.$$

From Lmma 2.11, we deduce that $\tilde{V}$ is relatively compact set in $C_{h^*}(\mathbb{X})$. Hence Γ_2 is completely continuous.

(VI) Take into account Lemmas 2.10, 3.2 and Definition 2.8, we obtain that $\Gamma_i(\overline{SAP_\omega(\mathbb{X})}) \subset \overline{SAP_\omega(\mathbb{X})}$, $i = 1, 2$. Hence, $\Gamma(\overline{SAP_\omega(\mathbb{X})}) \subset \overline{SAP_\omega(\mathbb{X})}$ and $\Gamma_2 : (\overline{SAP_\omega(\mathbb{X})}) \to \overline{SAP_\omega(\mathbb{X})}$ is completely continuous. Putting $B_r^* := B_r^*(\overline{SAP_\omega(\mathbb{X})})$, we claim that there is $r > 0$ such that $\Gamma(B_r^*) \subset B_r^*$. In fact, if we assume that this assertion is false, then for all $r > 0$ we can choose $u^r \in B_r^*$ and $t^r \ge 0$ such that $\|\Gamma u^r(t^r)\|/h(t^r) > r$. We observe that

$$\|\Gamma u^r(t^r)\| \le C(\theta,\alpha)M(\|u_0\| + \|g(0,u_0)\| + L_{g_1} r + \|g(\cdot,0)\|_\infty + C(\theta,\alpha)M \int_0^{t^r} \frac{W((1+\|\tilde{K}\|)rh^*(s)}{1+|\tilde{\omega}|(t^r-s)^\alpha} ds.$$

Thus, $1 \le L_{g_1} + \liminf_{r\to\infty} \frac{\beta(r)}{r}$, which is contrary to assumption (iv). We get that Γ_1 is a contraction on B_r^* and $\overline{\Gamma_2(B_r^*)}$ is a compact set. It follows from [34, Corollary 4.3.2] that Γ has a fixed point $u \in \overline{SAP_\omega(\mathbb{X})}$. Let $(u_n)_n$ be sequence in $SAP_\omega(\mathbb{X})$ that converges to u . We see that $(\Gamma u_n)_n$ converges to $\Gamma u = u$ uniformly in $[0, \infty)$. This implies that $u \in SAP_\omega(\mathbb{X})$, and this completes the proof.

□

Corollary 3.7. *Let condition* (H1) *hold. Assume that* $f : [0,\infty) \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ *satisfies assumption (H4) and the Hölder type condition*

$$\|f(t,x_1,x_2) - f(t,y_1,y_2)\| \le C_1(\|x_1 - y_1\|^{\varpi} + \|x_2 - y_2\|^{\varpi}),\ 0 < \varpi < 1$$

for all $t \in [0,\infty)$ *and* x_i, $y_i \in \mathbb{X}$ *for* $i = 1,2$, *where* C_1 *is a positive constant. Moreover, assume the following conditions:*

(a1) $f(t,0,0) = q$.

(a2) *For all* a, $b \in [0,\infty)$, $a < b$, *and* $r > 0$, *the set* $\{f(s,x,\tilde{K}x) : a \le s \le b, x \in C_{h^*}(\mathbb{X}), \|x\|_{h^*} \le r\}$ *is relatively compact in* $\mathbb{X}$.

(a3) $\sup_{t\in[0,\infty)} C(\theta,\alpha)M \int_0^t \frac{(1+\|\tilde{K}\|)h^*(s)^{\varpi}}{1+|\tilde{\omega}|(t-s)^{\alpha}} ds = C_2 < \infty$.

Then (1.1) *has an* S*-asymptotically* ω*-periodic mild solution.*

Proof. By the Hölder type condition, it is not difficult to see that (H4) hold. Let $C_0 = \|q\|$ and $W(\xi_1+\xi_2) = C_0 + C_1(\xi_1^{\varpi} + \xi_2^{\varpi})$, then (H6) is hold. By $(a3)$, it is easy to see that (i) in Theorem 3.5 is satisfied. To verify (iii) in Theorem 3.5, note that for each $\varepsilon > 0$, there exists $0 < \delta < (\frac{\varepsilon}{C_1C_2})^{1/\varpi}$ such that for each $u, v \in C_{h^*}([0,\infty),\mathbb{X})$,$\|u - v\|_{h^*} \le \delta$, one gets the following that

$$\int_0^t \frac{\|f(s,u(s),\tilde{K}u(s)) - f(s,v(s),\tilde{K}v(s))\|}{1+|\tilde{\omega}|(t-s)^{\alpha}} ds \le \int_0^t \frac{C_1h^*(s)^{\varpi}(\|u_1 - v_1\|^{\varpi} + \|u_2 - v_2\|)^{\varpi}}{1+|\tilde{\omega}|(t-s)^{\alpha}} ds$$
$$\le C_1C_2\delta^{\varpi} \le \varepsilon \qquad t \ge 0.$$

On the other hand, (iv) can be easily proved by applying the definition of W. Accordint to Theorem 3.5, we conclude that Eq. (1.1) has a mild solution, $u(t) \in SAP_{\omega}(\mathbb{X})$. □

Acknowledgment

The author is supported by Department of Education of Guangdong Province of China. The author is thankful to the referees for their insightful comments and suggestions, which helped to improve the quality of the manuscript considerably.

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