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On a family of surfaces with common asymptotic curve in the Galilean space $\mathbf{G}_3$

Zühal Küçükarslan Yüzbaşı

Fırat University, Faculty of Science, Department of Mathematics, 23119, Elazig, Turkey.

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Abstract

In this paper, we obtain the parametric representation for a family of surfaces through a given asymptotic curve by using the Frenet frame in the Galilean space $\mathbf{G}_3$. Necessary and sufficient conditions are given for that curve to be an isoasymptotic curve on the parametric surfaces. We also provide an example in support of our results.

Keywords: Asymptotic curve, parametric surface, Galilean space.
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1. Introduction

The classical theory of asymptotic curves on surfaces is one of the most important research topics in Differential Geometry. A curve on a surface is said to be asymptotic if the curve on a regular surface is given such that the normal curvature is zero in the asymptotic direction. This direction can only occur for non-positive (negative or zero) Gaussian curvature on the surface along the asymptotic curve [5, 14].

In [15], Wang et al. introduced the concept of family of surfaces in the Euclidean 3-space. Then, Şaffak et al. [12] defined the surfaces with a common asymptotic curve in the Minkowski 3-space. Recently, Özturk [9] gave the parametric representation of a surface through a pseudo null curve. More results about families of surfaces with common asymptotic curves can be found in [1, 6].

Galilean geometry is one of the real Cayley-Klein geometries whose motions are the Galilean transformations of classical kinematics [16]. Differential geometry of the Galilean space $\mathbf{G}_3$ was significantly developed in [11]. There are several works dealing with this topic. In [2], Dede analyzed tubular surfaces between the

Email address: `zuhal2387@yahoo.com.tr` (Zühal Küçükarslan Yüzbaşı)

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Galilean and Euclidean space. Dede et al. [3] studied parallel surfaces in the Galilean space. Ogrenmis et al. [8] obtained characterizations of the helix for a curve in the Galilean space.

The starting point of our study is to express the parametric representation of the surface through a given asymptotic curve in the Galilean space $\mathbf{G}_3$. First, we obtain the necessary and sufficient conditions for the given curve to be isoasymptotic on the parametric surface. After that, the family of parametric surfaces with common asymptotic curve in the Galilean space $\mathbf{G}_3$ is defined. Finally, we give an example to illustrate this family of surfaces.

2. Preliminaries

The Galilean space $\mathbf{G}_3$ is a Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, +)$, given in [7]. The absolute figure of the Galilean space consists of an ordered triple $\{\omega, f, I\}$ in which ω is the ideal (absolute) plane, f is the line (absolute line) in ω and I is the fixed elliptic involution of f.

A vector $\mathbf{x} = (x_1, x_2, x_3)$ is called non-isotropic if $x_1 \neq 0$. All unit isotropic vectors are of the form $\mathbf{x} = (1, x_2, x_3)$. For isotropic vectors we have $x_1 = 0$.

Definition 2.1 ([13]). Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be vectors in $\mathbf{G}_3$. The Galilean scalar product of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{cases} x_1 y_1, & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0 \\ x_2 y_2 + x_3 y_3, & \text{if } x_1 = 0 \text{ and } y_1 = 0 \end{cases}$$

Definition 2.2 ([13]). Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be vectors in $\mathbf{G}_3$. The Galilean vector product $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix} 0 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Let r be an admissible curve of the class $\mathbf{C}^\infty$ in $\mathbf{G}_3$, parameterized by the invariant parameter u, given by

$$r(u) = (u, f(u), g(u)).$$

Then the curvature $\kappa(u)$ and the torsion $\tau(u)$ of the curve r can be given by

$$\kappa(u) = \sqrt{f''(u)^2 + g''(u)^2},$$

and

$$\tau(u) = \frac{\det(r'(u), r''(u), r'''(u))}{\kappa^2(u)},$$

and the associated moving trihedron satisfies

$$\begin{cases} t(u) = r'(u) = (1, f'(u), g'(u)), \\ n(u) = \dfrac{r''(u)}{\kappa(u)} = \dfrac{1}{\kappa(u)} \left(0, f''(u), g''(u)\right), \\ b(u) = \dfrac{1}{\kappa(u)} \left(0, -g''(u), f''(u)\right), \end{cases}$$

where t, n and b are called the vectors of the tangent, principal normal and binormal of $r(u)$, respectively.

Frenet formulas are given by

$$\begin{cases} t' = \kappa n, \\ n' = \tau b, \\ b' = -\tau n, \end{cases}$$

(see [10]).

The equation of a surface in $\mathbf{G}_3$ can be given by the parametrization

$$\varphi\left(v^1, v^2\right) = \left(\varphi_1\left(v^1, v^2\right), \varphi_2\left(v^1, v^2\right), \varphi_3\left(v^1, v^2\right)\right), \ v^1, v^2 \in \mathbb{R},$$

where $\varphi_1\left(v^1, v^2\right), \varphi_2\left(v^1, v^2\right)$ and $\varphi_3\left(v^1, v^2\right) \in \mathbf{C}^3$ ([11]).

3. Surfaces with common asymptotic curve in Galilean space G_3

Let $\varphi = \varphi(u, v)$ be a parametric surface on the arc-length parametrized curve $r(u)$ in $\mathbf{G}_3$. The surface is defined by

$$\varphi(u, v) = r(u) + \left[x(u, v)\, t(u) + y(u, v)\, n(u) + z(u, v)\, b(u)\right], \tag{3.1}$$

$$L_1 \leq u \leq L_2 \text{ and } T_1 \leq v \leq T_2, \tag{3.2}$$

where $x(u, v), y(u, v)$ and $z(u, v)$ which are the values of the marching-scale functions indicate, respectively, the extension-like, flexion-like and retortion-like effects, by the point unit through time v, starting from $r(u)$, $\{t(u), n(u), b(u)\}$ is the frame associated with the curve $r(u)$ in $\mathbf{G}_3$, and the values of the marching-scale functions are C^1 functions.

Our first aim is to obtain the necessary and sufficient conditions for the curve $r(u)$ to be a parametric and asymptotic curve on the surface.

Since $r(u)$ is a parametric curve on the surface $\varphi(u, v)$, there exists a parameter $v_0 \in [T_1, T_2]$ such that

$$x(u, v_0) = y(u, v_0) = z(u, v_0) = 0, \ L_1 \leq u \leq L_2 \text{ and } T_1 \leq v \leq T_2.$$

Moreover, the curve $r(u)$ on the surface $\varphi(u, v)$ is asymptotic iff the binormal $b(u)$ of the curve $r(u)$ and the normal $\eta(u, v_0)$ of the surface $\varphi(u, v)$ at any point on the curve $r(u)$ are parallel to each other [4].

If the curve is both asymptotic and parametric on φ, then it is said to be isoasymptotic on φ.

Theorem 3.1. *The curve $r(u)$ is isoasymptotic on the surface $\varphi(u, v)$ iff the following relations are satisfied:*

$$x(u, v_0) = y(u, v_0) = z(u, v_0) = 0, \tag{3.3}$$

$$-\left(1 + x_u\right) z_v + z_u x_v = 0, \tag{3.4}$$

$$\left(1 + x_u\right) y_v - y_u x_v \neq 0. \tag{3.5}$$

Proof. Let $r(u)$ be a curve on the surface $\varphi(u, v)$ in $\mathbf{G}_3$. If $r(u)$ is parametric curve on this surface, then there exists a parameter $v = v_0$ such that $r(u) = \varphi(u, v_0)$, that is

$$x(u, v_0) = y(u, v_0) = z(u, v_0) = 0.$$

The normal $\eta(u, v)$ of the surface is given by

$$\eta(u, v) = \varphi_u \times \varphi_v, \tag{3.6}$$

where from (3.1),

$$\varphi_u = \left(1 + x_u\right) t + \left(kx + y_u - \tau z\right) n + \left(\tau y + z_u\right) b,$$

and

$$\varphi_v = x_v t + y_v n + z_v b.$$

Using (3.6), the normal $\eta(u, v)$ can be written as

$$\eta(u, v) = \left[-\left(1 + x_u\right) z_v + \left(\tau y + z_u\right) x_v\right] n + \left[\left(1 + x_u\right) y_v - \left(kx + y_u - \tau z\right) x_v\right] b,$$

and from (3.3) we get

$$\eta(u, v_0) = [-(1 + x_u) z_v + z_u x_v] n + [(1 + x_u) y_v - y_u x_v] b.$$

Given that $r(u)$ is asymptotic curve if and only if $b(u) \| \eta(u, v_0)$, we obtain

$$-(1 + x_u) z_v + z_u x_v = 0$$

and

$$(1 + x_u) y_v - y_u x_v \neq 0,$$

which completes the proof. □

The set of surfaces given by (3.1) and satisfying (3.3), (3.4) and (3.5) is called the family of surfaces with common isoasymptotic in $\mathbf{G}_3$. The marching-scale functions $x(u, v), y(u, v)$ and $z(u, v)$ can be given in two different forms:

Case 1. If we take

$$\begin{aligned} x(u, v) &= a(u) X(v), \\ y(u, v) &= b(u) Y(v), \\ z(u, v) &= c(u) Z(v), \end{aligned} \tag{3.7}$$

then the sufficient condition for which the curve $r(u)$ is isoasymptotic on the surface $\varphi(u, v)$ can be expressed as

$$\begin{aligned} &X(v_0) = Y(v_0) = Z(v_0) = 0, \\ &b(u) \neq 0, \ \frac{dY(v_0)}{dv} \neq 0, \\ &c(u) = 0, \ \frac{dZ(v_0)}{dv} = 0, \end{aligned} \tag{3.8}$$

where $a(u), b(u), c(u)$, $X(v), Y(v)$ and $Z(v)$ are C^1 functions and $a(u), b(u)$ and $c(u)$ are not identically zero.

Case 2. If we take

$$\begin{aligned} x(u, v) &= \mathbf{f}(a(u) X(v)), \\ y(u, v) &= \mathbf{g}(b(u) Y(v)), \\ z(u, v) &= \mathbf{h}(c(u) Z(v)), \end{aligned} \tag{3.9}$$

then the sufficient condition for which $r(u)$ is an isoasymptotic curve on the surface $\varphi(u, v)$ can be expressed as

$$\begin{aligned} &X(v_0) = Y(v_0) = Z(v_0) = 0, \\ &\mathbf{f}(0) = \mathbf{g}(0) = \mathbf{h}(0) = 0, \\ &b(u) \neq 0, \ \frac{dY(v_0)}{dv} \neq 0, \ \mathbf{g}'(0) \neq 0, \\ &c(u) = 0, \ \frac{dZ(v_0)}{dv} = 0, \ h'(0) = 0, \end{aligned}$$

where $a(u), b(u), c(u)$, $X(v), Y(v), Z(v)$, $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ are C^1 functions and $a(u), b(u)$ and $c(u)$ are not identically zero.

Then, we get the marching-scale functions in (3.7) and (3.9) which are general for expressing surfaces having a given curve as an isoasymptotic curve in $\mathbf{G}_3$. Also, different types of marching-scale functions can be chosen according to Theorem 3.1.

Example 3.2. Let r be parameterized by

$$r(u) = (u, \sin u, \cos u).$$

It is easy to show that

$$\begin{cases} t &= (1, \cos u, -\sin u), \\ n &= (0, -\sin u, -\cos u), \\ b &= (0, \cos u, -\sin u), \end{cases}$$

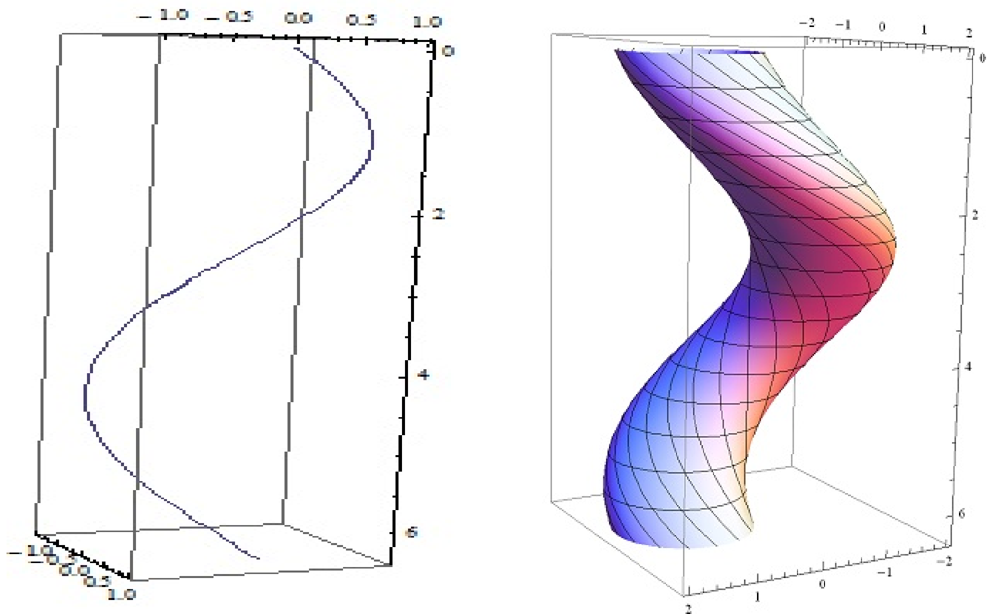
where $\kappa = 1$ is the curvature and $\tau = 1$ is the torsion of the curve in $\mathbf{G}_3$.

We will give the family of surfaces with this common isoasymptotic curve. If we choose

$$x(u,v) = 0 \ , y(u,v) = \sin v \text{ and } z(u,v) = \cos v,$$

and $v_0 = 0$ such that equation (3.8) is satisfied, a member of this family is obtained by

$$\varphi(u,v) = (u, \sin u - \sin v \sin u + \cos v \cos u, \cos u - \sin v \cos u - \cos v \sin u).$$



(a) (b)

Figure 1: (a) The curve $r(u)$, (b) A member of the family of surfaces having $r(u)$ as an asymptotic curve

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