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Existence and controllability of fractional stochastic neutral functional integro-differential systems with state-dependent delay in Fréchet spaces

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Abstract

This paper investigates the existence and uniqueness of solutions of mild solutions for a fractional stochastic neutral functional integro-differential equation with state-dependent delay in Fréchet spaces. The main techniques rely on the fractional calculus, properties of characteristic solution operators and fixed point theorems. Since we do not assume the characteristic solution operators are compact, our theorems guarantee the effectiveness of controllability results in the infinite dimensional spaces.

Keywords: Fractional neutral stochastic integro-differential equations, fractional derivatives and integrals, state-dependent delay, solution operator, fixed point theorem.
2010 MSC: 34A60, 34F05, 26A33, 93B05.

1. Introduction

In this paper we study the existence result of mild solutions for a class of fractional neutral stochastic integro-differential equations with state-dependent delay

$$dD(t, x_t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} AD(s, x_s)dsdt + f\left(t, x_{\rho(t,x_t)}\right) dw(t), \quad t \in J = [0, \infty), \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

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where the state $x(\cdot)$ takes values in a separable real Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, $1<\alpha<2$, $D(t,\varphi)=\varphi(0)-g(t,\varphi)$, $A: D(A)\subset H\to H$ is a linear densely defined operator of sectorial type on H. Suppose $\{w(t): t\geq 0\}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q>0$ defined on a complete probability space $(\Omega,\mathcal{F},P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by the Wiener process w. We are also employing the same notation $\|\cdot\|$ for the norm $L(K;H)$, where $L(K;H)$ denotes the space of all bounded linear operators from K into H. the time history $x_t: (-\infty,0]\to X$, $x(t+\theta),\theta\leq 0$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; The initial data $\{\varphi(t): -\infty<t\leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathcal{B}$-valued random variable independent of the Wiener process w with finite second moment, and f, g, ρ are functions subject to some additional conditions.

Fractional differential equations have received considerable attention in the recent years due to their wide applications in engineering, economy and other fields. Many papers on fractional calculus, fractional differential equations have appeared. It has seen considerable development in the last decade; see the monographs [31], [40], [42], [44], the papers [6], [7], [15], [16], [20], [31], [33], [34], [39], [47] and the references therein. We notice that the convolution integral in (1.1) is known as the Riemann-Liouville fractional integral (see [10], [11]). Recently, Cuevas and Souza [12] studied S-asymptotically ω-periodic solutions for the Cauchy problem involving Riemann-Liouville fractional integral. In [13], the authors established the existence of S-asymptotically ω-periodic solutions for fractional order functional integro-differential equations with infinite delay. However, on the one hand, there has been an increasing interest in extending certain classical deterministic results to stochastic differential equations. This is due to the fact that most problems in a real life situation to which mathematical models are applicable are basically stochastic rather than deterministic. Stochastic differential equations arise naturally in characterizing many problems in physics, biology, mechanics and so on; see [14], [22], [38] and the references therein. The existence, uniqueness, stability, invariant measures, and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by many authors; see, for example, Ichikawa [29], Kotelenez [32], Govindan [21], El-Borai *et al.* [17] and Taniguchi *et al.* [46], Mahmudov [36] and the references therein. Very recently, Chang *et al.* [9] investigated the global existence of mild solutions defined on the stochastic integro-differential equation in Fréechet spaces. Some interesting in the global existence of uniqueness results for functional differential equations evolution systems with finite delay have been presented by Ouahab [41], Henderson and Ouahab [25]. Baghli and Benchohra [3], [4] considered the global existence of uniqueness results for functional differential evolution systems with infinite delay. Benchohra and Ouahab [8] studied controllability results for functional semilinear differential evolution systems in Fréchet spaces. Agarwal *et al.* [1] established the controllability of mild solutions defined on the semi-infinite positive real interval for first order semilinear functional and neutral functional differential evolution equations with infinite delay in Fréchet spaces. Motivated by the works, Our purpose in this paper is to establish the global existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with state-dependent delay in Fréchet spaces. Functional differential equations with state-dependent delay appear frequently in applications as model equations and for this reason the study of such equations has received great attention in the last few years. The existence results for partial functional differential equations with state-dependent delay; see among another works [2], [26], [27] and the references therein.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give our main results. In Section 4, an example is given to illustrate our results. Finally in Section 5, we apply the preceding technique to a control problem.

2. Problem formulation and preliminaries

In this section, we introduce some basic definitions, notations and lemmas which are used throughout this paper.

Let $(\Omega,\mathcal{F},P;\mathbb{F})(\mathbb{F}=\{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. An H-valued random variable is an $\mathcal{F}$-measurable function $x(t):\Omega\to H$ and the collection of random variables $S = \{x(t,w) : \Omega \to H | t \in J\}$ is called a stochastic process. Generally, we just write $x(t)$ instead of $x(t,w)$ and $x(t) : J \to H$ in the space of S. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $\mathrm{Tr}(Q) = \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $w(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} w_i(t) e_i$, where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w.

Let $L(K;H)$ denote the space of all bounded linear operators from K into H. For $h_1, h_2 \in L(K;H)$, we define $(h_1, h_2) = \mathrm{Tr}(h_1 Q h_2^*)$ where h_2^* is the adjoint of the operator h_2 and Q is the nuclear operator associated with the Wiener process, where $Q \in L_n^+(K)$, the space of positive nuclear operator in K. For $\psi \in L(K;H)$ we define

$$\| \psi \|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{i=1}^{\infty} \| \sqrt{\lambda_i} \psi e_i \|^2 .$$

If $\| \psi \|_Q < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K;H)$ denote the space of all Q-Hilbert-Schmidt operators ψ. The completion $L_Q(K;H)$ of $L(K;H)$ with respect to the topology induced by the norm $\| \cdot \|_Q$ where $\| \psi \|_Q^2 = (\psi, \psi)$ is a Hilbert space with the above norm topology. For more details, we refer the reader to Da Prato and Zabczyk [14].

In this paper, we assume that the phase space $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into H, and satisfying the following fundamental axioms due to Hale and Kato (see e.g., in [24]).

(A) If $x : (-\infty, \sigma + b] \to H$, $b > 0$, is such that $x|_{[\sigma,\sigma+b]} \in C([\sigma, \sigma + b], H)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:
   (i) x_t is in $\mathcal{B}$;
   (ii) $\| x(t) \| \leq \tilde{H} \| x_t \|_{\mathcal{B}}$;
   (iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$, where $\tilde{H} \geq 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous and M is locally bounded; $\tilde{H}$, K, M are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma + b]$ into $\mathcal{B}$.

(C) The space $\mathcal{B}$ is complete.

Remark 2.1. In the rest of this section, M_n and K_n are the constants $K_n = \sup\{K(t) : 0 \leq t \leq n\}$, $M_n = \sup\{M(t) : 0 \leq t \leq n\}$ for each $n \in \mathbb{N}$.

A closed and linear operator A is said to be sectorial of type ω if there exist $0 < \theta < \pi/2, M > 0$ and $\omega \in \mathbb{R}$ such that its resolvent exists outside the sector $\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C} | \arg(-\lambda) < \theta\}$ and $\| (\lambda - A)^{-1} \| \leq \frac{M}{|\lambda - \omega|}, \lambda \notin \omega + S_\theta$. Sectorial operator are well studied in the literature. For a recent reference including several examples and properties we refer the reader to Haase [23]. In order to give an operator theoretical approach we recall the following definition (cf. [12], [13]).

Definition 2.2. Let A be a closed and linear operator with domain $D(A)$ defined on a Hilbert space X. We call A the generator of a solution operator if there exist $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(H)$ such that $\{\lambda^\alpha : \ \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) dt, \mathrm{Re}(\lambda) > \omega, x \in H$. In this case, $S_\alpha(\cdot)$ is called the solution operator generated by A.

We note that, if A is sectorial of type ω with $0 < \theta < \pi(1 - \frac{\alpha}{2})$ then A is the generator of a solution operator given by

$$S_\alpha(t) = \frac{1}{2\pi} \int_{\sum} e^{-\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} d\lambda, \tag{2.1}$$

where $\sum$ is a suitable path lying outside the sector $\omega + S_\alpha$.

Cuesta [10] has proved that, if A is a sectorial operator of type $\omega < 0$, for some $M_0 > 0$ and $0 < \theta < \pi(1 - \frac{\alpha}{2})$, there is $C > 0$ such that

$$\| S_\alpha(t) \| \leq \frac{CM_0}{1 + |\omega| t^\alpha}, \ t \geq 0. \tag{2.2}$$

Let $C([0, +\infty), H)$ be the space of continuous functions from $[0, +\infty)$ into H and $B(H)$ be the space of all bounded linear operators from H into H, with the norm $\| N \| = \sup\{\| N(y) \| : \| x \| = 1\}$.

A measurable function $x : [0, +\infty) \to H$ is Bochner integrable if $\| x \|$ is Lebesgue integrable. (For details on the Bochner integral properties, see Yosida [48]). $L^1([0, +\infty), H)$ denotes the space of measurable functions $x : [0, +\infty) \to H$ which are Bochner integrable, equipped with the norm $\| x \|_{L^1} = \int_0^{+\infty} \| x(t) \| \, dt$ for all $x \in L^1(J, H)$.

Consider the space

$$B_{+\infty} = \{x : (-\infty, +\infty) \to H : x|_J \in C_{\mathcal{F}_t}(J, H) : x_0 \in L_0^2(\Omega, H)\}. \tag{2.3}$$

Let X be a Fréchet space with a family of semi-norms $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$. Let $Y \subset X$, we say that F is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M_n} > 0$ such that

$$\| y \|_n \leq \overline{M_n} \ \text{ for all } y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X_n, \| \cdot \|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by $x \sim_n y$ if and only if $\| x - y \|_n = 0$ for all $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \| \cdot \|_n)$ the quotient space, the completion of X^n with respect to $\| \cdot \|_n$. To every $Y \subset X$, we associate a sequence the $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $\text{int}_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\| \cdot \|_n$ in X^n. We assume that the family of semi-norms $\{\| \cdot \|_n\}$ verifies:

$$\| x \|_1 \leq \| x \|_2 \leq \| x \|_3 \leq \cdots \ \text{ for every } x \in X.$$

Definition 2.3. A function $f : J \times H \to L_Q(K, H)$ is said to be an L^2-Carathéodory function if it satisfies:

(i) for each $t \in J$ the function $f(t, \cdot) : H \to L_Q(K, H)$ is continuous;
(ii) for each $x \in H$ the function $f(\cdot, x) : J \to L_Q(K, H)$ is $\mathcal{F}_t$-measurable;
(iii) for every positive integer k there exists $h_k \in L^1_{\text{loc}}(J, \mathbb{R}^+)$ such that

$$E \| f(t, x) \|^2 \leq h_k(t) \quad \text{for all } E \| x \|^2 \leq k$$

and for almost all $t \in J$.

The next result is a consequence of the phase space axioms.

Lemma 2.4. *Let $x : (-\infty, n] \to H$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(t) \in L_2^0(\Omega, \mathcal{B})$ and $x|_J \in B_{+\infty}$, then*

$$\| x_s \|_{\mathcal{B}} \leq M_n E \| \varphi \|_{\mathcal{B}} + K_n \sup_{0 \leq s \leq n} E \| x(s) \| .$$

Lemma 2.5 (Nonlinear Alternative of Granas-Frigon, [18])**.** *Let X be a Fréchet space and $Y \subset X$ a closed subset and $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements hold:*

(a) N has a fixed point;
(b) there exists $\lambda \in [0, 1)$, $n \in \mathbb{N}$, and $x \in \partial_n Y^n$ such that $\| x - \lambda N(x) \|_n = 0$.

3. Existence results

Definition 3.1. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, +\infty) \to H$ is called a mild solution of the system (1.1)–(1.2) if $x_0 = \varphi(t), x_{\rho(s,x_s)} \in \mathcal{B}$ satisfying $x_0 \in L_2^0(\Omega, H)$, and the restriction of $x(\cdot)$ to the interval J is continuous and satisfies the following integral equation

$$x(t) = S_\alpha(t)[\varphi(0) - g(0,\varphi)] + g(t, x_t) + \int_0^t S_\alpha(t-s) f\left(s, x_{\rho(s,x_s)}\right) dw(s), \quad t \in J.$$

Assume that $\rho : [0,n] \times \mathcal{B} \to (-\infty, n]$ is continuous. In addition, we assume the following hypotheses:

(H1) There exists $M > 0$ such that
$$\| S_\alpha(t) \|^2 \leq M \text{ for each } t \geq 0.$$

(H2) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-) = \{\rho(s,\psi) \leq 0, (s,\psi) \in [0,n] \times \mathcal{B}\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\| \varphi_t \| \leq J^\varphi(t) \| \varphi \|_{\mathcal{B}}$ for each $t \in \mathcal{R}(\rho^-)$.

(H3) The multifunction $F : J \times \mathcal{B} \to \mathcal{P}(L_Q(K,H))$ is L^2_{loc}-Carathéodory with compact and convex values for each $x \in \mathcal{B}$ and there exist a function $p \in L^1_{\text{loc}}(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : J \to (0,\infty)$ such that
$$E \| f(t,x) \|^2 \leq p(t)\psi(\| x \|^2_{\mathcal{B}})$$
for a.e. $t \in J$ and each $x \in \mathcal{B}$.

(H4) For all $R > 0$, there exists $l_R \in L^1_{\text{loc}}(J, \mathbb{R}^+)$ such that
$$E \| f(t,x) - f(t,y) \|^2 \leq l_R(t) E \| x - y \|^2_{\mathcal{B}}$$
for each $t \in J$ and for all $x, y \in \mathcal{B}$ with $E \| x \|^2_{\mathcal{B}} \leq R$ and $E \| y \|^2_{\mathcal{B}} \leq R$.

(H5) For all $R > 0$, there exists $\Gamma_R \in L^1_{\text{loc}}(J, \mathbb{R}^+)$ such that
$$E \| g(t,x) - g(t,y) \|^2 \leq \Gamma_R(t) E \| x - y \|^2_{\mathcal{B}}$$
for each $t \in J$ and for all $x, y \in \mathcal{B}$ with $E \| x \|^2_{\mathcal{B}} \leq R$ and $E \| y \|^2_{\mathcal{B}} \leq R$.

(H6) There exist constants $c_1 \geq 0$, and $c_2 > 0$ such that
$$\| g(t,x) \| \leq c_1 \| x \|_{\mathcal{B}} + c_2$$
for $t \in J, x \in \mathcal{B}$.

(H7) For each $n \in \mathbb{N}$, there exists a constant $\beta_n > 0$ such that
$$\frac{(1 - 6K_n^2 c_1)\beta_n}{\mu_1 + 6\text{Tr}(Q) M K_n^2 \psi(\beta_n) \| p \|_{L^1_{[0,n]}}} > 1,$$
where $\mu_1 = 12MK_n^2[\tilde{H}^2 \| \varphi \|^2_{\mathcal{B}} + (c_1 \| \varphi \|^2_{\mathcal{B}} + c_2)] + 2((M_n + J_0^\varphi) \| \varphi \|_{\mathcal{B}})^2 + 6K_n^2 c_2$.

Lemma 3.2 ([26], [27]). *Let $x : (-\infty, n] \to H$ be continuous on $[0,n]$ and let $x_0 = \varphi$ If (H2) is satisfied, then*
$$\| x_s \|_{\mathcal{B}} \leq (M_n + J_0^\varphi) \| \varphi \|_{\mathcal{B}} + K_n \sup\{\| x(\theta) \|; \theta \in [0, \max\{0,s\}]\}, s \in \mathcal{R}(\rho^-) \cup [0,n],$$
where $J_0^\varphi = \sup_{t \in \mathcal{R}(\rho^-)} J^\varphi(t)$.

Remark 3.3 ([26]). Let $\varphi \in \mathcal{B}$ and $t \leq 0$. The notation φ_t represents the function defined by $\varphi_t = \varphi(t+\theta)$. Consequently, if the function $x(\cdot)$ in axiom A is such that $x_0 = \varphi$, then $x_t = \varphi_t$. We observe that φ_t is well-defined for $t < 0$ since the domain of φ is $(-\infty, 0]$.

The main result of the paper is the following theorem.

Theorem 3.4. *Let $\varphi \in L_2^0(\Omega, H)$. If the assumptions (H1)-(H7) are satisfied and*

$$2K_n^2 \sup_{t\in[0,n]} \Gamma_n(t) < 1 \tag{3.1}$$

for each $n \in \mathbb{N}$, then the problem (1.1)–(1.2) *has a unique mild solution on J.*

Proof. Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms

$$\| x \|_n := \sup\{e^{\tau L_n^*}(t) E \| x(t) \|^2 : t \in [0, n]\}, \tag{3.2}$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)ds$, and $\bar{l}_n(t) = 2\mathrm{Tr}(Q)MK_b^2 l_n(s)$ and l_n is the function from (H4). Then $B_{+\infty}$ is a Fréchet space with the family of semi-norms $\| \cdot \|_{n\in\mathbb{N}}$.

Consider the space $\mathbb{Y} = \{x \in B_{+\infty} : x(0) = \varphi(0)\}$ endowed with the uniform convergence topology ($\| \cdot \|_\infty$) and define the map $\Phi : \mathbb{Y} \to \mathbb{Y}$ by

$$(\Phi x)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ S_\alpha(t)[\varphi(0) - g(0, \varphi)] + g(t, \bar{x}_t) + \int_0^t S_\alpha(t - s)f(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s), & t \in J, \end{cases}$$

where $\bar{x} : (-\infty, 0] \to H$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on $[0, n]$.

Let $\bar{\varphi} : (-\infty, 0) \to H$ be the extension of $(-\infty, 0]$ such that $\bar{\varphi}(\theta) = \varphi(0) = 0$ on $[0, n]$ and $J_0^\varphi = \sup\{J^\varphi(s) : s \in \mathcal{R}(\rho^-)\}$. We show that Φ has a fixed point, which in turn is a mild solution of the problem (1.1)–(1.2).

Let x be a possible solution of problem (1.1)–(1.2). Given $n \in \mathbb{N}$ and $t \in [0, n]$, then

$$x(t) = S_\alpha(t)[\varphi(0) - g(0, \varphi)] + g(t, \bar{x}_t) + \int_0^t S_\alpha(t - s)f(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s).$$

This implies by the hypotheses (H3) and (H6) that, for each $t \in [0, n]$, we have

$$\begin{aligned} E \| x(t) \|^2 &\le 3E \| S_\alpha(t)[\varphi(0) - g(0, \varphi)] \|^2 + 3E \| g(t, \bar{x}_t) \|^2 \\ &\quad + 3E \left\| \int_0^t S_\alpha(t - s)f(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s) \right\|^2 \\ &\le 6M[\tilde{H}^2 \| \varphi \|_{\mathcal{B}}^2 + (c_1 \| \varphi \|_{\mathcal{B}}^2 + c_2)] + 3(c_1 \| \bar{x}_t \|_{\mathcal{B}}^2 + c_2) \\ &\quad + 3\mathrm{Tr}(Q)M \int_0^t p(s)\psi(\| \bar{x}_{\rho(s,\bar{x}_s)} \|_{\mathcal{B}}^2)ds. \end{aligned}$$

By Lemma 2.4 and Lemma 3.2, it follows that $\rho(s, \bar{x}_s) \le s, s \in [0, n]$ and

$$\| \bar{x}_{\rho(s,\bar{x}_s)} \|_{\mathcal{B}}^2 \le 2[(M_n + J_0^\varphi) \| \varphi \|_{\mathcal{B}}]^2 + 2K_n^2 \sup_{s\in[0,n]} E \| x(s) \|^2 . \tag{3.3}$$

For each $t \in [0, n]$, we have

$$\begin{aligned} E \| x(t) \|^2 &\le 6M[\tilde{H}^2 \| \varphi \|_{\mathcal{B}}^2 + (c_1 \| \varphi \|_{\mathcal{B}}^2 + c_2)] \\ &\quad + 3\left[c_1\left(2((M_n + J_0^\varphi) \| \varphi \|_{\mathcal{B}})^2 + 2K_n^2 \sup_{s\in[0,n]} E \| x(s) \|^2 \right) + c_2\right] \\ &\quad + 3\mathrm{Tr}(Q)M \int_0^t p(s)\psi\left(2((M_n + J_0^\varphi) \| \varphi \|_{\mathcal{B}})^2 + 2K_n^2 \sup_{s\in[0,n]} E \| x(s) \|^2\right)ds. \end{aligned}$$

Consider the norm of the function γ defined by

$$\gamma(t) := 2[(M_n + J_0^\varphi) \| \varphi \|_{\mathcal{B}}]^2 + 2K_n^2 \sup_{0\le s\le t} E \| x(s) \|^2,$$

with $\| \gamma \|_\infty = \sup_{0\le t\le b} \gamma(t)$. By the previous inequality, we have for $t \in [0, n]$

$$\begin{aligned}\| \gamma \|_\infty \le 12MK_n^2[\tilde{H}^2 \| \varphi \|_{\mathcal{B}}^2 + (c_1 \| \varphi \|_{\mathcal{B}}^2 + c_2)] + 2((M_n + J_0^\varphi) \| \varphi \|_{\mathcal{B}})^2 \\ + 6K_n^2(c_1 \| \gamma \|_\infty + c_2) + 6\mathrm{Tr}(Q)MK_n^2 \int_0^t p(s)\psi(\| \gamma \|_\infty)ds,\end{aligned}$$

which implies that

$$\begin{aligned}\| \gamma \|_\infty \le 12MK_n^2[\tilde{H}^2 \| \varphi \|_{\mathcal{B}}^2 + (c_1 \| \varphi \|_{\mathcal{B}}^2 + c_2)] + 2((M_n + J_0^\varphi) \| \varphi \|_{\mathcal{B}})^2 + 6K_n^2 c_2 \\ + 6K_n^2 c_1 \| \gamma \|_\infty + 6\mathrm{Tr}(Q)MK_n^2\psi(\| \gamma \|_\infty) \int_0^n p(s)ds.\end{aligned}$$

Consequently,

$$\frac{(1 - 6K_n^2 c_1) \| \gamma \|_\infty}{\mu_1 + 6\mathrm{Tr}(Q)MK_n^2\psi(\| \gamma \|_\infty) \int_0^n p(s)ds} \le 1.$$

Then by the condition (H7), there exists β_n such that $\| \gamma \|_\infty \le \beta_n$. Since $\| x \|_{B_{+\infty}} \le \| \gamma \|_\infty$ we have $\| x \|_n \le \beta_n$. Set

$$\mathbb{X} = \{x \in B_{+\infty} : \sup\{E \| x(t) \|^2 : 0 \le t \le n\} \le \beta_n + 1 \text{ for all } n \in \mathbb{N}\}.$$

Clearly, $\mathbb{X}$ is a closed subset of $B_{+\infty}$. We shall show that $\Phi : \mathbb{X} \to B_{+\infty}$, is a contraction operator. Indeed, consider $x^1, x^2 \in B_{+\infty}$. Then, by using Lemma 2.4, Lemma 3.2 and (H4), (H5) for each $t \in [0, n]$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
&E \| \Phi_1\overline{x^1}(t) - \Phi_1\overline{x^2}(t) \|^2 \\
&\le 2E \| g(t, \overline{x^1}_t) - g(t, \overline{x^2}_t) \|^2 \\
&\quad + 2E \left\| \int_0^t S_\alpha(t-s)\left[f\left(s, \overline{x^1}_{\rho(s,\overline{x^1}_s)}\right) - f\left(s, \overline{x^2}_{\rho(s,\overline{x^2}_s)}\right)\right]dw(s) \right\|^2 \\
&\le 2\Gamma_n(t) \| \overline{x^1}_t - \overline{x^2}_t \|_{\mathcal{B}}^2 + 2\mathrm{Tr}(Q)M \int_0^t l_n(s)\left[\left\| \overline{x^1}_{\rho(s,\overline{x^1}_s)} - \overline{x^2}_{\rho(s,\overline{x^2}_s)} \right\|_{\mathcal{B}}^2\right]ds \\
&\le 2\Gamma_n(t)K_n^2 \sup_{t\in[0,n]} E \| \overline{x^1}(t) - \overline{x^2}(t) \|^2 \\
&\quad + 2\mathrm{Tr}(Q)MK_b^2 \int_0^t l_n(s)E \| \overline{x^1}(s) - \overline{x^2}(s) \|^2 ds \\
&\le 2K_n^2[e^{\tau\overline{L}_n(t)}][e^{-\tau\overline{L}_n(t)} \sup_{t\in[0,n]} E \| \overline{x^1}(t) - \overline{x^2}(t) \|^2] \\
&\quad + \int_0^t [\overline{l}_n(s)e^{\tau\overline{L}_n(s)}][e^{-\tau\overline{L}_n(s)} E \| \overline{x^1}(s) - \overline{x^2}(s) \|^2]ds \\
&\le 2K_n^2[e^{\tau\overline{L}_n(t)}] \sup_{t\in[0,n]} \Gamma_n(t) \| \overline{x^1} - \overline{x^2} \|_n + \int_0^t \frac{1}{\tau}\left[e^{\tau\overline{L}_n(s)}\right]' ds \| \overline{x^1} - \overline{x^2} \|_n \\
&\le e^{\tau\overline{L}_n(t)}\left[2K_n^2 \sup_{t\in[0,n]} \Gamma_n(t) + \frac{1}{\tau}\right] \| \overline{x^1} - \overline{x^2} \|_n,
\end{aligned}$$

by using $\bar{x} = x$ on $[0, n]$. Taking supremum over t,

$$\| \Phi x^1 - \Phi x^2 \|_n \le \left[2K_n^2 \sup_{t\in[0,n]} \Gamma_n(t) + \frac{1}{\tau}\right] \| x^1 - x^2 \|_n,$$

showing that the operator Φ is a contraction for all $n \in \mathbb{N}$. From the choice of $\mathbb{X}$ there is no $x \in \partial\mathbb{X}^n$ such that $x = \Phi(x)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Frigon and Granas shows that the operator Φ has a unique fixed point x, which is the unique mild solution of the problem (1.1)–(1.2). The proof is completed. □

4. Example

Consider the following fractional stochastic neutral functional integrodifferential equation of the form

$$d\Big[z(t,x)-\int_{-\infty}^{t}a_1(t)a_2(t-s)z(t,x)\Big]=J_t^{\alpha-1}\Big(\frac{\partial^2}{\partial x^2}-y\Big)\Big[z(t,x)-\int_{-\infty}^{t}a_1(t)a_2(t-s)z(t,x)\Big]$$
$$+\int_{-\infty}^{t}a_3(t)a_4(s-t)z(s-\rho_1(t)\rho_2(\parallel z(t)\parallel),x))ds,\ t\geq 0, 0\leq x\leq\pi, \tag{4.1}$$

$$z(t,0)=z(t,\pi)=0,\ t\geq 0, \tag{4.2}$$
$$z(t,x)=\varphi(t,x),\quad -\infty\leq t\leq 0, 0\leq x\leq\pi, \tag{4.3}$$

where φ is continuous, $w(t)$ denotes a standard cylindrical Wiener process in H defined on a stochastic space $(\Omega,\mathcal{F},P)$ and let $H=L^2([0,\pi])$ with the norm $\|\cdot\|$ and define the operators $A:D(A)\subset H\to H$ is the operator given by $A\omega=\omega''-y\omega$ with the domain $D(A):=\{\omega\in H:\omega''\in H,\omega(0)=\omega(\pi)=0\}$. Then

$$A\omega=\sum_{n=1}^{\infty}n^2\langle\omega,\omega_n\rangle\omega_n,\quad \omega\in D(A),$$

where $\omega_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx), n=1,2,\ldots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $T(t), t\geq 0$ in H and A is sectorial of type $\mu=-y<0$.

Let $r\geq 0, 1\leq p<\infty$ and let $h:(-\infty,-r]\to\mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (h-5), (h-6) in the terminology of Hino *et al.* [28]. Briefly, this means that h is locally integrable and there is a non-negative, locally bounded function γ on $(-\infty,0]$ such that $h(\xi+\theta)\leq\gamma(\xi)h(\theta)$ for all $\xi\leq 0$ and $\theta\in(-\infty,-r)\setminus N_\xi$, where $N_\xi\subseteq(-\infty,-r)$ is a set whose Lebesgue measure zero. We denote by $C_r\times L^p(h,H)$ the set consists of all classes of functions $\varphi:(-\infty,0]\to H$ such that φ is continuous on $[-r,0]$, Lebesgue-measurable, and $h\parallel\varphi\parallel^p$ is Lebesgue integrable on $(-\infty,-r)$. The seminorm is given by $\parallel\varphi\parallel_{\mathcal{B}}=\sup_{-r\leq\theta\leq 0}\parallel\varphi(\theta)\parallel+(\int_{-\infty}^{-r}h(\theta)\parallel\varphi\parallel^p d\theta)^{1/p}$. The space $\mathcal{B}=\mathcal{C}_r\times L^p(h,H)$ satisfies axioms (A)-(C). Moreover, when $r=0$ and $p=2$, we can take $\tilde{H}=1$, $M(t)=\gamma(-t)^{1/2}$ and $K(t)=1+(\int_{-t}^{0}h(\tau)d\tau)^{1/2}$ for $t\geq 0$ (see [28], Theorem 1.3.8 for details).

For the phase space $\mathcal{B}=\mathcal{C}_0\times L^2(h,H)$, we have identified $\varphi(\theta)(x)=\varphi(\theta,x), (\theta,x)\in(-\infty,0]\times[0,\pi]$, let $z(t)(x)=z(t,x)$. Additionally, we will assume that

(i) The functions $\rho_i:\mathbb{R}\to[0,\infty), i=1,2$, are continuous.

(ii) The functions $a_i:\mathbb{R}^+\to\mathbb{R}, i=1,2,3$, are continuous functions with

$$L_g=\parallel a_1\parallel_\infty\Big(\int_{-\infty}^{0}\frac{(a_2(-s))^2}{h(s)}\Big)^{\frac{1}{2}}<\infty, L_f=\parallel a_3\parallel_\infty\Big(\int_{-\infty}^{0}\frac{(a_4(s))^2}{h(s)}\Big)^{\frac{1}{2}}<\infty.$$

By defining the maps $g,f:\mathbb{R}\times\mathcal{B}\to H$ by

$$\rho(t,\varphi)=\rho_1(t)\rho_2(\parallel\varphi(0)\parallel),$$

$$g(t,\varphi)(x)=\int_{-\infty}^{0}a_1(t)a_2(-s)\varphi(s)(x)ds,\ D(t,\varphi)(x)=\varphi(0)x-g(t,\varphi)(x),$$

$$f(t,\psi)(x)=\int_{-\infty}^{0}a_3(t)a_4(-s)\varphi(s)(x)ds,\ J_t^{\alpha-1}g(t)=\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}g(s)ds.$$

From these definitions, it follows that $E\parallel g\parallel^2_{L(\mathcal{B},H)}\leq L_g^2$ and $E\parallel f\parallel^2_{L(\mathcal{B},H)}\leq L_f^2$. Then the problem (4.1)–(4.3) can be written as system (1.1)–(1.2). Hence by we can conclude that the system (4.1)–(4.3) admits a unique mild solution on $[0,+\infty)$.

5. Applications to control theory

This section is devoted to an application of the argument used in previous sections to the controllability of a class of fractional neutral stochastic integro-differential equations with state-dependent delay in a Hilbert H. More precisely we consider the following problem:

$$dD(t,x_t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} AD(s,x_s)dsdt + (Bu)(t)dt + f(t,x_{\rho(t,x_t)})dw(t), \quad t \in J = [0,\infty), \tag{5.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{5.2}$$

where A, f and D are as in Section 3. Also, the control function u belongs to the spaces $L^2(J,U)$, a Banach space of admissible control functions with U, a Banach space. Further, B is a bounded linear operator from U to H. Several authors have established the controllability results for stochastic semilinear differential and integrodifferential systems in Hilbert spaces, such as [19], [35], [37] and the references therein. In case the fractional integrodifferential system has been recently studied by Balachandran and Park [5], Tai and Wang [45].

Definition 5.1. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty,+\infty) \to H$ is called a mild solution of the system (5.1)–(5.2) if $x_0 = \varphi(t), x_{\rho(s,x_s)} \in \mathcal{B}$ satisfying $x_0 \in L_2^0(\Omega,H)$, and the restriction of $x(\cdot)$ to the interval J is continuous and satisfies the following integral equation

$$x(t) = S_\alpha(t)[\varphi(0) - g(0,\varphi)] + g(t,x_t) + \int_0^t S_\alpha(t-s)(Bu)(s)ds$$
$$+ \int_0^t S_\alpha(t-s)F(s,x_{\rho(s,x_s)})dw(s), \quad t \in J.$$

Definition 5.2. The system (5.1)–(5.2) is said to be controllable on the interval J if for every initial random variable $x_0, x_1 \in L_2^0(\Omega,H)$, there exists a stochastic control $u \in L^2(J,U)$, which is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that the mild solution $x(t)$ of system (5.1)–(5.2) satisfies $x(n) = x_1$.

We give the following assumptions:

(B1) The linear operator $W : L^2([0,n],U) \to X$ defined by

$$Wu = \int_0^n S_\alpha(n-s)Bu(s)ds$$

has an induced inverse operator W^{-1} which takes values in $L^2([0,n],U)\backslash KerW$ and there exist positive constants M_1 such that $\| BW^{-1} \| \leq M_1$.

(B2) For each $n\mathbb{N}$, there exists a constant $\beta_n^* > 0$ such that

$$\frac{(1-4c_1K_n^2(1+4MM_1n^2))\beta_n^*}{\mu_2 + 8\text{Tr}(Q)MK_n^2(1+2MM_1n^2)\psi(\beta_n^*) \| p \|_{L^1_{[0,n]}}} > 1,$$

where

$$\begin{aligned}\mu_2 = {} & 8MK_n^2[\tilde{H}^2 \| \varphi \|_{\mathcal{B}}^2 + (c_1 \| \varphi \|_{\mathcal{B}}^2 + c_2)] \\ & + 8K_n^2c_2 + 32MM_1n^2K_n^2[E \| x_1 \|^2 + 2M[\tilde{H}^2 \| \varphi \|_{\mathcal{B}}^2 + (c_1 \| \varphi \|_{\mathcal{B}}^2 + c_2)] + 4c_2] \\ & + 2((M_n + J_0^\varphi) \| \varphi \|_{\mathcal{B}})^2.\end{aligned}$$

Remark 5.3. The construction of the operator W and its inverse is studied by Quinn and Carmichael in Ref. [43].

Theorem 5.4. *Let $\varphi \in L_2^0(\Omega, H)$. If the assumptions (H1)-(H7), (B1) and (B2) are satisfied and*

$$3K_n^2\left(1+2M^2M_1n^2\right)\sup_{t\in[0,n]}\Gamma_n(t)<1 \tag{5.3}$$

for each $n \in \mathbb{N}$, then the problem (5.1)–(5.2) *has a unique mild solution on J.*

Proof. Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms

$$\| x \|_n := \sup\{e^{\tau L_n^*}(t)E \| x(t) \|^2 : t \in [0,n]\}, \tag{5.4}$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)ds$, and $\bar{l}_n(t) = 3\mathrm{Tr}(Q)MK_b^2(1+2MM_1n^2)$ and l_n is the function from (H4). Then $B_{+\infty}$ is a Fréchet space with the family of semi-norms $\| \cdot \|_{n\in\mathbb{N}}$. Using the condition (B1) for each $x(\cdot)$ and each $n \in \mathbb{N}$ define the control

$$u_x^n(t) = W^{-1}\left[x_1 - S_\alpha(n)[\varphi(0)-g(0,\varphi)] - g(t,\bar{x}_n) - \int_0^n S_\alpha(n-s)f(t,x_{\rho(s,x_t)}))ds\right](t).$$

Consider the space $\mathbb{Y} = \{x \in B_{+\infty} : x(0) = \varphi(0)\}$ endowed with the uniform convergence topology ($\| \cdot \|_\infty$) and define the map $\Phi : \mathbb{Y} \to \mathbb{Y}$ by

$$(\Phi x)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ S_\alpha(t)[\varphi(0)-g(0,\varphi)] + g(t,\bar{x}_t) + \int_0^t S_\alpha(t-s)(Bu_x^n)(s)ds & \\ \quad + \int_0^t S_\alpha(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s), & t \in J, \end{cases}$$

where $\bar{x} : (-\infty, 0] \to H$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on $[0,n]$.

Let $\bar{\varphi} : (-\infty, 0) \to H$ be the extension of $(-\infty, 0]$ such that $\bar{\varphi}(\theta) = \varphi(0) = 0$ on $[0,n]$ and $J_0^\varphi = \sup\{J^\varphi(s) : s \in \mathcal{R}(\rho^-)\}$. We show that Φ has a fixed point, which in turn is a mild solution of the problem (5.1)–(5.2).

Let x be a possible solution of problem (5.1)–(5.2). Given $n \in \mathbb{N}$ and $t \in [0,n]$, then

$$\begin{aligned} x(t) = {} & S_\alpha(t)[\varphi(0)-g(0,\varphi)] + g(t,\bar{x}_t) + \int_0^t S_\alpha(t-s)(Bu_x^n)(s)ds \\ & + \int_0^t S_\alpha(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s). \end{aligned}$$

This implies by the hypotheses (H3) and (H6) that, for each $t \in [0,n]$, we have

$$\begin{aligned} E \| x(t) \|^2 \le {} & 4E \| S_\alpha(t)[\varphi(0)-g(0,\varphi)] \|^2 + 4E \| g(t,\bar{x}_t) \|^2 \\ & + 4E\left\| \int_0^t S_\alpha(t-s)(Bu_x^n)(s)ds \right\|^2 \\ & + 4E\left\| \int_0^t S_\alpha(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s) \right\|^2 \\ \le {} & 4M[\tilde{H}^2 \| \varphi \|_{\mathcal{B}}^2 + (c_1 \| \varphi \|_{\mathcal{B}}^2 + c_2)] + 4(c_1 \| \bar{x}_t \|_{\mathcal{B}}^2 + c_2) \\ & + 16MM_1n\int_0^t \Big[E \| x_1 \|^2 + 2M[\tilde{H}^2 \| \varphi \|_{\mathcal{B}}^2 + (c_1 \| \varphi \|_{\mathcal{B}}^2 + c_2)] \\ & + 4(c_1 \| \bar{x}_n \|_{\mathcal{B}}^2 + c_2) + \mathrm{Tr}(Q)M\int_0^n p(\eta)\psi(\| \bar{x}_{\rho(\eta,\bar{x}_\eta)} \|_{\mathcal{B}}^2)d\eta\Big]ds \\ & + 4\mathrm{Tr}(Q)M\int_0^t p(s)\psi(\| \bar{x}_{\rho(s,\bar{x}_s)} \|_{\mathcal{B}}^2)ds \\ \le {} & 4M[\tilde{H}^2 \| \varphi \|_{\mathcal{B}}^2 + (c_1 \| \varphi \|_{\mathcal{B}}^2 + c_2)] + 4c_2 + 16MM_1n^2\Big[E \| x_1 \|^2 + 2M[\tilde{H}^2 \| \varphi \|_{\mathcal{B}}^2 \end{aligned}$$

$$+ (c_1 \parallel \varphi \parallel_{\mathcal{B}}^2 + c_2)] + 4c_2\Big] + 4c_1 \parallel \bar{x}_t \parallel_{\mathcal{B}}^2 + 16MM_1n^2c_1 \parallel \bar{x}_n \parallel_{\mathcal{B}}^2$$
$$+ 16MM_1n^2\mathrm{Tr}(Q)M \int_0^n p(s)\psi(\parallel \bar{x}_{\rho(s,\bar{x}_s)} \parallel_{\mathcal{B}}^2)ds$$
$$+ 4\mathrm{Tr}(Q)M \int_0^t p(s)\psi(\parallel \bar{x}_{\rho(s,\bar{x}_s)} \parallel_{\mathcal{B}}^2)ds.$$

It follows that we have for each $t \in [0, n]$,

$$E \parallel x(t) \parallel^2$$
$$\leq 4M[\tilde{H}^2 \parallel \varphi \parallel_{\mathcal{B}}^2 + (c_1 \parallel \varphi \parallel_{\mathcal{B}}^2 + c_2)] + 4c_2 + 16MM_1n^2\Big[E \parallel x_1 \parallel^2 + 2M[\tilde{H}^2 \parallel \varphi \parallel_{\mathcal{B}}^2$$
$$+ (c_1 \parallel \varphi \parallel_{\mathcal{B}}^2 + c_2)] + 4c_2\Big] + 4c_1\left(2((M_n + J_0^{\varphi}) \parallel \varphi \parallel_{\mathcal{B}})^2 + 2K_n^2 \sup_{s\in[0,n]} E \parallel x(s) \parallel^2\right)$$
$$+ 16MM_1n^2c_1\left(2((M_n + J_0^{\varphi}) \parallel \varphi \parallel_{\mathcal{B}})^2 + 2K_n^2 \sup_{s\in[0,n]} E \parallel x(s) \parallel^2\right)$$
$$+ 16MM_1n^2\mathrm{Tr}(Q)M \int_0^n p(s)\psi\left(2((M_b + J_0^{\varphi}) \parallel \varphi \parallel_{\mathcal{B}})^2 + 2K_n^2 \sup_{s\in[0,n]} E \parallel x(s) \parallel^2\right)ds$$
$$+ 4\mathrm{Tr}(Q)M \int_0^t p(s)\psi\left(2((M_n + J_0^{\varphi}) \parallel \varphi \parallel_{\mathcal{B}})^2 + 2K_n^2 \sup_{s\in[0,n]} E \parallel x(s) \parallel^2\right)ds.$$

Defined γ be as in Theorem 3.4 with $\parallel \gamma \parallel_\infty = \sup_{0\leq t\leq n} \gamma(t)$. By the previous inequality, we have for $t \in [0, n]$

$$\parallel \gamma \parallel_\infty \leq 8MK_b^2[\tilde{H}^2 \parallel \varphi \parallel_{\mathcal{B}}^2 + (c_1 \parallel \varphi \parallel_{\mathcal{B}}^2 + c_2)] + 8K_n^2c_2 + 32MM_1n^2K_n^2\Big[E \parallel x_1 \parallel^2$$
$$+ 2M[\tilde{H}^2 \parallel \varphi \parallel_{\mathcal{B}}^2 + (c_1 \parallel \varphi \parallel_{\mathcal{B}}^2 + c_2)] + c_2\Big] + 2((M_n + J_0^{\varphi}) \parallel \varphi \parallel_{\mathcal{B}})^2$$
$$+ 4c_1K_n^2 \parallel \gamma \parallel_\infty + 16MM_1n^2c_1K_n^2 \parallel \gamma \parallel_\infty$$
$$+ 16MM_1n^2\mathrm{Tr}(Q)MK_n^2 \int_0^n p(s)\psi(\parallel \gamma \parallel_\infty)ds$$
$$+ 8\mathrm{Tr}(Q)MK_n^2 \int_0^t p(s)\psi(\parallel \gamma \parallel_\infty)ds,$$

which implies that

$$\parallel \gamma \parallel_\infty \leq \mu_2 + 4c_1K_n^2(1 + 4MM_1n^2) \parallel \gamma \parallel_\infty + 6K_n^2c_1 \parallel \gamma \parallel_\infty$$
$$+ 8\mathrm{Tr}(Q)MK_n^2(1 + 2MM_1n^2)\psi(\parallel \gamma \parallel_\infty) \int_0^n p(s)ds.$$

Consequently,

$$\frac{(1 - 4c_1K_b^2(1 + 4MM_1n^2)) \parallel \gamma \parallel_\infty}{\mu_2 + 8\mathrm{Tr}(Q)MK_n^2(1 + 2MM_1n^2)\psi(\parallel \gamma \parallel_\infty) \int_0^n p(s)ds} \leq 1.$$

Then by the condition (B2), there exists β_n^* such that $\parallel \gamma \parallel_\infty \leq \beta_n$. Since $\parallel x \parallel_{B_{+\infty}} \leq \parallel \gamma \parallel_\infty$ we have $\parallel x \parallel_n \leq \beta_n^*$. Set

$$\mathbb{X} = \{x \in B_{+\infty} : \sup\{E \parallel x(t) \parallel^2 : 0 \leq t \leq n\} \leq \beta_n + 1 \text{ for all } n \in \mathbb{N}\}.$$

Clearly, $\mathbb{X}$ is a closed subset of $B_{+\infty}$. We shall show that $\Phi : \mathbb{X} \to B_{+\infty}$, is a contraction operator. Indeed, consider $x^1, x^2 \in B_{+\infty}$. Then, by using Lemma 2.4 and Lemma 3.2 and (H4), (H5) for each $t \in [0, n]$ and

$n \in \mathbb{N}$ we have

$$
\begin{aligned}
&E \parallel \Phi_1\overline{x^1}(t) - \Phi_1\overline{x^2}(t) \parallel^2 \\
&\quad \leq 3E \parallel g(t, \overline{x^1}_t) - g(t, \overline{x^2}_t) \parallel^2 + 3E \left\| \int_0^t S_\alpha(t-s)BW^{-1}\Big[x_1 - S_\alpha(n)[\varphi(0) - g(0,\varphi)] \right. \\
&\quad\quad - g(n, \overline{x^1}_n) - \int_0^n S_\alpha(n-\eta) f(\eta, \overline{x^1}_{\rho(\eta,\overline{x^1}_\eta)}))dw(\eta)\Big] \\
&\quad\quad \left. - \Big[x_1 - S_\alpha(n)[\phi(0) - g(0,\varphi)] - g(n, \overline{x^2}_n) - \int_0^n S_\alpha(n-\eta) f(\tau, \overline{x^2}_{\rho(\eta,\overline{x^2}_\eta)}))dw(\eta)\Big](\eta) \right\|^2 \\
&\quad\quad + 3E \left\| \int_0^t S_\alpha(t-s)\Big[f\Big(s, \overline{x^1}_{\rho(s,\overline{x^1}_s)}\Big) - f\Big(s, \overline{x^2}_{\rho(s,\overline{x^2}_s)}\Big)\Big]dw(s) \right\|^2 \\
&\quad \leq 3\Gamma_n(t) \parallel \overline{x^1}_t - \overline{x^2}_t \parallel_{\mathcal{B}}^2 + 6MM_1 n \int_0^t \Big[E \parallel g(n, \overline{x^1}_n) - g(n, \overline{x^2}_n) \parallel^2 \\
&\quad\quad + \mathrm{Tr}(Q)M \int_0^n E \parallel f(\tau, \overline{x^1}_{\rho(\tau,\overline{x^1}_\tau)})) - f(\tau, \overline{x^2}_{\rho(\tau,\overline{x^2}_\tau)})) \parallel^2 d\tau\Big]ds \\
&\quad\quad + 3\mathrm{Tr}(Q)M \int_0^t l_n(s)\Big[\Big\| \overline{x^1}_{\rho(s,\overline{x^1}_s)} - \overline{x^2}_{\rho(s,\overline{x^2}_s)} \Big\|_{\mathcal{B}}^2\Big]ds \\
&\quad \leq 3\Gamma_n(t)K_b^2 \sup_{t\in[0,n]} E \parallel \overline{x^1}(t) - \overline{x^2}(t) \parallel^2 + 6MM_1 n^2 \Gamma_n(t) K_b^2 \sup_{t\in[0,n]} E \parallel \overline{x^1}(t) - \overline{x^2}(t) \parallel^2 \\
&\quad\quad + 6M^2 M_1 n^2 \mathrm{Tr}(Q) K_n^2 \int_0^t l_n(s) E \parallel \overline{x^1}(s) - \overline{x^2}(s) \parallel^2 ds \\
&\quad\quad + 3\mathrm{Tr}(Q) M K_n^2 \int_0^t l_n(s) E \parallel \overline{x^1}(s) - \overline{x^2}(s) \parallel^2 ds \\
&\quad \leq 3\Gamma_n(t)K_n^2(1 + 2MM_1n^2)[e^{\tau \overline{L}_n(t)}][e^{-\tau \overline{L}_n(t)} \sup_{t\in[0,n]} E \parallel \overline{x^1}(t) - \overline{x^2} \parallel^2 (t)] \\
&\quad\quad + \int_0^t [\bar{l}_n(s) e^{\tau \overline{L}_n(s)}][e^{-\tau \overline{L}_n(s)} E \parallel \overline{x^1}(s) - \overline{x^2}(s) \parallel^2] ds \\
&\quad \leq 3\Gamma_n(t)K_n^2(1 + 2MM_1n^2)[e^{\tau \overline{L}_n(s)}] \parallel \overline{x^1} - \overline{x^2} \parallel_n + \int_0^t \frac{1}{\tau}\Big[e^{\tau \overline{L}_n(s)}\Big]' ds \parallel \overline{x^1} - \overline{x^2} \parallel_n \\
&\quad \leq e^{\tau \overline{L}_n(t)} \Big[3 \sup_{t\in[0,n]} \Gamma_n(t) K_n^2 (1 + 2M^2 M_1 n^2) + \frac{1}{\tau}\Big] \parallel \overline{x^1} - \overline{x^2} \parallel_n,
\end{aligned}
$$

by using $\bar{x} = x$ on $[0, n]$. Taking supremum over t,

$$
\parallel \Phi x^1 - \Phi x^2 \parallel_n \leq \Big[3 \sup_{t\in[0,n]} \Gamma_n(t) K_n^2 (1 + 2M^2 M_1 n^2) + \frac{1}{\tau}\Big] \parallel x^1 - x^2 \parallel_n,
$$

showing that the operator Φ is a contraction for all $n \in \mathbb{N}$. From the choice of $\mathbb{X}$ there is no $x \in \partial \mathbb{X}^n$ such that $x = \Phi(x)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Frigon and Granas shows that the operator Φ has a unique fixed point x, which is the unique mild solution of the problem (5.1)–(5.2). The proof is completed. □

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References

[1] R. P. Agarwal, S. Baghli, M. Benchohra, *Controllability for semilinear functional and neutral functional evolution equations with infinite delay in Fréchet spaces*, Appl. Math. Optim., **60** (2009), 253–274. 1

[2] A. Anguraj, M. M. Arjunan, E. Hernández, *Existence results for an impulsive neutral functional differential equation with state-dependent delay*, Appl. Anal., **86** (2007), 861–872. 1

[3] S. Baghli, M. Benchohra, *Multivalued evolution equations with infinite delay in Fréchet spaces*, Electron. J. Qual. Theory Differ. Equ., **2008** (2008), 24 pages. 1

[4] S. Baghli, M. Benchohra, *Global uniqueness results for partial functional and neutral functional evolution equations with infinite delay*, Differential Integral Equations, **23** (2010), 31–50. 1

[5] K. Balachandran, J. Y. Park, *Controllability of fractional integrodifferential systems in Banach spaces*, Nonlinear Anal. Hybrid Syst., **3** (2009), 363–367. 5

[6] K. Balachandran, J. J. Trujillo, *The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces*, Nonlinear Anal., **72** (2010), 4587–4593. 1

[7] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl., **338** (2008), 1340–1350. 1

[8] M. Benchohra, A. Ouahab, *Controllability results for functional semilinear differential inclusions in Fréchet spaces*, Nonlinear Anal., **61** (2005), 405–423. 1

[9] Y.-K. Chang, Z.-H. Zhao, J. J. Nieto, *Global existence and controllability to a stochastic integro-differential equation*, Electron. J. Qual. Theory Differ. Equ., **2010** (2010), 15 pages. 1

[10] E. Cuesta, *Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations*, Discrete Contin. Dyn. Syst., **2007** (2007), 277–285. 1, 2

[11] E. Cuesta, C. Palencia, *A numerical method for an integro-differential equation with memory in Banach spaces: qualitative properties*, SIAM J. Numer. Anal., **41** (2003), 1232–1241. 1

[12] C. Cuevas, J. de Souza, *S-asymptotically ω-periodic solutions of semilinear fractional integro-differential equations*, Appl. Math. Lett., **22** (2009), 865–870. 1, 2

[13] C. Cuevas, J. de Souza, *Existence of S-asymptotically ω-periodic solutions for fractional order functional integro-differential equations with infinite delay*, Nonlinear Anal., **72** (2010), 1683–1689. 1, 2

[14] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge, (2014). 1, 2

[15] M. M. El-Borai, *Some probability densities and fundamental solutions of fractional evolution equations*, Chaos Solitons Fractals, **14** (2002), 433–440. 1

[16] M. M. El-Borai, *Semigroups and some nonlinear fractional differential equations*, Appl. Math. Comput., **149** (2004), 823–831. 1

[17] M. M. El-Borai, O. L. Mostafa, H. M. Ahmed, *Asymptotic stability of some stochastic evolution equations*, Appl. Math. Comput., **144** (2003), 273–286. 1

[18] M. Frigon, A. Granas, *Resultats de type Leray-Schauder pour des contractions sur des espaces de Fréchet*, Ann. Sci. Math. Quebec, **22** (1998), 161–168. 2.5

[19] X. Fu, K. Mei, *Approximate controllability of semilinear partial functional differential systems*, J. Dyn. Control Syst., **15** (2009), 425–443. 5

[20] W. G. Glockle, T. F. Nonnemacher, *A fractional calculus approach of self-similar protein dynamics*, Biophys. J., **68** (1995), 46–53. 1

[21] T. E. Govindan, *Stability of mild solutions of stochastic evolution equations with variable delay*, Stochastic Anal. Appl., **21** (2003), 1059–1077. 1

[22] W. Grecksch, C. Tudor, *Stochastic evolution equations: a Hilbert space approach*, Akademic-Verlag, Berlin, (1995). 1

[23] M. Haase, *The functional calculus for sectorial operators*, *Birkhäuser*-Verlag, Basel, (2006). 2

[24] J. K. Hale, J. Kato, *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac., **21** (1978), 11–41. 2

[25] J. Henderson, A. Ouahab, *Existence results for nondensely defined semilinear functional differential inclusions in Fréchet spaces*, Electron. J. Qual. Theory Differ. Equ., **2005** (2005), 17 pages. 1

[26] E. Hernández, L. Ladeira, A. Prokopczyk, *A note on state dependent partial functional differential equations with unbounded delay*, Nonlinear Anal., **7** (2006), 510–519. 1, 3.2, 3.3

[27] E. Hernández, M. McKibben, *On state-dependent delay partial neutral functional-differential equations*, Appl. Math. Comput., **186** (2007), 294–301. 1, 3.2

[28] Y. Hino, S. Murakami, T. Naito, *Functional-differential equations with infinite delay*, Springer-Verlag, Berlin, (1991). 4

[29] A. Ichikawa, *Stability of semilinear stochastic evolution equations*, J. Math. Anal. Appl., **90** (1982), 12–44. 1

[30] O. K. Jaradat, A. Al-Omari, S. Momani, *Existence of the mild solution for fractional semilinear initial value problems*, Nonlinear Anal., **69** (2008), 3153–3159.

[31] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B.V., Amsterdam, (2006). 1

[32] P. Kotelenez, *On the semigroup approach to stochastic evolution equations, in stochastic space time models and limit theorems*, Springer, Netherlands, (1985). 1
[33] V. Lakshmikantham, *Theory of fractional functional differential equations*, Nonlinear Anal., **69** (2008), 3337–3343. 1
[34] V. Lakshmikantham, A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal., **69** (2008), 2677–2682. 1
[35] N. I. Mahmudov, *Controllabiltiy of linear stochastic systems in Hilbert spaces*, J. Math. Anal. Appl., **259** (2001), 64–82. 5
[36] N. I. Mahmudov, *Existence and uniqueness results for neutral SDEs in Hilbert spaces*, Stoch. Anal. Appl., **24** (2006), 79–95. 1
[37] N. I. Mahmudov, A. Denker, *On controllability of linear stochastic systems*, Internat. J. Control, **73** (2000), 144–151. 5
[38] X. Mao, *Stochastic differential equations and their applications*, Horwood Publishing Limited, Chichester, (1997). 1
[39] R. Metzler, W. Schick, H. G. Kilian, T. F. Nonnemacher, *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phys., **103** (1995), 7180–7186. 1
[40] K. S. Miller, B. Ross, *An introduction to the fractional calculus and differential equations*, John Wiley & Sons, Inc., NewYork, (1993). 1
[41] A. Ouahab, *Local and global existence and uniqueness results for impulsive functional differential equations with multiple delay*, J. Math. Anal. Appl., **323** (2006), 456–472. 1
[42] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, (1999). 1
[43] M. D. Quinn, N. Carmichael, *An approach to nonlinear control problems using fixed-point methods, degree theory and pseudo-inverses*, Numer. Funct. Anal. Optim., **7** (1984), 197–219. 5.3
[44] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, Yverdon, (1993). 1
[45] Z. Tai, X. Wang, *Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces*, Appl. Math. Lett., **22** (2009), 1760–1765. 5
[46] T. Taniguchi, K. Liu, A. Truman, *Existence, uniqueness and asymptotic behavior of mild solutions to stochastic functional differential equations in Hilbert spaces*, J. Differential Equations, **181** (2002), 72–91. 1
[47] Z. Yan, *Existence results for fractional functional integrodifferential equations with nonlocal conditions in Banach spaces*, Ann. Polon. Math., **97** (2010), 285–299. 1
[48] K. Yosida, *Functional analysis*, Springer-Verlag, Berlin, (1995). 2