



Permanence and almost periodic solutions for a single-species system with impulsive effects on time scales

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Abstract

In this paper, we first propose a single-species system with impulsive effects on time scales and by establishing some new comparison theorems of impulsive dynamic equations on time scales, we obtain sufficient conditions to guarantee the permanence of the system. Then we prove a Massera type theorem for impulsive dynamic equations on time scales and based on this theorem, we establish a criterion for the existence and uniformly asymptotic stability of a unique positive almost periodic solution of the system. Finally, we give an example to show the feasibility of our main results. Our example also shows that the continuous time system and its corresponding discrete time system have the same dynamics. Our results of this paper are completely new. ©2016 All rights reserved.

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1. Introduction

In 1978, Ludwig et al. [8] introduced the following single-species system:

$$x'(t) = x(t)[a - bx(t)] - h(x),$$

where $x(t)$ is the density of species x at time t , a and b are the intrinsic growth rate and self-inhibition rate, respectively. The $h(x)$ -term represents predation. Predation is an increasing function and usually saturates for large enough x . If the density of species x is small, then the predation term $h(x)$ drops rapidly. To

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investigate the effects of other specific forms of $h(x)$, Murray [10] took $h(x) = \frac{\alpha x^2(t)}{\beta + x^2(t)}$ and the authors of [13, 14] took $h(x) = \frac{cx(t)}{d+x(t)}$.

Since, in reality, many natural and man-made factors (e.g., fire, drought, flooding deforestation, hunting, harvesting, breeding etc.) always lead to rapid decrease or increase of population number at fixed times—such sudden changes can often be characterized mathematically in the form of impulses, the authors of [14] considered the following single-species system governed by the impulsive differential equation:

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t)] - \frac{c(t)x(t)}{d(t) + x(t)}, & t \neq t_k, \\ x(t_k^+) = (1 + \lambda_k)x(t_k), & t = t_k, \quad k \in \mathbb{N}, \end{cases} \quad (1.1)$$

where $x(0) > 0$, t_k is an impulsive point for every k and $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots$, and \mathbb{N} is the set of positive integers, the coefficients $a(t)$, $b(t)$, $c(t)$, $d(t)$ are positive continuous T -periodic functions for $t \geq 0$, the jump condition reflects the possibility of impulsive perturbations on species x , $\{\lambda_k\}$ is assumed to be a real sequence with $\lambda_k > -1$ and there exists an integer $q > 0$ such that $\lambda_{k+q} = \lambda_k$, $t_{k+q} = t_k + T$. By using Brouwer's fixed point theorem and the Lyapunov function, sufficient conditions for the existence and global asymptotic stability of positive periodic solutions of the system were derived.

It is well known that biological and environmental parameters are naturally subject to fluctuation in time, the effects of a periodically or almost periodically varying environment are considered as important selective forces on systems in a fluctuating environment. Therefore, on the one hand, models should take into account the seasonality of the periodically changing environment. However, on the other hand, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. For this reason, the assumption of almost periodicity is more realistic, more important and more general when we consider the effects of the environmental factors. Also, at present, few results are available for the existence of positive almost periodic solutions to population models with impulses.

Meanwhile, discrete time models governed by difference equations are very important in implementation and applications, so it is significant to study the discrete time models. As we know, the study of dynamical systems on time scales can unify and extend continuous and discrete analysis [2], which is now an active area of research. In recent years, a variety of dynamic equations on time scales have been investigated (see [1, 2, 3, 4, 5, 6, 11, 12, 17]). However, only few papers [7, 15, 16] published on the permanence for dynamic equation models on time scales, and up to now, there is no paper published on the permanence for impulsive dynamic equation models on time scales. Thus, it is worthwhile continuing to study the single-species system with impulsive effects on time scales.

Motivated by the above reasons, in this paper, we are concerned with the following single-species system with impulsive effects on time scales:

$$\begin{cases} x^\Delta(t) = a(t) - b(t)e^{x(t)} - \frac{c(t)}{d(t) + m(t)e^{x(t)}}, & t \neq t_k, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) = x(t_k) \ln(1 + \lambda_k), & t = t_k, \quad k \in \mathbb{N}, \end{cases} \quad (1.2)$$

where \mathbb{T} is an almost periodic time scale, $0 \leq t_0 \in \mathbb{T}$.

Remark 1.1. Let $y(t) = e^{x(t)}$. If $\mathbb{T} = \mathbb{R}$, then system (1.2) is reduced to the following system:

$$\begin{cases} y'(t) = y(t)[a(t) - b(t)y(t)] - \frac{c(t)y(t)}{d(t) + m(t)y(t)}, & t \neq t_k, \quad t \in [t_0, +\infty), \\ y(t_k^+) = (1 + \lambda_k)y(t_k), & t = t_k, \quad k \in \mathbb{N}, \end{cases} \quad (1.3)$$

let $m(t) \equiv 1$, then system (1.2) is reduced to system (1.1) and if $\mathbb{T} = \mathbb{Z}$, then system (1.2) is reduced to the

following system:

$$\begin{cases} y(n+1) = y(n) \exp \left\{ a(n) - b(n)y(n) - \frac{c(n)}{d(n) + m(n)y(n)} \right\}, & n \neq n_k, \\ y(n_k^+) = (1 + \lambda_k)y(n_k), & n = n_k, \quad k \in \mathbb{N}, \quad n \in [t_0, +\infty)_{\mathbb{Z}}. \end{cases} \quad (1.4)$$

From the point of view of biology, we focus our discussion on the positive solutions of system (1.2). So it is assumed that the initial condition of system (1.2) is of the form $x(t_0) > 0$.

For convenience, we denote

$$f^l = \inf_{t \in \mathbb{T}} f(t), \quad f^u = \sup_{t \in \mathbb{T}} f(t),$$

where f is an almost periodic function on \mathbb{T} .

Throughout this paper, we assume that

- (H₁) $a(t), b(t), c(t), d(t), m(t)$ are all bounded nonnegative almost periodic functions on \mathbb{T} such that $a^l > 0$, $b^l > 0$, $c^l > 0$, $d^l \geq 1$ and $m^l > 0$;
- (H₂) $\{\lambda_k\}$ is an almost periodic sequence and there exists positive constant r such that $0 < r \leq \prod_{t_0 < t_k < t} \ln(1 + \lambda_k) \leq 1$ for $t \geq t_0$ and $0 < \lambda_k \leq e - 1$ for $k \in \mathbb{N}$;
- (H₃) the set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k, j \in \mathbb{N}$ is uniformly almost periodic and $\inf_k t_k^1 = \theta > 0$;
- (H₄) $a^u > b^l$, $(a^l - c^u)r > b^u$, $-a^l + c^u \in \mathcal{R}^+$.

The main purpose of this paper is to discuss the permanence of system (1.2) by establishing some new comparison theorems of impulsive dynamic equations on time scales and based on the permanence result, by establishing a Massera [9] type theorem of impulsive dynamic equations on time scales, we obtain the existence and uniformly asymptotic stability of a unique positive almost periodic solution of system (1.2) on time scales. To the best of our knowledge, this is the first time to study the permanence and almost periodicity of system (1.3) and system (1.4), and is the first time to study the permanence of impulsive dynamic equations on time scales.

The organization of this paper is as follows: In Section 2, we introduce some notations and definitions, state some preliminary results which are needed in later sections and establish some new comparison theorems. In Section 3, we establish some sufficient conditions for the permanence of (1.2). In Section 4, we prove a Massera type theorem for impulsive dynamic equations on time scales and apply this theorem to obtain some sufficient conditions for the existence and uniformly asymptotic stability of a unique positive almost periodic solution of (1.2). In Section 5, we give an example to illustrate the feasibility and effectiveness of our results obtained in previous sections. We draw a conclusion in Section 6.

2. Preliminaries and comparison theorems

In this section, we shall recall some basic definitions and lemmas which are used in what follows.

A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers, the forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ and the forward graininess $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\} \text{ and } \mu(t) = \sigma(t) - t.$$

A point t is said to be left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. If \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} \setminus m$; otherwise $\mathbb{T}^k = \mathbb{T}$. If \mathbb{T} has a right-scattered minimum m , then $\mathbb{T}_k = \mathbb{T} \setminus m$; otherwise $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is right-dense continuous or rd-continuous provided it is continuous at right-dense points in \mathbb{T} and its left-sided limits exist (finite) at left-dense points in \mathbb{T} . If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on \mathbb{T} .

For $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$, then f is called delta differentiable at $t \in \mathbb{T}$ if there exists $c \in \mathbb{R}$ such that for given any $\varepsilon > 0$, there is an open neighborhood U of t satisfying

$$|[f(\sigma(t)) - f(s)] - c[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. In this case, c is called the delta derivative of f at $t \in \mathbb{T}$, and is denoted by $c = f^\Delta(t)$. For $\mathbb{T} = \mathbb{R}$, we have $f^\Delta = f'$, the usual derivative, and for $\mathbb{T} = \mathbb{Z}$ we have the backward difference operator, $f^\Delta(t) = \Delta f(t) := f(t+1) - f(t)$.

Let f be right-dense continuous, if $F^\Delta(t) = f(t)$, then we define the delta integral by $\int_r^s f(t)\Delta t = F(s) - F(r), r, s \in \mathbb{T}$.

Lemma 2.1 ([2]). *Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}$. Then*

- (i) $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$;
- (ii) $(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t)$;
- (iii) *If f and f^Δ are continuous, then $(\int_a^t f(t, s)\Delta s)^\Delta = f(\sigma(t), t) + \int_a^t f^\Delta(t, s)\Delta s$.*

A function $p : \mathbb{T} \rightarrow \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$.

If $r \in \mathcal{R}$, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp \left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\text{Log}(1 + hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \rightarrow \mathbb{R}$ be two regressive functions. We define

$$p \oplus q = p + q + \mu pq, \quad \ominus p = -\frac{p}{1 + \mu p}, \quad p \ominus q = p \oplus (\ominus q) = \frac{p - q}{1 + \mu q}.$$

Then the generalized exponential function has the following properties.

Lemma 2.2 ([2]). *Assume that $p, q : \mathbb{T} \rightarrow \mathbb{R}$ are two regressive functions. Then*

- (i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- (ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (iii) $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$;
- (iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- (v) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$;
- (vi) $e_p(t, s)/e_q(t, s) = e_{p \ominus q}(t, s)$;
- (vii) $(\frac{1}{e_p(t, s)})^\Delta = \frac{-p(t)}{e_p^\sigma(t, s)}$.

Lemma 2.3. *Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a continuous function, $f(t) > 0$ and $f^\Delta(t) \geq 0$ for $t \in \mathbb{T}$. Then*

$$\frac{f^\Delta(t)}{f^\sigma(t)} \leq [\ln(f(t))]^\Delta \leq \frac{f^\Delta(t)}{f(t)}.$$

If $f(t) > 0$ and $f^\Delta(t) < 0$ for $t \in \mathbb{T}$, then

$$\frac{f^\Delta(t)}{f(t)} \leq [\ln(f(t))]^\Delta \leq \frac{f^\Delta(t)}{f^\sigma(t)}.$$

Proof. If $f^\Delta(t) \geq 0$ for $t \in \mathbb{T}$, by use of Chain Rule, we can obtain

$$[\ln(f(t))]^\Delta = \begin{cases} \frac{f^\Delta(t)}{f(t)}, & \mu(t) = 0, \\ \left(\int_0^1 \frac{dh}{f(t) + h\mu(t)f^\Delta(t)} \right) f^\Delta(t), & \mu(t) \neq 0. \end{cases}$$

If $\mu(t) \neq 0$, then

$$\begin{aligned} [\ln(f(t))]^\Delta &= \left(\int_0^1 \frac{dh}{f(t) + h\mu(t)f^\Delta(t)} \right) f^\Delta(t) = \frac{1}{\mu(t)f^\Delta(t)} \int_{f(t)}^{f(t)+\mu(t)f^\Delta(t)} \frac{ds}{s} f^\Delta(t) \\ &= \frac{1}{\mu(t)} \ln \left(\frac{f(t) + \mu(t)f^\Delta(t)}{f(t)} \right) = \frac{1}{\mu(t)} \ln \left(\frac{f^\sigma(t)}{f(t)} \right). \end{aligned} \tag{2.1}$$

Let $g_1(r) = r - 1 - \ln r$. Then $g_1'(r) = \frac{r-1}{r} \geq 0$ for $r \geq 1$. Hence, g_1 is an increasing function. By use of $g_1(1) = 0$, we have $g_1(r) \geq 0$ for $r \geq 1$. That is, $r - 1 \geq \ln r$ for $r \geq 1$. Since

$$\begin{aligned} \frac{f^\Delta(t)}{f(t)} &= \frac{1}{\mu(t)} \left(\frac{f(t) + \mu(t)f^\Delta(t)}{f(t)} - 1 \right) \\ &\geq \frac{1}{\mu(t)} \ln \left(\frac{f(t) + \mu(t)f^\Delta(t)}{f(t)} \right) \\ &= [\ln(f(t))]^\Delta, \end{aligned}$$

and so,

$$[\ln(f(t))]^\Delta \leq \frac{f^\Delta(t)}{f(t)}.$$

Similarly, by use of Chain Rule, if $\mu(t) = 0$, then $f(t) = f^\sigma(t)$, and we have

$$[\ln(f(t))]^\Delta = \frac{f^\Delta(t)}{f(t)} = \frac{f^\Delta(t)}{f^\sigma(t)}.$$

If $\mu(t) \neq 0$, let $g_2(r) = \ln r + \frac{1}{r} - 1$, then $g_2'(r) = \frac{r-1}{r^2} \geq 0$ for $r \geq 1$. Hence, g_2 is an increasing function. By use of $g_2(1) = 0$, we have $g_2(r) \geq 0$ for $r \geq 1$. That is, $\ln r \geq 1 - \frac{1}{r}$ for $r \geq 1$. By use of (2.1), we have

$$\begin{aligned} \frac{f^\Delta(t)}{f^\sigma(t)} &= \frac{1}{\mu(t)} \left(\frac{\mu(t)f^\Delta(t) - f^\sigma(t)}{f^\sigma(t)} + 1 \right) = \frac{1}{\mu(t)} \left(1 - \frac{f(t)}{f^\sigma(t)} \right) \\ &\leq \frac{1}{\mu(t)} \ln \left(\frac{f^\sigma(t)}{f(t)} \right) = [\ln(f(t))]^\Delta, \end{aligned}$$

and so

$$\frac{f^\Delta(t)}{f^\sigma(t)} \leq [\ln(f(t))]^\Delta.$$

Similarly, if $f^\Delta(t) \leq 0$ for $t \in \mathbb{T}$, we can prove that

$$\frac{f^\Delta(t)}{f(t)} \leq [\ln(f(t))]^\Delta \leq \frac{f^\Delta(t)}{f^\sigma(t)}.$$

The proof is completed. □

Definition 2.4 ([6]). A time scale \mathbb{T} is called an almost periodic time scale if

$$\Pi = \{ \tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T} \} \neq \{0\}.$$

Definition 2.5. Let \mathbb{T} be an almost periodic time scale. A function $f \in C(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic function if the ϵ -translation set of f

$$E\{\epsilon, f\} = \{t \in \Pi : |f(t + \tau) - f(t)| < \epsilon, \forall t \in \mathbb{T}\}$$

is relatively dense for all $\epsilon > 0$; that is, for any given $\epsilon > 0$, there exists a constant $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains a $\tau(\epsilon) \in E\{\epsilon, f\}$ such that

$$|f(t + \tau) - f(t)| < \epsilon, \forall t \in \mathbb{T}.$$

τ is called the ϵ -translation number of f .

Remark 2.6. Definition 2.5 is a slightly modified version of Definition 3.9 in [6].

Lemma 2.7 ([6]). If $f, g \in C(\mathbb{T}, \mathbb{R}^n)$ are almost periodic, then $fg, f + g$ are almost periodic.

Lemma 2.8 ([6]). If $f \in C(\mathbb{T}, \mathbb{R}^n)$ is almost periodic and $F(\cdot)$ is uniformly continuous on the value field of $f(t)$, then $F \circ f$ is almost periodic.

Lemma 2.9 ([6]). If $f \in C(\mathbb{T}, \mathbb{R}^n)$ is almost periodic, then $F(t)$ is almost periodic if and only if $F(t)$ is bounded on \mathbb{T} , where $F(t) = \int_0^t f(s)\Delta s$.

Definition 2.10 ([7]). $\varphi \in C(\mathbb{T}, \mathbb{R})$ is said to be asymptotically almost periodic, if

$$\varphi(t) = p(t) + q(t), \tag{2.2}$$

where $p(t)$ is an almost periodic function on \mathbb{T} and $q(t)$ is continuous on \mathbb{T} , $\lim_{t \rightarrow \infty} q(t) = 0$.

Lemma 2.11 ([7]). Let $\varphi \in C(\mathbb{T}, \mathbb{R})$ is asymptotically almost periodic. Then the decomposition (2.2) is unique.

Lemma 2.12 ([7]). Let $\varphi \in C(\mathbb{T}, \mathbb{R})$, the following propositions are equivalent:

- (i) $\varphi(t)$ is asymptotically almost periodic;
- (ii) for any $\epsilon > 0$, there exist constants $l(\epsilon) > 0$ and $k(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains at least one τ such that

$$|\varphi(t + \tau) - \varphi(t)| < \epsilon, \forall t, t + \tau \geq k(\epsilon).$$

Lemma 2.13 ([2]). Assume that $a \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. If $a \in \mathcal{R}^+$ on \mathbb{T}^k , then $e_a(t, t_0) > 0$ for all $t \in \mathbb{T}$.

Lemma 2.14 ([5]). Assume that $x \in PC^1[\mathbb{T}, \mathbb{R}]$ and

$$\begin{cases} x^\Delta(t) \leq (\geq) p(t)x(t) + q(t), & t \neq t_k, t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) \leq (\geq) d_k x(t_k) + b_k, & t = t_k, k \in \mathbb{N}. \end{cases}$$

Then for $t \geq t_0 \geq 0$,

$$\begin{aligned} x(t) \leq (\geq) & x(t_0) \prod_{t_0 < t_k < t} d_k e_p(t, t_0) + \sum_{t_0 < t_k < t} \left(\prod_{t_0 < t_j < t} d_j e_p(t, t_k) \right) b_k \\ & + \int_{t_0}^t \prod_{s < t_k < t} d_k e_p(t, \sigma(s)) q(s) \Delta s, \end{aligned}$$

where $PC^1 = \{y : [0, \infty)_{\mathbb{T}} \rightarrow \mathbb{R} \text{ which is rd-continuous except at } t_k, k = 1, 2, \dots, \text{ for which } y(t_k^-), y(t_k^+), y^\Delta(t_k^-), y^\Delta(t_k^+) \text{ exist with } y(t_k^-) = y(t_k), y^\Delta(t_k^-) = y^\Delta(t_k)\}$.

Lemma 2.15. Assume that $x \in PC^1[\mathbb{T}, \mathbb{R}]$, $-a \in \mathcal{R}^+$, $\alpha \leq \prod_{t_0 < t_k < t} d_k \leq \beta$ for $t \geq t_0$.

(i) If

$$\begin{cases} x^\Delta(t) \leq b - ax(t), & t \neq t_k, t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) \leq d_k x(t_k) + b_k, & t = t_k, k \in \mathbb{N}, \end{cases} \tag{2.3}$$

then for $t \geq t_0$,

$$x(t) \leq x(t_0)\beta e_{(-a)}(t, t_0) + \sum_{t_0 < t_k < t} \beta e_{(-a)}(t, t_k)b_k + \frac{b\beta}{a}[1 - e_{(-a)}(t, t_0)].$$

Especially, if $b > 0$, $a > 0$, we have $\limsup_{t \rightarrow +\infty} x(t) \leq \frac{b\beta}{a}$.

(ii) If

$$\begin{cases} x^\Delta(t) \geq b - ax(t), & t \neq t_k, t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) \geq d_k x(t_k) + b_k, & t = t_k, k \in \mathbb{N}, \end{cases} \tag{2.4}$$

then for $t \geq t_0$,

$$x(t) \geq x(t_0)\alpha e_{(-a)}(t, t_0) + \sum_{t_0 < t_k < t} \alpha e_{(-a)}(t, t_k)b_k + \frac{b\alpha}{a}[1 - e_{(-a)}(t, t_0)].$$

Especially, if $b > 0$, $a > 0$, we have $\liminf_{t \rightarrow +\infty} x(t) \geq \frac{b\alpha}{a}$.

Proof. Because the proof of (ii) is similar to that of (i), we only prove (i). By Lemma 2.14 and (2.3), we have

$$\begin{aligned} x(t) &\leq x(t_0) \prod_{t_0 < t_k < t} d_k e_{(-a)}(t, t_0) + \sum_{t_0 < t_k < t} \left(\prod_{t_0 < t_j < t} d_j e_{(-a)}(t, t_j) \right) b_k \\ &\quad + \int_{t_0}^t \prod_{s < t_k < t} d_k e_{(-a)}(t, \sigma(s)) b \Delta s. \end{aligned}$$

In view of $\prod_{t_0 < t_k < t} d_k \leq \beta$, we have

$$\begin{aligned} x(t) &\leq x(t_0)\beta e_{(-a)}(t, t_0) + \sum_{t_0 < t_k < t} \beta e_{(-a)}(t, t_k)b_k + \beta \int_{t_0}^t e_{(-a)}(t, \sigma(s)) b \Delta s \\ &\leq x(t_0)\beta e_{(-a)}(t, t_0) + \sum_{t_0 < t_k < t} \beta e_{(-a)}(t, t_k)b_k + \frac{b\beta}{a}[1 - e_{(-a)}(t, t_0)]. \end{aligned}$$

In particular, if $a > 0$, then $e_{(-a)}(t, t_0) < 1$. We obtain $\limsup_{t \rightarrow +\infty} x(t) \leq \frac{b\beta}{a}$. The proof is completed. \square

Similarly, we can easily obtain the following results:

Lemma 2.16. Assume that $-b \in \mathcal{R}^+$, $x \in PC^1[\mathbb{T}, \mathbb{R}]$ and $x(t) > 0$ for $t \in \mathbb{T}$ and $\alpha \leq \prod_{t_0 < t_k < t} d_k \leq \beta$ for $t \geq t_0$.

(i) If

$$\begin{cases} x^\Delta(t) \leq x^\sigma(t)(b - ax(t)), & t \neq t_k, t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) \leq d_k x(t_k), & t = t_k, k \in \mathbb{N}, \end{cases} \tag{2.5}$$

then for $t \geq t_0$,

$$x(t) \leq \frac{b\beta}{a} \left[1 + \left(\frac{b}{ax(t_0)} - 1 \right) e_{(-b)}(t, t_0) \right]^{-1}.$$

Especially, if $b > 0$, $a > 0$, we have $\limsup_{t \rightarrow +\infty} x(t) \leq \frac{b\beta}{a}$.

(ii) If

$$\begin{cases} x^\Delta(t) \geq x^\sigma(t)(b - ax(t)), & t \in [t_0, +\infty)_{\mathbb{T}}, t \neq t_k, \\ x(t_k^+) \geq d_k x(t_k), & t = t_k, k \in \mathbb{N}, \end{cases} \tag{2.6}$$

then for $t \geq t_0$,

$$x(t) \geq \frac{b\alpha}{a} \left[1 + \left(\frac{b}{ax(t_0)} - 1 \right) e_{(-b)}(t, t_0) \right]^{-1}.$$

Especially, if $b > 0, a > 0$, we have $\liminf_{t \rightarrow +\infty} x(t) \geq \frac{b\alpha}{a}$.

Proof. Because the proof of (ii) is similar to that of (i), we only prove (i). Noticing that

$$\frac{x^\Delta(t)}{x(t)x^\sigma(t)} \leq \frac{b}{x(t)} - a$$

and

$$\left(\frac{1}{x(t)} \right)^\Delta = -\frac{x^\Delta(t)}{x(t)x^\sigma(t)}.$$

Let $y(t) = \frac{1}{x(t)}$, we have

$$\begin{cases} -y^\Delta(t) \leq by(t) - a, & t \neq t_k, t \in [t_0, +\infty)_{\mathbb{T}}, \\ y(t_k^+) \geq \frac{1}{d_k} y(t_k), & t = t_k, k \in \mathbb{N}, \end{cases}$$

that is,

$$\begin{cases} y^\Delta(t) \geq a - by(t), & t \neq t_k, t \in [t_0, +\infty)_{\mathbb{T}}, \\ y(t_k^+) \geq \frac{1}{d_k} y(t_k), & t = t_k, k \in \mathbb{N}. \end{cases}$$

By Lemma 2.15 (ii), for $t > t_0$, we have

$$y(t) \geq y(t_0) \frac{1}{\beta} e_{(-b)}(t, t_0) + \frac{a}{b\beta} [1 - e_{(-b)}(t, t_0)],$$

that is,

$$x(t) \leq \frac{b\beta}{a} \left[1 + \left(\frac{b}{ax(t_0)} - 1 \right) e_{(-b)}(t, t_0) \right]^{-1}.$$

In particular, if $b > 0$, then $e_{(-b)}(t, t_0) < 1$. We obtain $\limsup_{t \rightarrow +\infty} x(t) \leq \frac{b\beta}{a}$. The proof is completed. \square

Lemma 2.17. Assume that $-b \in \mathcal{R}^+, a > 0, x \in PC^1[\mathbb{T}, \mathbb{R}]$ and $x(t) > 0, \alpha \leq \prod_{t_0 < t_k < t} d_k \leq \beta$ for $t \geq t_0, \bar{\mu} = \sup_{t \in \mathbb{T}} \mu(t)$. If

$$\begin{cases} x^\Delta(t) \geq x(t)(b - ax(t)), & t \neq t_k, t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) \geq d_k x(t_k), & t = t_k, k \in \mathbb{N}, \end{cases} \tag{2.7}$$

then for $t \geq t_0$,

$$x(t) \geq \frac{b\alpha}{a} \left[1 + \left(\frac{b}{ax(t_0)} - 1 \right) e_{(-\frac{b}{1+\bar{\mu}b})}(t, t_0) \right]^{-1}.$$

Especially, if $b > 0$, we have $\liminf_{t \rightarrow +\infty} x(t) \geq \frac{b\alpha}{a}$.

Proof. Since

$$x(t) = x^\sigma(t) - \mu(t)x^\Delta(t),$$

we have

$$x^\Delta(t) \geq x(t)(b - ax(t)) = (x^\sigma(t) - \mu(t)x^\Delta(t))(b - ax(t)),$$

this is,

$$[1 + \mu(t)(b - ax(t))]x^\Delta(t) \geq x^\sigma(t)(b - ax(t))$$

and so

$$x^\Delta(t) \geq x^\sigma(t) \left(\frac{b}{1 + \bar{\mu}b} - \frac{a}{1 + \bar{\mu}b}x(t) \right).$$

Thus we have

$$\begin{cases} x^\Delta(t) \geq x^\sigma(t) \left(\frac{b}{1 + \bar{\mu}b} - \frac{a}{1 + \bar{\mu}b}x(t) \right), & t \neq t_k, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) \geq d_k x(t_k), & t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{2.8}$$

From Lemma 2.16 and (2.8), we have

$$x(t) \geq \frac{b\alpha}{a} \left[1 + \left(\frac{b}{ax(t_0)} - 1 \right) e_{(-\frac{b}{1+\bar{\mu}b})}(t, t_0) \right]^{-1}.$$

In particular, if $b > 0$, then $e_{(-\frac{b}{1+\bar{\mu}b})}(t, t_0) < 1$. We obtain $\liminf_{t \rightarrow +\infty} x(t) \geq \frac{b\alpha}{a}$. The proof is completed. □

3. Permanence

In this section, we will give our main results about the permanence of system (1.2). For convenience, we introduce the following notations:

$$x^* = \frac{a^u - b^l}{b^l}, \quad x_* = \ln \left(\frac{(a^l - c^u)r}{b^u} \right).$$

Lemma 3.1. *Assume that (H_1) - (H_4) hold. Let $x(t)$ be any solution of system (1.2). Then*

$$x_* \leq \liminf_{t \rightarrow +\infty} x(t) \leq \limsup_{t \rightarrow +\infty} x(t) \leq x^*.$$

Proof. Let $x(t)$ be any solution of system (1.2). From (1.2), it follows that

$$\begin{aligned} x^\Delta(t) &\leq a^u - b^l e^{x(t)} \leq a^u - b^l [1 + x(t)] \\ &= (a^u - b^l) - b^l x(t) \end{aligned}$$

and so

$$\begin{cases} x^\Delta(t) \leq (a^u - b^l) - b^l x(t), & t \neq t_k, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) \leq x(t_k) \ln(1 + \lambda_k), & t = t_k, \quad k \in \mathbb{N}. \end{cases}$$

In view of Lemma 2.15 (i), we have

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{a^u - b^l}{b^l} = x^*.$$

By system (1.2), we arrive at

$$x^\Delta(t) \geq a^l - c^u - b^u e^{x(t)}, \quad t \in [t_0, +\infty)_{\mathbb{T}}.$$

Let $N(t) = e^{x(t)}$. Then it is obvious that $N(t) > 0$. The above inequality yields that

$$[\ln(N(t))]^\Delta \geq a^l - c^u - b^u N(t).$$

If $N^\Delta(t) \geq 0$, in view of Lemma 2.3, we have

$$\frac{N^\Delta(t)}{N(t)} \geq a^l - c^u - b^u N(t),$$

and so

$$N^\Delta(t) \geq N(t)[a^l - c^u - b^u N(t)].$$

Thus

$$\begin{cases} N^\Delta(t) \geq N(t)[a^l - c^u - b^u N(t)], & t \neq t_k, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ N(t_k^+) \geq (1 + \lambda_k)N(t_k), & t = t_k, \quad k \in \mathbb{N}. \end{cases}$$

By applying Lemma 2.17, $(a^l - c^u)r > b^u$ and $-a^l + c^u \in \mathcal{R}^+$, we have

$$\liminf_{t \rightarrow +\infty} N(t) \geq \frac{(a^l - c^u)r}{b^u}.$$

If $N^\Delta(t) < 0$, in view of Lemma 2.3, we have

$$\frac{N^\Delta(t)}{N^\sigma(t)} \geq a^l - c^u - b^u N(t)$$

and so

$$N^\Delta(t) \geq N^\sigma(t)[a^l - c^u - b^u N(t)].$$

Thus

$$\begin{cases} N^\Delta(t) \geq N^\sigma(t)[a^l - c^u - b^u N(t)], & t \neq t_k, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ N(t_k^+) \geq (1 + \lambda_k)N(t_k), & t = t_k, \quad k \in \mathbb{N}. \end{cases}$$

By applying Lemma 2.17, $(a^l - c^u)r > b^u$ and $-a^l + c^u \in \mathcal{R}^+$, we have

$$\liminf_{t \rightarrow +\infty} N(t) \geq \frac{(a^l - c^u)r}{b^u}.$$

That is,

$$\liminf_{t \rightarrow +\infty} x(t) \geq \ln \left(\frac{(a^l - c^u)r}{b^u} \right) = x_*.$$

The proof is complete. □

Theorem 3.2. *Assume that (H_1) - (H_4) hold. Then system (1.2) is permanence.*

4. A Massera type theorem and almost periodic solutions

In this section, we will prove a Massera type theorem and use it to study the existence of almost periodic solutions of (1.2). Consider the following equation

$$\begin{cases} x^\Delta(t) = f(t, x), & t \neq t_k, t \in \mathbb{T}^+, \\ \Delta x(t_k) = I_k(x(t_k)), & t = t_k, k \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $f : \mathbb{T}^+ \times S_B \rightarrow \mathbb{R}$, $S_B = \{x \in \mathbb{R} : \|x\| < B\}$, $\|x\| = \sup_{t \in \mathbb{T}} |x(t)|$, the functions $I_k \in C[\mathbb{R}, \mathbb{R}]$, $k \in \mathbb{N}$ are almost periodic uniformly with respects to $x \in S_B$ and are Lipschitz continuous in x , $f(t, x)$ is almost periodic in t uniformly for $x \in S_B$ and is continuous in x . The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k, j \in \mathbb{N}$ is uniformly almost periodic and $\inf_k t_k^1 = \theta > 0$. To find the solution of (4.1), we consider the product system of (4.1) as follows

$$\begin{cases} x^\Delta(t) = f(t, x), & t \neq t_k, t \in \mathbb{T}^+, \\ \Delta x(t_k) = I_k(x(t_k)), & t = t_k, k \in \mathbb{N}, \end{cases} \quad \begin{cases} y^\Delta(t) = f(t, y), & t \neq t_k, t \in \mathbb{T}^+, \\ \Delta y(t_k) = I_k(y(t_k)), & t = t_k, k \in \mathbb{N}. \end{cases}$$

Define

$$V_1 = \left\{ V : \mathbb{T}^+ \times S_B \times S_B \rightarrow \mathbb{R}^+, V \text{ is rd - continuous in } (t_{k-1}, t_k]_{\mathbb{T}^+} \times S_B \times S_B \text{ and } \lim_{(t,x,y) \rightarrow (t_k, x_0, y_0), t > t_k} V(t, x, y) = V(t_k^+, x_0, y_0) \right\}.$$

Lemma 4.1. *Suppose that there exists a Lyapunov functional $V(t, x, y) \in V_1$ satisfying the following conditions*

- (i) $a(\|x - y\|) \leq V(t, x, y) \leq b(\|x - y\|)$, where $(t, x, y) \in \mathbb{T}^+ \times S_B \times S_B$, $a, b \in \kappa$, $\kappa = \{a \in C(\mathbb{R}^+, \mathbb{R}^+) : a(0) = 0 \text{ and } a \text{ is increasing}\}$;
- (ii) $|V(t, x, y) - V(t, x_1, y_1)| \leq L(\|x - x_1\| + \|y - y_1\|)$, where $(t, x, y) \in \mathbb{T}^+ \times S_B \times S_B$, $L > 0$ is a constant;
- (iii) $V(t_k^+, x + I_k(x), y + I_k(y)) \leq V(t, x, y)$, $x, y \in S_B$, $t = t_k$, $k \in \mathbb{N}$;
- (iv) $D^+ V_{(4.1)}^\Delta(t, x, y) \leq -cV(t, x, y)$, where $c > 0$, $-c \in \mathcal{R}^+$, $x, y \in S_B$, $t \neq t_k$, $k \in \mathbb{N}$.

Moreover, if there exists a solution $x(t) \in S$ of (4.1) for $t \in \mathbb{T}^+$, where $S \subset S_B$ is a compact set, then there exists a unique almost periodic solution $p(t) \in S$ of (4.1), which is uniformly asymptotically stable. In particular, if $f(t, x)$ is ω -periodic in t uniformly for $x \in S_B$ and there exists a positive integer q such that $t_{k+q} = t_k + \omega$, $I_{k+q}(x) = I_k(x)$ with $t_k \in \mathbb{T}^+$, then $p(t)$ is also periodic.

Proof. Take $\{\omega_n\} \subset \Pi$ such that $\omega_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Suppose that $\varphi(t) \in S$ is a solution of (4.1) for $t \in \mathbb{T}^+$, then $\varphi(t + \omega_n) \in S$ is a solution of the following equation

$$\begin{cases} x^\Delta(t) = f(t + \omega_n, x), & t \neq t_k - \omega_n, \\ \Delta x(t_k + \omega_n) = I_k(x(t_k + \omega_n)), & t = t_k - \omega_n. \end{cases}$$

For any $\varepsilon > 0$, take large enough integer $n_0(\varepsilon, \beta)$ such that when $m \geq l \geq n_0(\varepsilon)$, we have

$$b(2B)e_{(-c)}(\omega_l, 0) < \frac{a(\varepsilon)}{2}$$

and

$$|f(t + \omega_m, x) - f(t + \omega_l, x)| < \frac{ca(\varepsilon)}{2L}.$$

Then for (iv), we have

$$\begin{aligned} D^+V^\Delta(t, \varphi(t), \varphi(t + \omega_m - \omega_l)) &\leq -cV(t, \varphi(t), \varphi(t + \omega_m - \omega_l)) + L|f(t + \omega_m - \omega_l, \varphi(t + \omega_m - \omega_l)) \\ &\quad - f(t, \varphi(t + \omega_m - \omega_l))|, \end{aligned}$$

for $t \neq t_k - (\omega_m - \omega_l)$.

On the other hand, from $t = t_k - (\omega_m - \omega_l)$ and (iii) it follows that

$$V(t, \varphi(t) + I_k(\varphi(t)), \varphi(t + \omega_m - \omega_l) + I_k(\varphi(t + \omega_m - \omega_l))) \leq V(t, \varphi(t), \varphi(t + \omega_m - \omega_l))$$

and so

$$D^+V^\Delta(t, \varphi(t), \varphi(t + \omega_m - \omega_l)) \leq -cV(t, \varphi(t), \varphi(t + \omega_m - \omega_l)) + \frac{ca(\varepsilon)}{2}.$$

When $m \geq l \geq n_0(\varepsilon)$, we have

$$\begin{aligned} V(t + \omega_l, \varphi(t + \omega_l), \varphi(t + \omega_m)) &\leq e_{(-c)}(t + \omega_l, 0)V(0, \varphi(0), \varphi(\omega_m - \omega_l)) + \frac{a(\varepsilon)}{2}(1 - e_{(-c)}(t + \omega_l, 0)) \\ &\leq e_{(-c)}(t + \omega_l, 0)V(0, \varphi(0), \varphi(\omega_m - \omega_l)) + \frac{a(\varepsilon)}{2} \\ &\leq a(\varepsilon). \end{aligned}$$

By (i), for $m \geq l \geq n_0(\varepsilon)$ and $t \in \mathbb{T}^+$, we obtain

$$|\varphi(t + \omega_m) - \varphi(t + \omega_l)| < \varepsilon,$$

which implies that $\varphi(t)$ is asymptotically almost periodic. Then $\varphi(t) = p(t) + q(t)$, where $p(t)$ is almost periodic and $q(t) \rightarrow 0$, as $t \rightarrow \infty$. Therefore $p(t) \in S$ is an almost periodic solution of (4.1). It is easy to verify that $p(t)$ is uniformly asymptotically stable and every solution in S_B tends to $p(t)$, which means that $p(t)$ is unique. In particular, if $f(t, x)$ is ω -periodic in t uniformly for $x \in S_B$ and there exists a positive integer q such that $t_{k+q} = t_k + \omega$, $I_{k+q}(x) = I_k(x)$ with $t_k \in \mathbb{T}^+$, then $p(t + \omega) \in S$ is also a solution. By the uniqueness, we have $p(t + \omega) = p(t)$. The proof is complete. \square

Let $x(t)$ be any solution of system (1.2), $\Omega = \{x(t) : 0 < x_* \leq x(t) \leq x^*\}$. It is easy to verify that under the conditions of Theorem 3.2, Ω is an invariant set of (1.2).

Lemma 4.2. *Assume that (H_1) - (H_4) hold. Then $\Omega \neq \emptyset$.*

Proof. By the almost periodicity of $a(t)$, $b(t)$, $c(t)$, $d(t)$ and $m(t)$, there exists a sequence $\omega = \{\omega_p\} \subseteq \mathbb{II}$ with $\omega_p \rightarrow +\infty$ as $p \rightarrow +\infty$ such that for $t \neq t_k$, we have

$$\begin{aligned} a(t + \omega_p) &\rightarrow a(t), \quad b(t + \omega_p) \rightarrow b(t), \quad c(t + \omega_p) \rightarrow c(t) \\ d(t + \omega_p) &\rightarrow d(t), \quad m(t + \omega_p) \rightarrow m(t), \quad p \rightarrow +\infty, \end{aligned}$$

and there exists a subsequence $\{k_l\}$ of $\{p\}$, $k_l \rightarrow +\infty$, $l \rightarrow +\infty$, such that $t_{k_l} \rightarrow t_k$, $\lambda_{k_l} \rightarrow \lambda_k$.

In view of Lemma 3.1, for all $\varepsilon > 0$ then there exists a $t_1 \in \mathbb{T}$ and $t_1 \geq t_0$ such that

$$x_* - \varepsilon \leq x(t) \leq x^* + \varepsilon, \text{ for } t \geq t_1.$$

Write $x_p(t) = x(t + \omega_p)$ for $t \geq t_1$, $p = 1, 2, \dots$. For any positive integer q , it is easy to see that there exist sequences $\{x_p(t) : p \geq q\}$ such that the sequences $\{x_p(t)\}$ has subsequences, denoted by $\{x_p(t)\}$ again,

converging on any finite interval of \mathbb{T} as $p \rightarrow +\infty$, respectively. Thus, there is a function $y(t)$ defined on \mathbb{T} such that

$$x_p(t) \rightarrow y(t), \text{ for } t \in \mathbb{T}, \text{ as } p \rightarrow +\infty.$$

Since

$$\begin{cases} x_p^\Delta(t) = a(t + \omega_p) - b(t + \omega_p)e^{x(t + \omega_p)} \\ \quad - \frac{c(t + \omega_p)}{d(t + \omega_p) + m(t + \omega_p)e^{x(t + \omega_p)}}, \quad t \neq t_k - \omega_p, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ x_p(t_k^+) = x(t_k + \omega_p) \ln(1 + \lambda_{k_1}), \quad t = t_k - \omega_p, \quad k \in \mathbb{N}, \end{cases}$$

we have

$$\begin{cases} y^\Delta(t) = a(t) - b(t)e^{y(t)} - \frac{c(t)}{d(t) + m(t)e^{y(t)}}, \quad t \neq t_k, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ y(t_k^+) = y(t_k) \ln(1 + \lambda_k), \quad t = t_k, \quad k \in \mathbb{N}. \end{cases}$$

We can easily see that $y(t)$ is a solution of system (1.2) and $x_* - \epsilon \leq y(t) \leq x^* + \epsilon$ for $t \in \mathbb{T}$. Since ϵ is an arbitrary small positive number, it follows that $x_* \leq y(t) \leq x^*$ for $t \in \mathbb{T}$. \square

Theorem 4.3. *Assume that (H_1) - (H_4) hold. Suppose further that*

(H_5) $\gamma > 0$ and $-\gamma \in \mathcal{R}^+$, where

$$\gamma = 2b^l e^{x_*} + \frac{2c^l m^l e^{x_*}}{(d^u + m^u e^{x_*})^2} - \bar{\mu}(b^u)^2 e^{2x_*} - \frac{\bar{\mu}(c^u)^2 (m^u)^2 e^{2x_*}}{(d^l + m^l e^{x_*})^4} - \frac{2\bar{\mu}b^u c^u m^u e^{2x_*}}{(d^l + m^l e^{x_*})^2}.$$

Then (1.2) has a unique almost periodic solution $x(t)$, which is uniformly asymptotically stable.

Proof. From Lemma 4.2, there exists $x(t)$ such that $x_* \leq x(t) \leq x^*$. Hence, $|x(t)| \leq K$, where $K = \max\{|x^*|, |x_*|\}$. Denote $\|x\| = \sup_{t \in \mathbb{T}} |x(t)|$. Suppose that $x = x(t)$, $y = y(t)$ are any two positive solutions of system (1.2), then $\|x\| \leq K$, $\|y\| \leq K$. In view of system (1.2), we have

$$\begin{cases} x^\Delta(t) = a(t) - b(t)e^{x(t)} - \frac{c(t)}{d(t) + m(t)e^{x(t)}}, \quad t \neq t_k, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) = x(t_k) \ln(1 + \lambda_k), \quad t = t_k, \quad k \in \mathbb{N}, \\ y^\Delta(t) = a(t) - b(t)e^{y(t)} - \frac{c(t)}{d(t) + m(t)e^{y(t)}}, \quad t \neq t_k, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ y(t_k^+) = y(t_k) \ln(1 + \lambda_k), \quad t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{4.2}$$

Consider the Lyapunov function $V(t, x, y)$ on $\mathbb{T}^+ \times \Omega \times \Omega$ defined by

$$V(t, x, y) = (x(t) - y(t))^2.$$

It is easy to see that there exist two constants $C_1 > 0$, $C_2 > 0$ such that $(C_1\|x - y\|)^2 \leq V(t, x, y) \leq (C_2\|x - y\|)^2$. Let $a, b \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a(x) = C_1^2 x^2$, $b(x) = C_2^2 x^2$, so the condition (i) of Lemma 4.1 is satisfied. Besides,

$$\begin{aligned} |V(t, x, y) - V(t, \bar{x}, \bar{y})| &= |(x(t) - y(t))^2 - (\bar{x}(t) - \bar{y}(t))^2| \\ &\leq |(x(t) - y(t)) - (\bar{x}(t) - \bar{y}(t))| |(x(t) - y(t)) + (\bar{x}(t) - \bar{y}(t))| \\ &\leq |(x(t) - y(t)) - (\bar{x}(t) - \bar{y}(t))| (|x(t)| + |y(t)| + |\bar{x}(t)| + |\bar{y}(t)|) \\ &\leq 4K [|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)|] \\ &= L(\|x - \bar{x}\| + \|y - \bar{y}\|), \end{aligned}$$

where $L = 4K$, so condition (ii) of Lemma 4.1 is also satisfied.

On the other hand for $t = t_k$, we have

$$\begin{aligned} V(t_k^+, x(t_k^+), y(t_k^+)) &= (x(t_k^+) - y(t_k^+))^2 \\ &= [\ln(1 + \lambda_k)]^2 (x(t_k) - y(t_k))^2 \\ &\leq (x(t_k) - y(t_k))^2 \\ &= V(t_k, x(t_k), y(t_k)), \end{aligned}$$

then condition (iii) of Lemma 4.1 is also satisfied.

In view of system (4.2), we have

$$\begin{cases} (x(t) - y(t))^\Delta = -b(t)(e^{x(t)} - e^{y(t)}) \\ \quad + c(t) \left(\frac{1}{d(t) + m(t)e^{x(t)}} - \frac{1}{d(t) + m(t)e^{y(t)}} \right), \quad t \neq t_k, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) - y(t_k^+) = (x(t_k) - y(t_k)) \ln(1 + \lambda_k), \quad t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{4.3}$$

For convenience, we denote $u(t) = x(t) - y(t)$. Using the mean value theorem we get

$$e^{x(t)} - e^{y(t)} = e^{\xi(t)}(x(t) - y(t)), \tag{4.4}$$

$$\frac{1}{d(t) + m(t)e^{x(t)}} - \frac{1}{d(t) + m(t)e^{y(t)}} = -\frac{m(t)e^{\zeta(t)}}{(d(t) + m(t)e^{\zeta(t)})^2}(x(t) - y(t)), \tag{4.5}$$

where $\xi(t)$ and $\zeta(t)$ lie between $x(t)$ and $y(t)$. Then, by use of (4.4) and (4.5), (4.3) can be written as

$$\begin{cases} u^\Delta(t) = -b(t)e^{\xi(t)}u(t) - \frac{c(t)m(t)e^{\zeta(t)}}{(d(t) + m(t)e^{\zeta(t)})^2}u(t), \quad t \neq t_k, \quad t \in [t_0, +\infty)_{\mathbb{T}}, \\ u(t_k^+) = u(t_k) \ln(1 + \lambda_k), \quad t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{4.6}$$

Calculating the right derivative D^+V^Δ of V along the solution of (4.6) for $t \neq t_k$,

$$\begin{aligned} D^+V^\Delta(t, x, y) &= [2(x(t) - y(t)) + \mu(t)(x(t) - y(t))^\Delta](x(t) - y(t))^\Delta \\ &= [2u(t) + \mu(t)u^\Delta(t)]u^\Delta(t) \\ &= \left\{ 2u(t) + \mu(t) \left[-b(t)e^{\xi(t)}u(t) - \frac{c(t)m(t)e^{\zeta(t)}}{(d(t) + m(t)e^{\zeta(t)})^2}u(t) \right] \right\} \\ &\quad \times \left[-b(t)e^{\xi(t)}u(t) - \frac{c(t)m(t)e^{\zeta(t)}}{(d(t) + m(t)e^{\zeta(t)})^2}u(t) \right] \\ &= -2b(t)e^{\xi(t)}u^2(t) - \frac{2c(t)m(t)e^{\zeta(t)}}{(d(t) + m(t)e^{\zeta(t)})^2}u^2(t) + \mu(t)b^2(t)e^{2\xi(t)}u^2(t) \\ &\quad + \frac{\mu(t)c^2(t)m^2(t)e^{2\zeta(t)}}{(d(t) + m(t)e^{\zeta(t)})^4}u^2(t) + \frac{2\mu(t)b(t)m(t)c(t)e^{(\xi(t)+\zeta(t))}}{(d(t) + m(t)e^{\zeta(t)})^2}u(t)^2 \\ &\leq -2b^l e^{x^*}u^2(t) - \frac{2c^l m^l e^{x^*}}{(d^u + m^u e^{x^*})^2}u^2(t) + \bar{\mu}(b^u)^2 e^{2x^*}u^2(t) + \frac{\bar{\mu}(c^u)^2 (m^u)^2 e^{2x^*}}{(d^l + m^l e^{x^*})^4}u^2(t) \\ &\quad + \frac{2\bar{\mu}b^u c^u m^u e^{2x^*}}{(d^l + m^l e^{x^*})^2}u(t)^2 \\ &\leq -\left[2b^l e^{x^*} + \frac{2c^l m^l e^{x^*}}{(d^u + m^u e^{x^*})^2} - \bar{\mu}(b^u)^2 e^{2x^*} - \frac{\bar{\mu}(c^u)^2 (m^u)^2 e^{2x^*}}{(d^l + m^l e^{x^*})^4} - \frac{2\bar{\mu}b^u c^u m^u e^{2x^*}}{(d^l + m^l e^{x^*})^2} \right] u^2(t) \end{aligned}$$

$$\begin{aligned} &\leq -\gamma u^2(t) \\ &= -\gamma V(t, x, y). \end{aligned}$$

By (H_5) , we see that condition (iv) of Lemma 4.1 holds. Hence, according to Lemma 4.1, there exists a unique uniformly asymptotically stable almost periodic solution $x(t)$ of system (1.2), and $x(t) \in \Omega$. \square

5. An example

Consider the following single-species system with impulsive effects on time scale \mathbb{T} :

$$\begin{cases} x^\Delta(t) = a(t) - b(t)e^{x(t)} - \frac{c(t)}{d(t) + m(t)e^{x(t)}}, & t \neq t_k, t \in [0, +\infty)_{\mathbb{T}}, \\ x(t_k^+) = x(t_k) \ln(1 + \lambda_k), & t = t_k, k \in \mathbb{N}, \end{cases} \quad (5.1)$$

where $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, and

$$a(t) = 0.4 - 0.01 \sin(\sqrt{2}t), \quad b(t) = 0.34, \quad c(t) = 0.009 + 0.001 \cos(\sqrt{5}t),$$

$$d(t) = 1.05 + 0.05 \cos(\sqrt{5}t), \quad m(t) = 0.2 + 0.03 \sin(\sqrt{3}t), \quad \lambda_k = e^{(0.9)\frac{1}{2^k}} - 1, \quad t_k = k.$$

By calculating, we have

$$a^u = 0.41, \quad a^l = 0.39, \quad b^u = b^l = 0.34, \quad c^u = 0.01, \quad c^l = 0.008,$$

$$d^u = 1.1, \quad d^l = 1, \quad m^u = 0.23, \quad m^l = 0.17, \quad r \approx 0.949,$$

so we obtain

$$x^* = \frac{a^u - b^l}{b^l} \approx 0.206, \quad x_* = \ln \left(\frac{(a^l - c^u) \times 0.949}{b^u} \right) \approx 0.059.$$

When $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$ and $\gamma \approx 0.723$. When $\mathbb{T} = \mathbb{Z}$, $\mu(t) = 1$ and $\gamma \approx 0.547$. So, in both cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, all conditions in Theorem 4.3 are satisfied, and thus (5.1) has a unique positive almost periodic solution, which is uniformly asymptotically stable.

6. Conclusion

In this paper, some new comparison theorems and a Massera type theorem for impulsive dynamic equations on time scales are established. Based on these results, the existence and uniformly asymptotic stability of unique positive almost periodic solution of a single-species system with impulsive effects on time scales is obtained. Our results of this paper are completely new and can be used to study other types impulsive dynamic equation models on time scales.

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