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Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces using the CLRg property

Shuang Wang

School of Mathematical Sciences, Yancheng Teachers University, Yancheng, 224051, Jiangsu, P. R. China.

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Abstract

By means of weakening conditions of the gauge function ϕ and the CLRg property, some common fixed point theorems are established in fuzzy metric spaces. The two mappings considered here are assumed to be weakly compatible. Our results extend and improve very recent theorems in the related literature.

Keywords: Fuzzy metric spaces, weakly compatible mappings, common fixed points, common limit in the range property.
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1. Introduction and Preliminaries

Common fixed point theorems for weakly commuting pair of mappings were first studied in 1982 by Sessa [21]. Later on, Jungck [15] introduced the notion of compatible mappings which generalizes the concept of weakly commuting pair of mappings. Jungck [15] also showed that compatible pair of mappings commute on the set of coincidence points of the involved mappings. In 1996, Jungck and Rhoades [16] introduced the notion of weakly compatible mappings. Afterward, Aamri and Moutawakil [1] introduced the notion of property (E.A.) which is a special case of tangential property due to Sastry and Murthy [19]. In 2011, Sintunavarat and Kumam [22] obtained that the notions of property (E.A.) always requires the completeness (or closedness) of underlying subspaces for the existence of common fixed point. Hence they coined the idea of commom limit in the range property (called CLR) which relaxes the requirement of completeness (or closedness) of the underlying subspace.

Email address: `wangshuang19841119@yahoo.com` (Shuang Wang)

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In 2012, Jain et al. [14] extended the concept of (CLR) property in the coupled case and also established the following theorem (i.e., Theorem 3.2 in [14]).

Theorem 1.1 ([14]). *Let $(X, M, *)$ be a GV-FMS, $*$ being continuous t-norm of H-type. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ satisfying*

$$M(F(x,y), F(u,v), \phi(t)) \geq M(gx, gu, t) * M(gy, gv, t) \text{ for all } x, y, u, v \in X \text{ and all } t > 0,$$

with the following conditions: (i) the pair (F, g) is weakly compatible, (ii) the pair (F, g) satisfies CLRg property. Then F and g have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = F(x,x) = gx$.

Most recently, Hierro and Sintunavarat [6] generalized some results of Jain et al. [14] by using the generalized contractive conditions and the (CLR) property in fuzzy metric spaces. They obtained the following theorem (i.e., Theorem 21 in [6]).

Theorem 1.2 ([6]). *Let $(X, M, *)$ be a FMS such that $*$ is a t-norm of H-type and $f, g : X \to X$ be weakly compatible mappings having the CLRg property. Assume that there exist $\phi \in \Phi'$ and $N \in \mathbb{N}$ such that*

$$M(fx, fy, \phi(t)) \geq *^N M(gx, gy, t) \text{ for all } x, y \in X \text{ and all } t > 0.$$

Then f and g have a unique common fixed point (that is, there is a unique $w \in X$ such that $fw = gw = w$). In fact, if $z \in X$ is any coincidence point of f and g, then $w = fz = gz$ is their only common fixed point.

Under the CLRg property, it is not necessary to assume the completeness of the spaces in Theorems 1.1 and 1.2, which is an important advantage compared with the most of theorems in fixed point theory. But, many results are obtained under the assumption that ϕ satisfies $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$ and some other conditions (see, e.g. [4, 12, 13, 14, 18]). As Ćirić [3] has point out, the condition $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$ is very strong and difficult for testing in practice. In [6], Hierro and Sintunavarat weakened assumptions of gauge function ϕ in Theorem 1.1 by the condition (a): $\phi(t) > 0$ and $\lim_{k\to\infty} \phi^k(t) = 0$ for all $t > 0$. Can the condition (a) be weakened further? It goes without saying that this question is worth studying. In 2015, Fang [5] gave an affirmative answer to the question by introducing the condition (b): for each $t > 0$ there exists $r \geq t$ such that $\lim_{n\to\infty} \phi^n(r) = 0$ in the setup of complete Menger probabilistic metric spaces and fuzzy metric spaces.

Motivated and inspired by results of papers [5, 6, 14], we established some new common fixed point theorems for weakly compatible mappings in fuzzy metric spaces by using the CLRg property and the condition (b). Our results generalize and extend some recent results in [6] and the reference therein. In addition, we illustrate our main results with an example.

2. Preliminaries

In order to fix the framework needed to state our main results, we recall the following notions.

Throughout this paper, let $\mathbb{I} = [0,1]$, $\mathbb{R}^+ = [0, \infty)$, and $\mathbb{N}$ be the set of all natural numbers. For brevity, $f(x)$ and $g(x)$ will be denoted by fx and gx, respectively. In the sequel, t will be a positive real number.

Definition 2.1 ([6]). Given $f, g : X \to X$, we will say that a point $x \in X$ is a:
- fixed point of f if $fx = x$;
- coincidence point of f and g if $fx = gx$;
- common fixed point of f and g if $fx = gx = x$.

Following Gnana-Bhaskar and Lakshmikantham (see [8]), given $F : X \times X \to X$ and $g : X \to X$, we will say that a point $(x, y) \in X \times X$ is a
- coupled fixed point of F if $F(x,y) = x$ and $F(y,x) = y$;
- coupled coincidence point of F and g if $F(x,y) = gx$ and $F(y,x) = gy$;
- coupled common fixed point of F and g if $F(x,y) = gx = x$ and $F(y,x) = gy = y$.

Definition 2.2 ([20])**.** A triangular norm (also called a t-norm) is a map $* : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ that is associative, commutative, non-decreasing in both arguments and has 1 as identity. A t-norm is continuous if it is continuous in $\mathbb{I}^2$ as mapping. If $a_1, a_2, \dots, a_m \in \mathbb{I}$, then

$$*_{i=1}^{m} a_i = a_1 * a_2 * \cdots * a_m.$$

For each $a \in [0,1]$, the sequence $\{*^m a\}_{m=1}^{\infty}$ is defined inductively by $*^1 a = a$ and $*^{m+1} a = (*^m a) * a$ for all $m \geq 1$.

Remark 2.3 ([6]). If $m, n \in \mathbb{N}$, then $*^m(*^n a) = *^{mn} a$ for all $a \in \mathbb{I}$.

Definition 2.4 ([11])**.** A t-norm is said to be of H-type if the sequence $\{*^m a\}_{m=1}^{\infty}$ is equicontinuous at $a = 1$, i.e., for all $\varepsilon \in (0,1)$, there exists $\eta \in (0,1)$ such that if $a \in (1-\eta, 1]$, then $*^m a > 1 - \varepsilon$ for all $m \in \mathbb{N}$.

Definition 2.5 ([17])**.** A fuzzy metric space in the sense of Kramosil and Michálek (briefly, a FMS) is a triple $(X, M, *)$, where X is a non-empty set, $*$ is a continuous t-norm and $M : X \times X \times \mathbb{R}^+ \to \mathbb{I}$ is a fuzzy set satisfying the following conditions for all $x, y, z \in X$ and $t, s \geq 0$:

(FM-1) $M(x, y, 0) = 0$;
(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$;
(FM-3) $M(x, y, t) = M(y, x, t)$;
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(FM-5) $M(x, y, \cdot) : \mathbb{R}^+ \to \mathbb{I}$ is left continuous.

In this case, we also say that (X, M) is a FMS under $*$. In the sequel, we will only consider FMS verifying:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

In 1994, George and Veeramani introduced the notion of fuzzy metric space by modifying the previous concept due to Kramosil and Michálek.

Definition 2.6 ([7])**.** A triple $(X, M, *)$ is called a fuzzy metric space (in the sense of George and Veeramani) if X is an arbitrary non-empty set, $*$ is a continuous t-norm and $M : X \times X \times \mathbb{R}^+ \to \mathbb{I}$ is a fuzzy set satisfying, for each $x, y, z \in X$ and $t, s > 0$, conditions (FM-2), (FM-3) and (FM-4), and replacing (FM-1) and (FM-5) by the following properties:
(GV-1) $M(x, y, t) > 0$,
(GV-5) $M(x, y, \cdot) : (0, \infty) \to \mathbb{I}$ is continuous.

For short, we use GV-FMS to refer a fuzzy metric space in the sense of George and Veeramani. Obviously, every GV-FMS can be extended to a FMS in the sense of Kramosil and Michálek.

Definition 2.7 ([7])**.** Let $(X, M, *)$ be a FMS. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. A sequence $\{x_n\}$ in X is said to an M-Cauchy sequence, if for each $\varepsilon \in (0,1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $m, n \geq n_0$. A fuzzy metric space is called complete if every M-Cauchy sequence is convergent in X.

Lemma 2.8 ([9])**.** *If (X, M) is a FMS under some t-norm and $x, y \in X$, then $M(x, y, \cdot)$ is a non-decreasing function on $(0, \infty)$.*

Definition 2.9 ([2])**.** We will say that the maps $f, g : X \to X$ are weakly compatible (or the pair (f, g) is w-compatible) if $fgx = gfx$ for all $x \in X$ such that $fx = gx$.

Definition 2.10 ([6])**.** Let (X, M) be a FMS under some continuous t-norm. Two mappings $f, g : X \to X$ are said to have the CLRg property if there exists a sequence $\{x_n\} \subseteq X$ and a point $z \in X$ such that the sequences $\{fx_n\}$ and $\{gx_n\}$ are M-convergent and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gz$.

Throughout this paper, let Φ denote the family of all functions $\phi : (0, \infty) \to (0, \infty)$ such that the following properties are fulfilled:
(Φ_1) ϕ is non-decreasing;
(Φ_2) ϕ is upper semi-continuous from the right;
(Φ_3) $\sum_{k=1}^{\infty} \phi^k(t) < \infty$ for all $t > 0$ (where $\phi^{k+1}(t) = \phi(\phi^k(t))$ for all $k \in \mathbb{N}$ and all $t > 0$).
Let Φ' denote the family of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ verifying the condition (a): $\phi(t) > 0$ and $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$, and let Φ_w denote the class of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition (b), i.e., for each $t > 0$ there exists $r \geq t$ such that $\lim_{n\to\infty} \phi^n(r) = 0$.

Obviously, the condition $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$ implies the condition (b). The following example shows that the reverse is not true in general. Hence Φ' is a proper subclass of Φ_w.

Example 2.11 ([5])**.** Let the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\phi(t) = \begin{cases} \frac{t}{1+t}, \ if \ 0 \leq t < 1, \\ -\frac{t}{3} + \frac{4}{3}, \ if \ 1 \leq t \leq 2, \\ t - \frac{4}{3}, \ if \ 2 < t < \infty. \end{cases} \tag{2.1}$$

Notice that $\phi \in \Phi_w$ but $\phi \notin \Phi'$.

Lemma 2.12 ([5])**.** *Let $\phi \in \Phi_w$, then for each $t > 0$ there exists $r \geq t$ such that $\phi(r) < t$.*

3. Main Results

In this section, we state and prove some new common fixed point theorems for weakly compatible mappings in fuzzy metric spaces by using the CLRg property and the condition (b). In order to obtain our main results, we need the following lemma.

Lemma 3.1. *Let $(X, M, *)$ be a FMS and $f, g : X \to X$ be mappings having the CLRg property, that is, there is a sequence $\{x_n\} \subseteq X$ and $z \in X$ such that $\{fx_n\} \to gz$ and $\{gx_n\} \to gz$. Assume that there exist $N \in \mathbb{N}$ and $\phi \in \Phi_w$ such that*

$$M(fx, fy, \phi(t)) \geq *^N M(gx, gy, t) \text{ for all } x, y \in X \text{ and all } t > 0. \tag{3.1}$$

Then $fz = gz$, that is, f and g have a coincidence point.

Proof. As $\{gx_n\} \to gz$, we have that $\lim_{n\to\infty} M(gx_n, gz, t) = 1$ for all $t > 0$. Since $\phi \in \Phi_w$, by Lemma 2.12, for each $t > 0$ there exists $r \geq t$ such that $\phi(r) < t$. Notice that, for all $n \in \mathbb{N}$, as $M(fx_n, fz, \cdot)$ is a nondecreasing function, then

$$M(fx_n, fz, t) \geq M(fx_n, fz, \phi(r)) \geq *^N M(gx_n, gz, r) \geq *^N M(gx_n, gz, t) \text{ for all } t > 0.$$

As $*$ is a continuous mapping, we deduce that for all $t > 0$

$$\lim_{n\to\infty} M(fx_n, fz, t) \geq *^N (\lim_{n\to\infty} M(gx_n, gz, t)) = *^N 1 = 1.$$

Therefore, $\{fx_n\} \to fz$. Taking into account that $\{fx_n\} \to gz$, the uniqueness of the limit prove that $fz = gz$. □

Theorem 3.2. *Let $(X, M, *)$ be a FMS such that $*$ is a t-norm of H-type and $f, g : X \to X$ be weakly compatible mappings having the CLRg property. Assume that there exist $\phi \in \Phi_w$ and $N \in \mathbb{N}$ satisfying the condition* (3.1)*. Then f and g have a unique common fixed point. In fact, if $z \in X$ is any coincidence point of f and g, then $w = fz = gz$ is their only common fixed point.*

Proof. As f and g have the CLRg property, there exists a sequence $\{x_n\} \subseteq X$ and a point $z \in X$ such that $\{fx_n\} \to gz$ and $\{gx_n\} \to gz$. By Lemma 3.1, we have $fz = gz$. Throughout the rest of the proof, let z be any coincidence point of f and g. Denote $w = fz = gz$. We will prove that w is the unique common fixed point of f and g. Since f and g are weakly compatible mappings, we have

$$fz = gz \Rightarrow fgz = gfz \Rightarrow fw = gw.$$

It is not hard to see that the condition (3.1) implies that

$$\phi(t) > 0 \text{ for all } t > 0. \tag{3.2}$$

In fact, if there exists some $t_0 > 0$ such that $\phi(t_0) = 0$, by the condition (3.1), we have

$$0 = M(fx, fx, \phi(t_0)) \geq *^N M(gx, gx, t_0) = *^N 1 = 1,$$

which is a contraction. It is evident that (3.2) implies that $\phi^n(t) > 0$ for all $n \in \mathbb{N}$ and $t > 0$. We now prove that

$$M(gw, w, \phi^n(t)) \geq *^{N^n} M(gw, w, t) \text{ for all } t > 0 \text{ and } n \in \mathbb{N}. \tag{3.3}$$

That is obvious for $n = 1$, since

$$M(gw, w, \phi(t)) = M(gfz, fz, \phi(t)) = M(fw, fz, \phi(t)) \geq *^N M(gw, w, t) \text{ for all } t > 0.$$

Suppose that (3.3) holds for some k. Using Remark 2.3 and (3.1), we have

$$\begin{aligned} M(gw, w, \phi^{k+1}(t)) &= M(fgz, fz, \phi(\phi^k(t))) \\ &\geq *^N M(gw, w, \phi^k(t)) \\ &\geq *^N (*^{N^k} M(gw, w, t)) \\ &= *^{N^{k+1}} M(gw, w, t), \end{aligned}$$

which completes the induction. Hence (3.3) holds for all $n \in \mathbb{N}$.

Since $*$ is a t-norm of H-type, for any $\varepsilon \in (0, 1)$ there exists $\eta \in (0, 1)$ such that

$$\text{if } a \in (1 - \eta, 1], \text{then } *^m a > 1 - \varepsilon \text{ for all } m \in \mathbb{N}. \tag{3.4}$$

It follows from (FM-6) that $\lim_{t\to\infty} M(gw, w, t) = 1$. So, there exists $t_1 > 0$ such that $M(gw, w, t_1) > 1 - \eta$. Applying (3.4), we have that

$$*^m M(gw, w, t_1) > 1 - \varepsilon \text{ for all } m \in \mathbb{N}. \tag{3.5}$$

Since $\phi \in \Phi_w$, there exists $t_2 \geq t_1$ such that $\lim_{n\to\infty} \phi^n(t_2) = 0$. Thus, for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\phi^n(t_2) < t$ for all $n \geq n_0$. By (3.3), (3.5) and the monotonicity of $M(x, y, \cdot)$, we get

$$M(gw, w, t) \geq M(gw, w, \phi^n(t_2)) \geq *^{N^n} M(gw, w, t_2) \geq *^{N^n} M(gw, w, t_1) > 1 - \varepsilon$$

for all $n \geq n_0$. Taking into account that $\varepsilon, t > 0$ are arbitrary, we have that $M(gw, w, t) = 1$ for all $t > 0$, that is $gw = w$. So, w is a common fixed point of f and g.

To prove the uniqueness, let $y \in X$ be another common fixed point of f and g, that is $fy = gy = y$. Using (FM-6), we have that $\lim_{t\to\infty} M(w, y, t) = 1$. Therefore, there exists $t_3 > 0$ such that $M(w, y, t_3) > 1 - \eta$. Applying (3.4), we know that

$$*^m M(w, y, t_3) > 1 - \varepsilon \text{ for all } m \in \mathbb{N}. \tag{3.6}$$

Since $\phi \in \Phi_w$, there exists $t_4 \geq t_3$ such that $\phi^n(t_4) \to 0$ as $n \to \infty$. So, for any $t > 0$, there exists $n_1 \in \mathbb{N}$ such that $\phi^n(t_4) < t$ for all $n \geq n_1$. Notice that

$$M(w, y, \phi(t_4)) = M(fw, fy, \phi(t_4)) \geq *^N M(gw, gy, t_4) = *^N M(w, y, t_4).$$

Furthermore, by Remark 2.3, we have

$$\begin{aligned}M(w,y,\phi^2(t_4)) &= M(fw,fy,\phi(\phi(t_4)))\\ &\geq *^N M(gw,gy,\phi(t_4))\\ &= *^N M(w,y,\phi(t_4))\\ &\geq *^{N^2} M(w,y,t_4).\end{aligned}$$

It also is possible to prove, by induction, that

$$M(w,y,\phi^n(t_4)) \geq *^{N^n} M(w,y,t_4) \text{ for all } n \in \mathbb{N}. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$M(w,y,t) \geq M(w,y,\phi^n(t_4)) \geq *^{N^n} M(w,y,t_4) \geq *^{N^n} M(w,y,t_3) > 1-\varepsilon$$

for all $n \geq n_1$. Therefore, we deduce that $M(w,y,t) = 1$ for all $t > 0$, that is $w = y$. This proves that f and g have a unique common fixed point. □

Next, we particularize Theorem 3.2 to the case in which g is the identity mapping on X.

Theorem 3.3. *Let $(X,M,*)$ be a FMS such that $*$ is a t-norm of H-type and let $f : X \to X$ be a mapping such that there exists a sequence $\{x_n\} \subseteq X$ and $z \in X$ verifying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n = z$. Assume that there exist $\phi \in \Phi_w$ and $N \in \mathbb{N}$ such that*

$$M(fx,fy,\phi(t)) \geq *^N M(x,y,t) \textit{ for all } x,y \in X \textit{ and all } t > 0. \tag{3.8}$$

Then f has a unique fixed point.

Remark 3.4. Theorem 3.3 is an improvement and generalization of Theorem 4.1 in [5]. Our result shows that the completeness of $(X,M,*)$ is unnecessary, and the inequality (4.1) in [5] is a special case of (3.8).

Corollary 3.5. *Let $(X,M,*)$ be a complete FMS such that $*$ is a t-norm of H-type. Assume that there exists $\phi \in \Phi_w$ such that*

$$M(fx,fy,\phi(t)) \geq M(x,y,t) \textit{ for all } x,y \in X \textit{ and all } t > 0, \tag{3.9}$$

where f is a self-mapping on X. Then f has a unique fixed point.

Proof. Using Theorem 3.3, we only need to show that there exists a sequence $\{x_n\} \subseteq X$ and $z \in X$ verifying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n = z$. It is easy to see that the condition (3.9) implies that

$$\phi(t) > 0 \text{ for all } t > 0. \tag{3.10}$$

In fact, if there exists some $t_5 > 0$ such that $\phi(t_5) = 0$, by the condition (3.9), we have

$$0 = M(fx,fx,\phi(t_5)) \geq M(x,x,t_5) = 1,$$

which is a contraction. Let $x_0 \in X$ and $x_n = fx_{n-1}, n \in \mathbb{N}$. By the condition (3.9), we have

$$M(x_n,x_m,\phi(t)) = M(fx_{n-1},fx_{m-1},\phi(t)) \geq M(x_{n-1},x_{m-1},t) \tag{3.11}$$

for all $n,m \in \mathbb{N}$ and $t > 0$.

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence. We proceed with the following steps:

Step 1. We claim that for any $t > 0$,

$$M(x_n,x_{n+1},t) \to 1 \text{ as } n \to \infty. \tag{3.12}$$

Using (FM-6), we have that $M(x_0, x_1, t) \to 1$ as $t \to \infty$. Therefore, for any $\varepsilon \in (0, 1)$, there exists $t_6 > 0$ such that $M(x_0, x_1, t_6) > 1 - \varepsilon$. Since $\phi \in \Phi_w$, there exists $t_7 \geq t_6$ such that $\lim_{n\to\infty} \phi^n(t_7) = 0$. Thus, for each $t > 0$, there exists $n_2 \in \mathbb{N}$ such that $\phi^n(t_7) < t$ for all $n \geq n_2$. It is evident that (3.11) implies that

$$M(x_n, x_{n+1}, \phi(t)) \geq M(x_{n-1}, x_n, t) \text{ for all } n \in \mathbb{N} \text{ and } t > 0. \tag{3.13}$$

It follows from (3.10) that $\phi^n(t) > 0$ for all $n \in \mathbb{N}$ and $t > 0$. By induction, it follows from (3.13) that

$$M(x_n, x_{n+1}, \phi^n(t)) \geq M(x_0, x_1, t) \text{ for all } n \in \mathbb{N} \text{ and } t > 0. \tag{3.14}$$

So, by (3.14) and the monotonicity of $M(x, y, \cdot)$, we have

$$M(x_n, x_{n+1}, t) \geq M(x_n, x_{n+1}, \phi^n(t_7)) \geq M(x_0, x_1, t_7) \geq M(x_0, x_1, t_6) > 1 - \varepsilon$$

for all $n \geq n_2$. Taking into account that $\varepsilon, t > 0$ are arbitrary, we conclude that (3.12) holds.

Step 2. We claim that for any $t > 0$,

$$M(x_n, x_m, t) \geq *^{m-n} M(x_n, x_{n+1}, t - \phi(r)) \text{ for all } m \geq n + 1, \tag{3.15}$$

where $r \geq t$. Since $\phi \in \Phi_w$, by Lemma 2.12, for any $t > 0$, there exists $r \geq t$ such that $\phi(r) < t$. Since $M(x_n, x_{n+1}, t) \geq M(x_n, x_{n+1}, t - \phi(r)) = *^1 M(x_n, x_{n+1}, t - \phi(r))$, then (3.15) holds for $m = n + 1$. Suppose now that $M(x_n, x_m, t) \geq *^{m-n} M(x_n, x_{n+1}, t - \phi(r))$ holds for some fixed $m \geq n + 1$. By (FM-4), (3.11) and the monotonicity of $*$, we get

$$\begin{aligned}
M(x_n, x_{m+1}, t) &= M(x_n, x_{m+1}, t - \phi(r) + \phi(r)) \\
&\geq M(x_n, x_{n+1}, t - \phi(r)) * M(x_{n+1}, x_{m+1}, \phi(r)) \\
&\geq M(x_n, x_{n+1}, t - \phi(r)) * M(x_n, x_m, r) \\
&\geq M(x_n, x_{n+1}, t - \phi(r)) * M(x_n, x_m, t) \\
&\geq M(x_n, x_{n+1}, t - \phi(r)) * (*^{m-n} M(x_n, x_{n+1}, t - \phi(r))) \\
&= *^{m+1-n} M(x_n, x_{n+1}, t - \phi(r)).
\end{aligned}$$

Thus, we prove that if (3.15) holds for some $m \geq n + 1$, then it also holds for $m + 1$. By induction, we conclude that (3.15) holds for all $m \geq n + 1$.

Step 3. We claim that $\{x_n\}$ is a Cauchy sequence. Since $*$ is a t-norm of H-type, for any $\varepsilon \in (0, 1)$ there exists $\eta \in (0, 1)$ such that

$$\text{if } a \in (1 - \eta, 1], \text{then } *^l a > 1 - \varepsilon \text{ for all } l \in \mathbb{N}. \tag{3.16}$$

It follows from (3.12) that there exists $n_3 \in \mathbb{N}$ such that $M(x_n, x_{n+1}, t - \phi(r)) > 1 - \eta$ for all $n \geq n_3$. So, by (3.16), we have

$$*^{m-n} M(x_n, x_{n+1}, t - \phi(r)) > 1 - \varepsilon \tag{3.17}$$

for all $m > n \geq n_3$. By (3.15) and (3.17), we get for each $t > 0$ and $\varepsilon \in (0, 1)$,

$$M(x_n, x_m, t) > 1 - \varepsilon \text{ for all } m > n \geq n_3.$$

This shows that $\{x_n\}$ is a Cauchy sequence.

Since $(X, M, *)$ is complete, there exists $z \in X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n = z$. This completes the proof. □

4. An Example

The following example shows that Theorem 3.2 is more general than Theorem 1.2 (i.e., Theorem 21 in [6]).

Example 4.1. Let $X = [0, \infty)$ and define $M : X \times X \times \mathbb{R}^+ \to \mathbb{I}$ as follows:

$$M(x, y, t) = \begin{cases} 1, \ if \ |x-y| < t, \\ \frac{t}{|x-y|+t}, \ if \ |x-y| \geq t. \end{cases} \tag{4.1}$$

As Gregori et al. point out in [10], any FMS (X, M) is equivalent to Menger space in the sense that $M(x, y, t) = F_{x,y}(t)$ for all $x, y \in X$ and $t \geq 0$. Then (X, M) is a FMS under $* = \min$ (see Example in [3]).

Let $f, g : X \to X$ be given by $fx = \frac{x}{1+x}$ and $gx = 2x$ for $x \in X$. Then f and g are weakly compatible mappings. In fact, if $fx = gx$, then the equality has a unique solution $z = 0$. At this point, $fg(0) = gf(0)$. Furthermore, using that f and g are continuous and the sequence $\{x_n = \frac{1}{n}\}$, we have that $\{fx_n\} \to gz$ and $\{gx_n\} \to gz$, so f and g have the CLRg property.

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by (2.1). By Example 2.11, we know that $\phi \in \Phi_w$ but $\phi \notin \Phi'$.

Notice that f and g verify the condition (3.1) using $N = 1$. In fact, if $|fx - fy| < \phi(t)$, then $M(fx, fy, \phi(t)) = 1 \geq M(gx, gy, t)$, (3.1) holds when $N = 1$. Suppose that $|fx - fy| \geq \phi(t)$. From (2.1), it is evident that $\phi(t) \geq \frac{t}{1+t}$ for all $t > 0$ and so $|fx - fy| \geq \frac{t}{1+t}$. Notice that

$$|fx - fy| = \frac{|x-y|}{1 + x + y + xy} = \frac{|x-y|}{1 + |x-y| + 2\min\{x, y\} + xy} \leq \frac{|x-y|}{1 + |x-y|}.$$

Therefore, $\frac{t}{1+t} \leq \frac{|x-y|}{1+|x-y|}$, which implies that $|x - y| \geq t$ since the function $f(u) = \frac{u}{1+u}$ is strictly increasing on $[0, \infty)$. By (4.1), we have $M(gx, gy, t) = \frac{t}{t+2|x-y|}$. So,

$$M(fx, fy, \phi(t)) = \frac{\phi(t)}{|fx - fy| + \phi(t)} \geq \frac{\frac{t}{1+t}}{\frac{t}{1+t} + \frac{|x-y|}{1+|x-y|}} \geq \frac{t}{2|x-y| + t} = M(gx, gy, t)$$

for all $t > 0$, i.e., (3.1) holds when $N = 1$.

This shows that all the conditions of Theorem 3.2 are satisfied. By Theorem 3.2, we deduce that f and g have a unique common fixed point, which is $z = 0$. However, Theorem 1.2 cannot be applied to this example because the ϕ defined by (2.1) does not meet the condition $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$.

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