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Some identities of q-Euler polynomials under the symmetric group of degree n

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Abstract

In this paper, we investigate some new symmetric identities for the q-Euler polynomials under the symmetric group of degree n which are derived from fermionic p-adic q-integrals on $\mathbb{Z}_p$.

Keywords: Identities of symmetry, Carlitz-type q-Euler polynomial, symmetric group of degree n, fermionic p-adic q-integral.
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1. Introduction

Let p be a fixed prime number such that $p \equiv 1 \pmod 2$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$ and the q-analogue of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

As is well known, the Euler numbers are defined by

$$E_0 = 1, \quad (E+1)^n + E_n = 2\delta_{0,n}, \quad (n \in \mathbb{N} \cup \{0\}),$$

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with the usual convention about replacing E^n by E_n(see [1–14]).

The Euler polynomials are given by

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l = (E+x)^n, \quad (n \geq 0), \quad (\text{see [1, 3]}).$$

In [4], Kim introduced Carlitz-type q-Euler numbers as follows:

$$\mathcal{E}_{0,q} = 1, \quad q(q\mathcal{E}_q + 1)^n + \mathcal{E}_{n,q} = [2]_q \delta_{0,n}, \quad (n \geq 0), \quad (\text{see [4]}), \tag{1.1}$$

with the usual convention about replacing $\mathcal{E}_q^n$ by $\mathcal{E}_{n,q}$.

The Carlitz-type q-Euler polynomials are also defined as

$$\mathcal{E}_{n,q}(x) = \left(q^x \mathcal{E}_q + [x]_q\right)^n = \sum_{l=0}^{n} \binom{n}{l} q^{lx} \mathcal{E}_{l,q} [x]_q^{n-l}, \quad (\text{see [2, 4]}). \tag{1.2}$$

Let $C(\mathbb{Z}_p)$ be the space of all $\mathbb{C}_p$-valued continuous functions on $\mathbb{Z}_p$. Then, for $f \in C(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as

$$\begin{aligned} I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \\ &= \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad (\text{see [4–10]}). \end{aligned} \tag{1.3}$$

From (1.3), we note that

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad (n \in \mathbb{N}), \quad (\text{see [4]}). \tag{1.4}$$

The Carlitz-type q-Euler polynomials can be represented by the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\mathcal{E}_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_{-q}(y), \quad (n \geq 0), \quad (\text{see [4]}). \tag{1.5}$$

Thus, by (1.5), we get

$$\begin{aligned} \mathcal{E}_{n,q}(x) &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \int_{\mathbb{Z}_p} [y]_q^l \, d\mu_q(y) [x]_q^{n-l} \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \mathcal{E}_{l,q} [x]_q^{n-l}, \quad (\text{see [4]}). \end{aligned}$$

From (1.4), we can easily derive

$$q \int_{\mathbb{Z}_p} [x+1]_q^n \, d\mu_{-q}(x) + \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_{-q}(x) = [2]_q \delta_{0,n}, \quad (n \in \mathbb{N} \cup \{0\}). \tag{1.6}$$

The equation (1.6) is equivalent to

$$q\mathcal{E}_{n.q}(1) + \mathcal{E}_{n,q} = [2]_q \delta_{0,n}, \quad (n \geq 0). \tag{1.7}$$

The purpose of this paper is to give some new symmetric identities for the Carlitz-type q-Euler polynomials under the symmetric group of degree n which are derived from fermionic p-adic q-integrals on $\mathbb{Z}_p$.

2. Symmetric identities for $\mathcal{E}_{n,q}(x)$ under S_n

Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ such that $w_1 \equiv w_2 \equiv w_3 \equiv \cdots \equiv w_n \equiv 1 \pmod 2$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{-q^{w_1\cdots w_{n-1}}}(y) \qquad (2.1)\\
&= \lim_{N\to\infty} \frac{1}{[p^N]_{q^{w_1\cdots w_{n-1}}}} \\
&\quad\times \sum_{y=0}^{p^N-1} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} \left(-q^{w_1\cdots w_{n-1}}\right)^y \\
&=\frac{1}{2}\lim_{N\to\infty} [2]_{q^{w_1\cdots w_{n-1}}} \\
&\quad\times \sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)(m+w_n y)+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n}\left(\prod_{\substack{i=1\\ i\neq j}}^{n} w_i\right)k_j\right]_q t} \\
&\quad\times (-1)^{m+y}\, q^{w_1\cdots w_{n-1}(m+w_n y)}.
\end{aligned}$$

Thus, by (2.1), we get

$$\begin{aligned}
&\frac{1}{[2]_{q^{w_1\cdots w_{n-1}}}} \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1} (-1)^{\sum_{i=1}^{n-1} k_i}\, q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} \qquad (2.2)\\
&\quad\times \int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{-q^{w_1\cdots w_{n-1}}}(y) \\
&= \frac{1}{2}\lim_{N\to\infty} \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}\sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1} (-1)^{\sum_{i=1}^{n-1} k_i+m+y} \\
&\quad\times q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j+\left(\prod_{j=1}^{n-1} w_j\right)m+\left(\prod_{j=1}^{n} w_j\right)y} \\
&\quad\times e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)(m+w_n y)+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t}.
\end{aligned}$$

As this expression is invariant under any permutation $\sigma \in S_n$, we have the following theorem.

Theorem 2.1. *Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ such that $w_1 \equiv w_2 \equiv \cdots \equiv w_n \equiv 1 \pmod 2$. Then, the following expressions*

$$\begin{aligned}
&\frac{1}{[2]_{q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}} \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1} (-1)^{\sum_{i=1}^{n-1} k_i}\, q^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j} \\
&\times \int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1} w_{\sigma(j)}\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j\right]_q t} d\mu_{q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}(y)
\end{aligned}$$

are the same for any $\sigma \in S_n$, $(n \geq 1)$.

Now, we observe that

$$\left[\left(\prod_{j=1}^{n-1} w_j\right) y + \left(\prod_{j=1}^{n} w_j\right) x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_j\right) k_j\right]_q t \tag{2.3}$$

$$= \left[\prod_{j=1}^{n-1} w_j\right]_q \left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}}.$$

By (2.3), we get

$$\int_{\mathbb{Z}_p} e^{\left[(\prod_{j=1}^{n-1} w_j)y + (\prod_{j=1}^{n} w_j)x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{-q^{w_1\cdots w_{n-1}}}(y) \tag{2.4}$$

$$= \sum_{m=0}^{\infty} \left[\prod_{j=1}^{n-1} w_j\right]_q^m \int_{\mathbb{Z}_p} \left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}}^m d\mu_{-q^{w_1\cdots w_{n-1}}}(y) \frac{t^m}{m!}$$

$$= \sum_{m=0}^{\infty} \left[\prod_{j=1}^{n-1} w_j\right]_q^m \mathcal{E}_{m,q^{w_1\cdots w_{n-1}}} \left(w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}\right) \frac{t^m}{m!}.$$

For $m \geq 0$, from (2.4), we have

$$\int_{\mathbb{Z}_p} \left[\left(\prod_{j=1}^{n-1} w_j\right) y + \left(\prod_{j=1}^{n} w_j\right) x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_q^m d\mu_{-q^{w_1\cdots w_{n-1}}}(y) \tag{2.5}$$

$$= \left[\prod_{j=1}^{n-1} w_j\right]_q^m \mathcal{E}_{m,q^{w_1\cdots w_{n-1}}} \left(w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}\right), \quad (n \in \mathbb{N}).$$

Therefore, by Theorem 2.1 and (2.5), we obtain the following theorem.

Theorem 2.2. *Let $w_1, \ldots w_n \in \mathbb{N}$ be such that $w_1 \equiv w_2 \equiv \cdots \equiv w_n \equiv 1 \pmod 2$. For $m \geq 0$, the following expressions*

$$\frac{\left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]_q^m}{[2]_{q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_{\sigma(l)}-1} (-1)^{\sum_{i=1}^{n-1} k_i} q^{w_{\sigma(n)} \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j}$$

$$\times \mathcal{E}_{m,q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}} \left(w_{\sigma(n)} x + w_{\sigma(n)} \sum_{j=1}^{m-1} \frac{k_j}{w_{\sigma(j)}}\right)$$

are the same for any $\sigma \in S_n$.

It is not difficult to show that

$$\left[y + w_n x + w_n \sum_{j=0}^{n-1} \frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}} \tag{2.6}$$

$$= \frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_{q^{w_n}} + q^{w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} [y + w_n x]_{q^{w_1\cdots w_{n-1}}}.$$

Thus, by (2.6), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\left[y+w_nx+w_n\sum_{j=0}^{n-1}\frac{k_j}{w_j}\right]^m_{q^{w_1\cdots w_{n-1}}} d\mu_{q^{-w_1\cdots w_{n-1}}}(y) \qquad (2.7)\\
&=\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1}w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]^{m-l}_{q^{w_n}} q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\\
&\quad\times\int_{\mathbb{Z}_p}[y+w_nx]^l_{q^{w_1\cdots w_{n-1}}}d\mu_{-q^{w_1\cdots w_{n-1}}}(y)\\
&=\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1}w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]^{m-l}_{q^{w_n}}\\
&\quad\times q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\mathcal{E}_{l,q^{w_1\cdots w_{n-1}}}(w_nx).
\end{aligned}$$

From (2.7), we have

$$\begin{aligned}
&\frac{\left[\prod_{j=1}^{n-1}w_j\right]^m_q}{[2]_{q^{w_1\cdots w_{n-1}}}}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{l=1}^{n-1}k_l}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\\
&\quad\times\int_{\mathbb{Z}_p}\left[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]^n_{q^{w_1\cdots w_{n-1}}}d\mu_{-q^{w_1\cdots w_{n-1}}}(y) \qquad (2.8)\\
&=\sum_{l=0}^{m}\binom{m}{l}\frac{\left[\prod_{j=1}^{n-1}w_j\right]^l_q}{[2]_{q^{w_1\cdots w_{n-1}}}}[w_n]_q^{m-l}\mathcal{E}_{l,q^{w_1\cdots w_{n-1}}}(w_nx)\\
&\quad\times\prod_{s=1}^{n-1}\sum_{k_s=0}^{w_s-1}(-1)^{\sum_{j=1}^{n-1}k_j}q^{(l+1)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]^{m-l}_{q^{w_n}}\\
&=\frac{1}{[2]_{q^{w_1w_2\cdots w_{n-1}}}}\sum_{l=0}^{m}\binom{m}{l}\left[\prod_{j=1}^{n-1}w_j\right]^l_q[w_n]_q^{m-l}\mathcal{E}_{l,q^{w_1\cdots w_{n-1}}}(w_nx)\\
&\quad\times\hat{T}_{m,q^{w_n}}(w_1,w_2,\ldots,w_{n-1}\mid l),
\end{aligned}$$

where

$$\begin{aligned}
&\hat{T}_{m,q}(w_1,\ldots,w_{n-1}\mid l)\\
&=\prod_{s=1}^{n-1}\sum_{k_s=0}^{w_s-1}q^{(l+1)\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]^{m-l}_q(-1)^{\sum_{j=1}^{n-1}k_j}.
\end{aligned}$$

As this expression is invariant under any permutation in S_n, we have the following theorem.

Theorem 2.3. *Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ be such that $w_1 \equiv w_2 \equiv \cdots \equiv w_n \equiv 1 \pmod 2$. For $m \geq 0$, the following expressions*

$$\frac{1}{[2]_{q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}} \sum_{l=0}^{m} \binom{m}{l} \left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]_q^l [w_{\sigma(n)}]_q^{m-l}$$
$$\times \mathcal{E}_{l,q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}\left(w_{\sigma(n)}x\right) \hat{T}_{m,q^{w_{\sigma(n)}}}\left(w_{\sigma(1)}, w_{\sigma(2)}, \ldots, w_{\sigma(n-1)} \mid l\right)$$

are the same for any $\sigma \in S_n$.

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