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A new general algorithm for set-valued mappings and equilibrium problem

Javad Vahidi

Department of Mathematics, Iran University of Science and Technology, Tehran, Iran.

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Abstract

We consider a multi-step algorithm to approximate a common element of the set of solutions of monotone and Lipschitz-type continuous equilibrium problems, and the set of common fixed points of a finite family of set-valued mappings satisfying condition (E). We prove strong convergence theorems of such an iterative scheme in real Hilbert spaces. This common solution is the unique solution of a variational inequality problem and it satisfies the optimality condition for a minimization problem. The main result extends various results exiting in the literature.

Keywords: Equilibrium problem, variational inequality, set-valued mapping, condition (E).
2010 MSC: 74Q05, 58E35, 54C60.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let f be a bifunction from $C \times C$ into $\mathbb{R}$, such that $f(x,x) = 0$ for all $x \in C$. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x,y) \geq 0, \ \forall y \in C.$$

The set of solutions is denoted by $Sol(f,C)$. Such problems arise frequently in mathematics, physics, engineering, game theory, transportation, electricity market, economics and network. In the literature, many techniques and algorithms have been proposed to analyze the existence and approximation of a solution to equilibrium problem; see [6, 10, 13].

Email address: jvahidi@iust.ac.ir (Javad Vahidi)

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If $f(x, y) = \langle Fx, y - x\rangle$ for every $x, y \in C$, where F is a mapping from C into $\mathcal{H}$, then the equilibrium problem becomes the classical variational inequality problem which is formulated as finding a point $x^* \in C$ such that

$$\langle Fx^*, y - x^*\rangle \geq 0,\ \forall y \in C.$$

The set of solutions of this problem is denoted by $VI(F, C)$.

It is well known that variational inequalities cover many branches of mathematics, such as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance, see [16, 20].

We recall the following well-known definitions. A bifunction $f : C \times C \to \mathbb{R}$ is said to be (i) strongly monotone on C with $\alpha > 0$ iff $f(x, y) + f(y, x) \leq -\alpha\|x - y\|^2$, $\forall x, y \in C$; (ii) monotone on C iff $f(x, y) + f(y, x) \leq 0$, $\forall x, y \in C$; (iii) psedumonotone on C iff $f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0$, $\forall x, y \in C$; (iv) Lipschitz-type continuous on C with constants $c_1 > 0$ and $c_2 > 0$ iff $f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2$, for all $x, y, z \in C$.

A subset $C \subset \mathcal{H}$ is called proximal if for each $x \in \mathcal{H}$, there exists a $y \in C$ such that

$$\| x - y \| = dist(x, C) = \inf\{\| x - z \| : z \in C\}.$$

We denote by $CB(C)$, $K(C)$ and $P(C)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of C respectively. The Hausdorff metric $\mathfrak{h}$ on $CB(\mathcal{H})$ is defined by

$$\mathfrak{h}(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

for all $A, B \in CB(\mathcal{H})$.

Let $T : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping. An element $x \in \mathcal{H}$ is said to be a fixed point of T, if $x \in Tx$.

Definition 1.1. A set-valued mapping $T : \mathcal{H} \to CB(\mathcal{H})$ is called

(i) nonexpansive if

$$\mathfrak{h}(Tx, Ty) \leq \|x - y\|,\ x, y \in \mathcal{H}.$$

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $\mathfrak{h}(Tx, Tp) \leq \|x - p\|$ for all $x \in \mathcal{H}$ and all $p \in F(T)$.

Recently, J. Garcia-Falset, E. Llorens-Fuster and T. Suzuki [15] generalized the concept of a nonexpansive single valued mapping by introducing a new condition, called condition (E). Very recently, Abkar and Eslamian [1], modified the condition (E), for set-valued mappings as follows:

Definition 1.2. A set-valued mapping $T : \mathcal{H} \to CB(\mathcal{H})$ is said to satisfy condition (E) provided that

$$\mathfrak{h}(Tx, Ty) \leq \mu\, dist(x, Tx) + \|x - y\|,\ x, y \in \mathcal{H},$$

for some $\mu > 0$.

The theory of set-valued mappings has applications in control theory, convex optimization, differential equations and economics. Fixed point theory for set-valued mappings has been studied by many authors, see [1, 2, 11, 12, 14, 19, 22, 23, 32] and the references therein. In the resent years, iterative algorithms for finding a common element of a set of solutions of equilibrium problem and the set of fixed points of nonexpansive mappings in a real Hilbert space have been studied by many authors (see, e.g., [3, 4, 5, 7, 8, 9, 29, 30, 31, 33, 35, 36, 37]). The motivation for studying such a problem is in its possible application in mathematical models whose constraints can be expressed as fixed-point problems and/or equilibrium problems. This happens, in particular, in practical problems as: signal processing, network resource allocation, image recovery; see, for instance, [17, 18, 25, 26].

In 2007, Takahashi and Takahashi [35], introduced an iterative scheme, by the viscosity approximation method, for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of nonexpansive mappings in the setting of Hilbert spaces. They also studied strong convergence of the sequences generated by their algorithm for a solution of the equilibrium problem, which are also fixed points of a nonexpansive mapping defined on a closed convex subset of a Hilbert space.

Motivated by fixed point techniques of Takahashi and Takahashi in [35] and an improvement set of extragradient-type iteration methods in [21], Anh [3], introduced a new iteration algorithm for finding a common element of the solution set of equilibrium problems with a monotone and Lipschitz-type continuous bifunction and the set of fixed points of a single valued nonexpansive mapping.

Here we consider a multi-step iterative scheme to approximate a common element of the set of solutions of monotone and Lipschitz-type continuous equilibrium problems and the set of common fixed points of a finite family of set-valued mappings satisfying condition (E). We prove strong convergence theorems of such iterative scheme in a real Hilbert space. This common solution is the unique solution of a variational inequality problem and it satisfies the optimality condition for a minimization problem. Our results generalize and improve the results of Anh, Kim and Muu [5], Anh [3], and many others.

2. Preliminaries

Throughout the paper, we denote by $\mathcal{H}$ a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let $\{x_n\}$ be a sequence in $\mathcal{H}$ and $x \in \mathcal{H}$. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$, and strong convergence by $x_n \to x$. Let C be a nonempty closed convex subset of $\mathcal{H}$. The nearest point projection from $\mathcal{H}$ to C, denoted P_C, assigns to each $x \in \mathcal{H}$, the unique point $P_C x \in C$ with the property

$$\|x - P_C x\| := \inf\{\|x - y\|,\ \forall y \in C\}.$$

It is known that P_C is a nonexpansive mapping and that for each $x \in \mathcal{H}$

$$\langle x - P_C x, y - P_C x\rangle \leq 0,\ \forall y \in C.$$

Definition 2.1. A bounded linear operator $\mathcal{A}$ on $\mathcal{H}$ is called strongly positive if there exists $\overline{\gamma} > 0$ such that

$$\langle \mathcal{A}x, x\rangle \geq \overline{\gamma}\|x\|^2,\ (x \in \mathcal{H}).$$

For a nonexpansive mapping T from a nonempty subset C of $\mathcal{H}$ into itself, a typical problem is to minimize the quadratic function

$$\min_{x \in F(T)} \frac{1}{2}\langle \mathcal{A}x, x\rangle - \langle x, b\rangle,$$

over the set of all fixed points $F(T)$ of T (see [27]).

Lemma 2.2 ([27]). *Let $\mathcal{A}$ be a strongly positive linear bounded self-adjoint operator on $\mathcal{H}$ with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|\mathcal{A}\|^{-1}$. Then $\|I - \rho\mathcal{A}\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.3 ([34]). *For $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$, we have:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$, (subdifferential inequality);
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$.

Lemma 2.4 ([4]). *Let C be a nonempty closed convex subset of a real Hilbert spaces $\mathcal{H}$ and let $f : C \times C \to \mathbb{R}$ be a psedumonotone and Lipschitz-type continuous bifunction. For each $x \in C$, let $f(x, .)$ be convex and subdifferentiable on C. Let $\{x_n\}$, $\{z_n\}$, and $\{w_n\}$ be sequences generated by $x_0 \in C$ and by*

$$\begin{cases} w_n &= argmin\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 :\ w \in C\}, \\ z_n &= argmin\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 :\ z \in C\}. \end{cases}$$

Then for each $x^\star \in Sol(f, C)$,

$$\|z_n - x^\star\|^2 \leq \|x_n - x^\star\|^2 - (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2, \quad \forall n \geq 0.$$

Lemma 2.5 ([38]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\eta_n)a_n + \eta_n\delta_n,\ n \geq 0,$$

where $\{\eta_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \eta_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\eta_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([25]). *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\tau(n) \to \infty$ and*

$$t_{\tau(n)} \leq t_{\tau(n)+1},\ t_n \leq t_{\tau(n)+1}.$$

for all (sufficiently large) $n \in \mathbb{N}$. In fact

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}.$$

Lemma 2.7 ([2]). *Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$. Let $T : C \to CB(C)$ be a quasi-nonexpansive set-valued mapping. If $F(T) \neq \emptyset$, and $T(p) = \{p\}$ for all $p \in F(T)$, then $F(T)$ is closed and convex.*

Lemma 2.8 ([2]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $T : C \to K(C)$ be a set-valued mapping satisfying condition (E). If x_n converges weakly to x^* and $\lim_{n\to\infty} dist(x_n, Tx_n) = 0$, then $x^* \in Tx^*$.*

3. Algorithm and its convergence analysis

For solving the equilibrium problem, let a bifunction f satisfy:

(A1) f is Lipschitz-type continuous on C,
(A2) f is monotone on C,
(A3) $f(x, .)$ is subdifferentiable and convex on C for every $x \in C$.

Now we state our main result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)–(A3)*. Let $T_i : C \to K(C), (i = 1, 2, \ldots, m)$ be a finite family of set-valued mappings, each satisfying condition (E). Assume that $\Gamma = \bigcap_{i=1}^m F(T_i) \bigcap Sol(f, C) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, \ldots, m)$ for each $p \in \Gamma$. Let h be a k-contraction of C into itself and $\mathcal{A}$ be a strongly positive bounded linear self-adjoint operator on $\mathcal{H}$ with coefficient $\overline{\gamma} < 1$ and $0 < \gamma < \frac{\overline{\gamma}}{k}$. Let $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ be sequences generated by $x_0 \in C$ and by*

$$\begin{cases} w_n &= argmin\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 :\ w \in C\}, \\ z_n &= argmin\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 :\ z \in C\}, \\ y_{n,1} &= \alpha_{n,1}z_n + (1-\alpha_{n,1})u_{n,1}, \\ y_{n,2} &= \alpha_{n,2}z_n + (1-\alpha_{n,2})u_{n,2}, \\ &\vdots \\ y_{n,m} &= \alpha_{n,m}z_n + (1-\alpha_{n,m})u_{n,m}, \\ x_{n+1} &= \theta_n\gamma h(x_n) + (I - \theta_n\mathcal{A})y_{n,m}, \qquad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $u_{n,1} \in T_1z_n, u_{n,k} \in T_ky_{n,k-1}, (k = 2, \ldots, m)$ and the sequences $\{\alpha_{n,i}\}$, $\{\lambda_n\}$ and $\{\theta_n\}$ satisfy the following conditions:

(i) $\{\theta_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \theta_n = 0$, $\sum\limits_{n=1}^{\infty} \theta_n = \infty$,

(ii) $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$,

(iii) $\{\alpha_{n,i}\} \subset [c,d] \subset (0,1)$ *for all* $1 \le i \le m$.

Then, the sequence $\{x_n\}$ *converges strongly to* $x^\star \in \bigcap_{i=1}^m F(T_i) \bigcap Sol(f,C)$ *which solves the variational inequality*

$$\langle (\mathcal{A} - \gamma h)x^\star, x - x^\star\rangle \ge 0,\ \forall x \in \Gamma. \tag{3.2}$$

Proof. First, we note that $P_\Gamma(I - \mathcal{A} + \gamma h)$ is a contraction from C into itself. By the Banach contraction principle, there exists a unique element $x^\star \in C$ such that $x^\star = P_\Gamma(I - \mathcal{A} + \gamma h)x^\star$. Now, we show that $\{x_n\}$ is bounded. Since $\lim\limits_{n\to\infty} \theta_n = 0$, we can assume that $\theta_n \in (0, \|\mathcal{A}\|^{-1})$, for all $n \ge 0$. By Lemma 2.2, we have $\|I - \theta_n \mathcal{A}\| \le 1 - \theta_n \overline{\gamma}$. From Lemma 2.4, we have

$$\|z_n - x^\star\| \le \|x_n - x^\star\|.$$

By our assumption on T_i, we have

$$\begin{aligned}
\|y_{n,1} - x^\star\| &= \|\alpha_{n,1} z_n + (1-\alpha_{n,1})u_{n,1} - x^\star\| \\
&\le \alpha_{n,1}\|z_n - x^\star\| + (1-\alpha_{n,1})\|u_{n,1} - x^\star\| \\
&= \alpha_{n,1}\|z_n - x^\star\| + (1-\alpha_{n,1}) dist(u_{n,1}, T_1 x^\star) \\
&\le \alpha_{n,1}\|z_n - x^\star\| + (1-\alpha_{n,1})\mathfrak{h}(T_1 z_n, T_1 x^\star) \\
&\le \alpha_{n,1}\|z_n - x^\star\| + (1-\alpha_{n,1})\|z_n - x^\star\| \\
&\le \|x_n - x^\star\|,
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n,2} - x^\star\| &= \|\alpha_{n,2} z_n + (1-\alpha_{n,2})u_{n,2} - x^\star\| \\
&\le \alpha_{n,2}\|z_n - x^\star\| + (1-\alpha_{n,2})\|u_{n,2} - x^\star\| \\
&= \alpha_{n,2}\|z_n - x^\star\| + (1-\alpha_{n,2}) dist(u_{n,2}, T_2 x^\star) \\
&\le \alpha_{n,2}\|z_n - x^\star\| + (1-\alpha_{n,2})\mathfrak{h}(T_2 y_{n,1}, T_2 x^\star) \\
&\le \alpha_{n,2}\|z_n - x^\star\| + (1-\alpha_{n,2})\|y_{n,1} - x^\star\| \\
&\le \|x_n - x^\star\|.
\end{aligned}$$

By continuing this process we obtain

$$\|y_{n,m} - x^\star\| \le \|x_n - x^\star\|.$$

Consequently,

$$\begin{aligned}
\|x_{n+1} - x^\star\| &= \|\theta_n(\gamma h(x_n) - \mathcal{A}x^\star) + (I - \theta_n\mathcal{A})(y_{n,m} - x^\star)\| \\
&\le \theta_n\|\gamma h(x_n) - \mathcal{A}x^\star\| + \|I - \theta_n\mathcal{A}\|\|y_{n,m} - x^\star\| \\
&\le \theta_n\|\gamma h(x_n) - \mathcal{A}x^\star\| + (1 - \theta_n\overline{\gamma})\|x_n - x^\star\| \\
&\le \theta_n\gamma\|h(x_n) - h(x^\star)\| + \theta_n\|\gamma h(x^\star) - \mathcal{A}x^\star\| + (1-\theta_n)\overline{\gamma}\|x_n - x^\star\| \\
&\le \theta_n\gamma k\|x_n - x^\star\| + \theta_n\|\gamma h(x^\star) - \mathcal{A}p\| + (1-\theta_n)\overline{\gamma}\|x_n - x^\star\| \\
&\le (1 - \theta_n(\overline{\gamma} - \gamma k))\|x_n - x^\star\| + \theta_n\|\gamma h(x^\star) - \mathcal{A}x^\star\| \\
&= (1 - \theta_n(\overline{\gamma} - \gamma k))\|x_n - x^\star\| + \theta_n(\overline{\gamma} - \gamma k)\tfrac{\|\gamma h(x^\star) - \mathcal{A}x^\star\|}{\overline{\gamma} - \gamma k}
\end{aligned}$$

$$\begin{aligned}
&\leq \max\left\{\|x_n - x^\star\|,\ \tfrac{\|\gamma h x^\star - \mathcal{A}x^\star\|}{\overline{\gamma}-\gamma k}\right\}\\
&\vdots\\
&\leq \max\left\{\|x_0 - x^\star\|,\ \tfrac{\|\gamma h x^\star - \mathcal{A}x^\star\|}{\overline{\gamma}-\gamma k}\right\}.
\end{aligned}$$

This implies that $\{x_n\}$ is bounded and we also obtain that $\{z_n\}$, $\{h(x_n)\}$ and $\{u_{n,i}\}$ are bounded.

Next, we show that $\lim_{n\to\infty} dist(z_n, T_i z_n) = 0$ for each $1 \leq i \leq m$. Indeed, by Lemma 2.4, we have

$$\|z_n - x^\star\|^2 \leq \|x_n - x^\star\|^2 - (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2. \tag{3.3}$$

By our assumption, we have that

$$\|u_{n,1} - x^\star\| = dist(u_{n,1}, T_1 x^\star) \leq \mathfrak{h}(T_1 z_n, T_1 x^\star) \leq \|z_n - x^\star\|.$$

Also for $k = 2, 3, \ldots, m$, we have that

$$\|u_{n,k} - x^\star\| = dist(u_{n,k}, T_k x^\star) \leq \mathfrak{h}(T_k y_{n,k-1}, T_k x^\star) \leq \|y_{n,k-1} - x^\star\|.$$

Using Lemma 2.3, we get

$$\begin{aligned}
\|y_{n,1} - x^\star\|^2 &= \|\alpha_{n,1} z_n + (1-\alpha_{n,1})u_{n,1} - x^\star\|^2\\
&\leq \alpha_{n,1}\|z_n - x^\star\|^2 + (1-\alpha_{n,1})\|u_{n,1} - x^\star\|^2 - \alpha_{n,1}(1-\alpha_{n,1})\|z_n - u_{n,1}\|^2\\
&\leq \alpha_{n,1}\|z_n - x^\star\|^2 + (1-\alpha_{n,1})\|z_n - x^\star\|^2 - \alpha_{n,1}(1-\alpha_{n,1})\|z_n - u_{n,1}\|^2\\
&= \|z_n - x^\star\|^2 - \alpha_{n,1}(1-\alpha_{n,1})\|z_n - u_{n,1}\|^2,
\end{aligned}$$

consequently,

$$\begin{aligned}
\|y_{n,2} - x^\star\|^2 &= \|\alpha_{n,2} z_n + (1-\alpha_{n,2})u_{n,2} - x^\star\|^2\\
&\leq \alpha_{n,2}\|z_n - x^\star\|^2 + (1-\alpha_{n,2})\|u_{n,2} - x^\star\|^2 - \alpha_{n,2}(1-\alpha_{n,2})\|z_n - u_{n,2}\|^2\\
&\leq \alpha_{n,2}\|z_n - x^\star\|^2 + (1-\alpha_{n,2})\|y_{n,1} - x^\star\|^2 - \alpha_{n,2}(1-\alpha_{n,2})\|z_n - u_{n,2}\|^2\\
&\leq \|z_n - x^\star\|^2 - \alpha_{n,2}(1-\alpha_{n,2})\|z_n - u_{n,2}\|^2 - (1-\alpha_{n,2})\alpha_{n,1}(1-\alpha_{n,1})\|z_n - u_{n,1}\|^2.
\end{aligned}$$

By continuing this process and applying (3.3), we obtain that

$$\begin{aligned}
\|y_{n,m} - x^\star\|^2 &= \|\alpha_{n,m} z_n + (1-\alpha_{n,m})u_{n,m} - x^\star\|^2\\
&\leq \alpha_{n,m}\|z_n - x^\star\|^2 + (1-\alpha_{n,m})\|u_{n,m} - x^\star\|^2 - \alpha_{n,m}(1-\alpha_{n,m})\|z_n - u_{n,m}\|^2\\
&\leq \alpha_{n,m}\|z_n - x^\star\|^2 + (1-\alpha_{n,m})\|y_{n,m-1} - x^\star\|^2 - \alpha_{n,m}(1-\alpha_{n,m})\|z_n - u_{n,m}\|^2\\
&\leq \|z_n - x^\star\|^2 - \alpha_{n,m}(1-\alpha_{n,m})\|z_n - u_{n,m}\|^2 - (1-\alpha_{n,m})\alpha_{n,m-1}(1-\alpha_{n,m-1})\|z_n - u_{n,m-1}\|^2\\
&- \cdots - (1-\alpha_{n,m})(1-\alpha_{n,m-1})\cdots(1-\alpha_{n,1})\alpha_{n,1}\|z_n - u_{n,1}\|^2\\
&\leq \|x_n - x^\star\|^2 - \alpha_{n,m}(1-\alpha_{n,m})\|z_n - u_{n,m}\|^2 - (1-\alpha_{n,m})\alpha_{n,m-1}(1-\alpha_{n,m-1})\|z_n - u_{n,m-1}\|^2\\
&- \cdots - (1-\alpha_{n,m})(1-\alpha_{n,m-1})\cdots(1-\alpha_{n,1})\alpha_{n,1}\|z_n - u_{n,1}\|^2\\
&- (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2.
\end{aligned}$$

Consequently, utilizing Lemma 2.2, we conclude that

$$\begin{aligned}
\|x_{n+1} - x^\star\|^2 &= \|\theta_n(\gamma h(x_n) - \mathcal{A}x^\star) + (I - \theta_n\mathcal{A})(y_{n,m} - x^\star)\|^2\\
&\leq \theta_n^2\|\gamma h(x_n) - \mathcal{A}x^\star\|^2 + (1-\theta_n\overline{\gamma})^2\|y_{n,m} - x^\star\|^2 + 2\theta_n(1-\theta_n\overline{\gamma})\|\gamma h(x_n) - \mathcal{A}x^\star\|\|y_{n,m} - x^\star\|
\end{aligned}$$

$$\begin{aligned}&\leq \theta_n^2\|\gamma h(x_n)-\mathcal{A}x^\star\|^2+(1-\theta_n\overline{\gamma})^2\|x_n-x^\star\|^2+2\theta_n(1-\theta_n\overline{\gamma})\|\gamma h(x_n)-\mathcal{A}x^\star\|\|x_n-x^\star\|\\&-(1-\theta_n\overline{\gamma})^2(1-2\lambda_n c_1)\|x_n-w_n\|^2-(1-\theta_n\overline{\gamma})^2(1-2\lambda_n c_2)\|w_n-z_n\|^2\\&-(1-\theta_n\overline{\gamma})^2\alpha_{n,m}(1-\alpha_{n,m})\|z_n-u_{n,m}\|^2\\&-\cdots-(1-\theta_n\overline{\gamma})^2(1-\alpha_{n,m})(1-\alpha_{n,m-1})\cdots(1-\alpha_{n,1})\alpha_{n,1}\|z_n-u_{n,1}\|^2.\end{aligned}$$

It follows

$$\begin{aligned}(1-\theta_n\overline{\gamma})^2\alpha_{n,m}(1-\alpha_{n,m})\|z_n-u_{n,m}\|^2 &\leq \|x_n-x^\star\|^2-\|x_{n+1}-x^\star\|^2\\&+2\theta_n(1-\theta_n\overline{\gamma})\|\gamma h(x_n)-\mathcal{A}x^\star\|\|x_n-x^\star\|+\theta_n^2\|\gamma h(x_n)-\mathcal{A}x^\star\|^2.\end{aligned}\tag{3.4}$$

In order to prove that $x_n\to x^\star$ as $n\to\infty$, we consider the following two cases.

Case 1. Assume that $\{\|x_n-x^\star\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n-x^\star\|\}_{n\geq n_0}$ is either nondecreasing or nonincreasing. Since $\{\|x_n-x^\star\|\}$ is bounded, it is convergent. Since $\lim_{n\to\infty}\theta_n=0$ and $\{h(x_n)\}$ and $\{x_n\}$ are bounded, from (3.4) we obtain

$$\lim_{n\to\infty}(1-\theta_n\overline{\gamma})^2\alpha_{n,m}(1-\alpha_{n,m})\|z_n-u_{n,m}\|^2=0.$$

Since $\{\alpha_{n,m}\}\subset[c,d]\subset(0,1)$, we get that

$$\lim_{n\to\infty}\|z_n-u_{n,m}\|=0,\tag{3.5}$$

By similar argument, we can obtain that

$$\lim_{n\to\infty}\|z_n-u_{n,k}\|=0,\ (1\leq k\leq m-1)\tag{3.6}$$

and

$$\lim_{n\to\infty}\|x_n-w_n\|=\lim_{n\to\infty}\|w_n-z_n\|=0.$$

Hence

$$\|x_n-z_n\|\leq\|x_n-w_n\|+\|w_n-z_n\|\to 0,\ as\ n\to\infty.\tag{3.7}$$

From (3.6) we have

$$\lim_{n\to\infty}dist(z_n,T_1z_n)\leq\lim_{n\to\infty}\|z_n-u_{n,1}\|=0,$$

and

$$\lim_{n\to\infty}dist(z_n,T_ky_{n,k-1})\leq\lim_{n\to\infty}\|z_n-u_{n,k}\|=0,\ (k=2,\ldots,m).$$

From (3.1) and (3.6) we get

$$\lim_{n\to\infty}\|z_n-y_{n,k}\|\leq\lim_{n\to\infty}(1-\alpha_{n,k})\|z_n-u_{n,k}\|=0,\ 1\leq k\leq m.$$

Using the above inequality for $k=2,\ldots,m$, we have

$$\begin{aligned}dist(z_n,T_kz_n)&\leq dist(z_n,T_ky_{n,k-1})+\mathfrak{h}(T_ky_{n,k-1},T_kz_n)\\&\leq dist(z_n,T_ky_{n,k-1})+\mu\,dist(y_{n,k-1},T_ky_{n,k-1})+\|y_{n,k-1}-z_n\|\\&\leq(\mu+1)dist(z_n,T_ky_{n,k-1})+(\mu+1)\|y_{n,k-1}-z_n\|\\&\leq(\mu+1)\|z_n-u_{n,k}\|+(\mu+1)\|y_{n,k-1}-z_n\|\to 0,\ n\to\infty.\end{aligned}\tag{3.8}$$

Next, we show that

$$\limsup_{n\to\infty}\langle(\mathcal{A}-\gamma h)x^\star,x^\star-x_n\rangle\leq 0.$$

For that, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}(\langle(\mathcal{A}-\gamma h)x^\star, x^\star - x_{n_i}\rangle = \limsup_{n\to\infty}\langle(\mathcal{A}-\gamma h)x^\star, x^\star - x_n\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to x^*. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup x^*$. From (3.7) we have $z_{n_i} \rightharpoonup x^*$. Applying (3.8) and Lemma 2.8 we conclude that $x^* \in \bigcap_{i=1}^m F(T_i)$. Now we show that $x^* \in Sol(f,C)$. Since $f(x,.)$ is convex on C for each $x \in C$, we see that

$$w_n = argmin\left\{\lambda_n f(x_n,y) + \frac{1}{2}\|y-x_n\|^2 : \ y \in C\right\}$$

if and only if

$$o \in \partial_2\left(f(x_n,y) + \frac{1}{2}\|y-x_n\|^2\right)(w_n) + N_C(w_n),$$

where $N_C(x)$ is the (outward) normal cone of C at $x \in C$. This implies that

$$0 = \lambda_n v + w_n - x_n + u_n, ,$$

where $v \in \partial_2 f(x_n, w_n)$ and $u_n \in N_C(w_n)$. By the definition of the normal cone N_C, we have

$$\langle w_n - x_n, y - w_n\rangle \geq \lambda_n\langle v, w_n - y\rangle, \ \forall y \in C. \tag{3.9}$$

Since $f(x_n,.)$ is subdifferentiable on C, by the well-known Moreau–Rockafellar theorem [28], there exists $v \in \partial_2 f(x_n, w_n)$ such that

$$f(x_n,y) - f(x_n,w_n) \geq \langle v, y - w_n\rangle, \ \forall y \in C.$$

Combining this with (3.9), we get

$$\lambda_n(f(x_n,y) - f(x_n,w_n)) \geq \langle w_n - x_n, w_n - y\rangle, \ \forall y \in C.$$

Hence

$$f(x_{n_i},y) - f(x_{n_i},w_{n_i}) \geq \frac{1}{\lambda_{n_i}}\langle w_{n_i} - x_{n_i}, w_{n_i} - y\rangle, \ \forall y \in C.$$

Since $\lim_{n\to\infty}\|x_n - w_n\| = 0$, we have that $w_{n_i} \rightharpoonup x^*$. Now by continuity of f and the assumption that $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{L})$ we have

$$f(x^*,y) \geq 0, \ \forall y \in C.$$

This implies that $x^* \in Sol(f,C)$. Thus, it is clear that $x^* \in \Gamma$. Since $x^\star = P_\Gamma(I - \mathcal{A} + \gamma h)x^\star$ and $x^* \in \Gamma$, we have

$$\limsup_{n\to\infty}\langle(\mathcal{A}-\gamma h)x^\star, x^\star - x_n\rangle = \lim_{i\to\infty}(\langle(\mathcal{A}-\gamma h)x^\star, x^\star - x_{n_i}\rangle = (\langle(\mathcal{A}-\gamma h)x^\star, x^\star - x^*\rangle \leq 0.$$

Using Lemma 2.3 and our assumption, we have

$$\begin{aligned}
\|x_{n+1} - x^\star\|^2 &\leq \|(I - \theta_n\mathcal{A})(y_n - x^\star)\|^2 + 2\theta_n\langle\gamma h(x_n) - \mathcal{A}x^\star, x_{n+1} - x^\star\rangle \\
&\leq (1-\theta_n\overline{\gamma})^2\|x_n - x^\star\|^2 + 2\theta_n\gamma\langle h(x_n) - h(x^\star), x_{n+1} - x^\star\rangle + 2\theta_n\langle\gamma h(x^\star) - \mathcal{A}x^\star, x_{n+1} - x^\star\rangle \\
&\leq (1-\theta_n\overline{\gamma})^2\|x_n - x^\star\|^2 + 2\theta_n k\gamma\|x_n - x^\star\|\|x_{n+1} - x^\star\| + 2\theta_n\langle\gamma h(x^\star) - \mathcal{A}x^\star, x_{n+1} - x^\star\rangle \\
&\leq (1-\theta_n\overline{\gamma})^2\|x_n - x^\star\|^2 + \theta_n k\gamma(\|x_n - x^\star\|^2 + \|x_{n+1} - x^\star\|^2) + 2\theta_n\langle\gamma h(x^\star) - \mathcal{A}x^\star, x_{n+1} - x^\star\rangle \\
&\leq ((1-\theta_n\overline{\gamma})^2 + \theta_n k\gamma)\|x_n - x^\star\|^2 + \theta_n\gamma k\|x_{n+1} - x^\star\|^2 + 2\theta_n\langle\gamma h(q) - \mathcal{A}x^\star, x_{n+1} - x^\star\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1}-x^\star\|^2 &\le \tfrac{1-2\theta_n\overline{\gamma}+(\theta_n\overline{\gamma})^2+\theta_n\gamma k}{1-\theta_n\gamma k}\|x_n-x^\star\|^2+\tfrac{2\theta_n}{1-\theta_n\gamma k}\langle\gamma h(x^\star)-\mathcal{A}x^\star,x_{n+1}-x^\star\rangle\\ &=(1-\tfrac{2(\overline{\gamma}-\gamma k)\theta_n}{1-\theta_n\gamma k})\|x_n-x^\star\|^2+\tfrac{(\theta_n\overline{\gamma})^2}{1-\theta_n\gamma k}\|x_n-x^\star\|^2+\tfrac{2\theta_n}{1-\theta_n\gamma k}\langle\gamma h(x^\star)-\mathcal{A}x^\star,x_{n+1}-x^\star\rangle\\ &\le(1-\tfrac{2(\overline{\gamma}-\gamma k)\theta_n}{1-\theta_n\gamma k})\|x_n-x^\star\|^2+\tfrac{2(\overline{\gamma}-\gamma k)\theta_n}{1-\theta_n\gamma k}(\tfrac{(\theta_n\overline{\gamma}^2)M}{2(\overline{\gamma}-\gamma k)}+\tfrac{1}{\overline{\gamma}-\gamma k}\langle\gamma h(x^\star)-\mathcal{A}x^\star,x_{n+1}-x^\star\rangle)\\ &=(1-\eta_n)\|x_n-x^\star\|^2+\eta_n\delta_n,\end{aligned}$$

where

$$M=\sup\{\|x_n-x^\star\|^2:n\ge 0\},\ \ \eta_n=\frac{2(\overline{\gamma}-\gamma k)\theta_n}{1-\theta_n\gamma k},$$

and

$$\delta_n=\frac{(\theta_n\overline{\gamma}^2)M}{2(\overline{\gamma}-\gamma k)}+\frac{1}{\overline{\gamma}-\gamma k}\langle\gamma hq-\mathcal{A}x^\star,x_{n+1}-x^\star\rangle.$$

It is easy to see that $\eta_n\to 0$, $\sum\limits_{n=1}^{\infty}\eta_n=\infty$ and $\limsup\limits_{n\to\infty}\delta_n\le 0$. Hence, by Lemma 2.5, the sequence $\{x_n\}$ converges strongly to $x^\star$. Now, since $\lim\limits_{n\to\infty}\|x_n-w_n\|=\lim\limits_{n\to\infty}\|w_n-z_n\|=0$, we have that $\{w_n\}$ and $\{z_n\}$ converge strongly to $x^\star$.

Case 2. Assume that $\{\|x_n-x^\star\|\}$ is not a monotone sequence. Then, we can define a sequence $\{\tau(n)\}$ of integers for all $n\ge n_0$ (for some n_0 large enough) by

$$\tau(n):=\max\{k\in\mathbb{N};k\le n:\|x_k-x^\star\|<\|x_{k+1}-x^\star\|\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and for all $n\ge n_0$,

$$\|x_{\tau(n)}-x^\star\|<\|x_{\tau(n)+1}-x^\star\|.$$

From (3.4) we obtain

$$\lim_{n\to\infty}\|x_{\tau(n)}-z_{\tau(n)}\|=\lim_{n\to\infty}\|x_{\tau(n)}-w_{\tau(n)}\|=\lim_{n\to\infty}\|x_{\tau(n)}-T_ix_{\tau(n)}\|=0.$$

Following an argument similar to that in *Case* 1, we have

$$\|x_{\tau(n)+1}-x^\star\|^2\le(1-\eta_{\tau(n)})\|x_{\tau(n)}-x^\star\|^2+\eta_{\tau(n)}\delta_{\tau(n)},$$

where $\eta_{\tau(n)}\to 0$, $\sum\limits_{n=1}^{\infty}\eta_{\tau(n)}=\infty$ and $\limsup\limits_{n\to\infty}\delta_{\tau(n)}\le 0$. Therefore, by Lemma 2.5, we get that $\lim\limits_{n\to\infty}\|x_{\tau(n)}-x^\star\|=0$ and $\lim\limits_{n\to\infty}\|x_{\tau(n)+1}-x^\star\|=0$. Now Lemma 2.6 implies

$$0\le\|x_n-x^\star\|\le\max\{\|x_{\tau(n)}-x^\star\|,\|x_n-x^\star\|\}\le\|x_{\tau(n)+1}-x^\star\|.$$

Thus $\{x_n\}$ converges strongly to $x^\star=P_\Gamma(I-\mathcal{A}+\gamma h)x^\star$. This completes the proof. □

Now, following Shahzad and Zegeye [32], we remove the restriction $T(p)=\{p\}$ for all $p\in F(T)$. Let $T:C\to P(C)$ be a multivalued mapping and

$$P_T(x)=\{y\in Tx:\|x-y\|=dist(x,Tx)\}.$$

We have $P_T(p)=\{p\}$ for all $p\in F(T)$. Now, by using an argument similar to the one in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $f:C\times C\to\mathbb{R}$ be a bifunction satisfying* (A1)–(A3)*. Let $T_i:C\to K(C)$, $(i=1,2,\ldots,m)$ be a finite family of set-valued mappings, such that each P_{T_i} satisfies condition (E). Assume that $\Gamma=\bigcap_{i=1}^m F(T_i)\bigcap Sol(f,C)\neq\emptyset$. Let h be a k-contraction of C into itself and $\mathcal{A}$ be a strongly positive bounded linear self-adjoint operator on $\mathcal{H}$ with coefficient $\overline{\gamma}<1$, and $0<\gamma<\frac{\overline{\gamma}}{k}$. Let $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ be sequences generated by $x_0\in C$ and by:*

$$
\begin{cases}
w_n & = argmin\left\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : \ w \in C\right\}, \\
z_n & = argmin\left\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 : \ z \in C\right\}, \\
y_{n,1} & = \alpha_{n,1} z_n + (1-\alpha_{n,1}) u_{n,1}, \\
y_{n,2} & = \alpha_{n,2} z_n + (1-\alpha_{n,2}) u_{n,2}, \\
 & \vdots \\
y_{n,m} & = \alpha_{n,m} z_n + (1-\alpha_{n,m}) u_{n,m}, \\
x_{n+1} & = \theta_n \gamma h(x_n) + (I - \theta_n \mathcal{A}) y_{n,m}, \ \forall n \geq 0,
\end{cases} \tag{3.10}
$$

where $u_{n,1} \in P_{T_1}(z_n)$, $u_{n,k} \in P_{T_k}(y_{n,k-1}), (k = 2, \ldots, m)$. *Let* $\{\alpha_{n,i}\}, \{\lambda_n\}$ *and* $\{\theta_n\}$ *satisfy*

(i) $\{\theta_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \theta_n = 0$, $\sum\limits_{n=1}^{\infty} \theta_n = \infty$,

(ii) $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$,

(iii) $\{\alpha_{n,i}\} \subset [c,d] \subset (0,1)$ *for all* $1 \leq i \leq m$.

Then, the sequence $\{x_n\}$ *converges strongly to* $x^\star \in \bigcap_{i=1}^m F(T_i) \bigcap Sol(f,C)$, *which solves the variational inequality*

$$
\langle (\mathcal{A} - \gamma h)x^\star, x - x^\star \rangle \geq 0, \ \forall x \in \Gamma. \tag{3.11}
$$

As a consequence of Theorem 3.1, for a family of single valued mappings we have the following result:

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$ *and let* $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)–(A3). *Let* $T_i : C \to C, (i = 1, 2, \ldots, m)$ *be a finite family of single valued mappings, satisfying condition (E) such that* $\Gamma = \bigcap_{i=1}^m F(T_i) \bigcap Sol(f,C) \neq \emptyset$. *Let* h *be a* k*-contraction of* C *into itself and* $\mathcal{A}$ *be a strongly positive bounded linear self-adjoint operator on* $\mathcal{H}$ *with coefficient* $\overline{\gamma} < 1$, *and* $0 < \gamma < \frac{\overline{\gamma}}{k}$. *Let* $\{x_n\}$, $\{w_n\}$ *and* $\{z_n\}$ *be sequences generated by* $x_0 \in C$ *and by*

$$
\begin{cases}
w_n & = argmin\left\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : \ w \in C\right\}, \\
z_n & = argmin\left\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 : \ z \in C\right\}, \\
y_{n,1} & = \alpha_{n,1} z_n + (1-\alpha_{n,1}) T_1 z_n, \\
y_{n,2} & = \alpha_{n,2} z_n + (1-\alpha_{n,2}) T_2 y_{n,1}, \\
 & \vdots \\
y_{n,m} & = \alpha_{n,m} z_n + (1-\alpha_{n,m}) T_m y_{n,m-1}, \\
x_{n+1} & = \theta_n \gamma h(x_n) + (I - \theta_n \mathcal{A}) y_{n,m}, \ \forall n \geq 0.
\end{cases} \tag{3.12}
$$

Let $\{\alpha_{n,i}\}, \{\lambda_n\}$ *and* $\{\theta_n\}$ *satisfy*

(i) $\{\theta_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \theta_n = 0$, $\sum\limits_{n=1}^{\infty} \theta_n = \infty$,

(ii) $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$,

(iii) $\{\alpha_{n,i}\} \subset [c,d] \subset (0,1)$ *for all* $1 \leq i \leq m$.

Then, the sequence $\{x_n\}$ *converges strongly to* $x^\star \in \bigcap_{i=1}^m F(T_i) \bigcap Sol(f,C)$, *which solves the variational inequality*

$$
\langle (\mathcal{A} - \gamma h)x^\star, x - x^\star \rangle \geq 0, \ \forall x \in \Gamma. \tag{3.13}
$$

4. Application

In this section, we consider the particular equilibrium problem corresponding to the function f defined for every $x, y \in C$ by $f(x,y) = \langle F(x), y - x \rangle$, with $F : C \to \mathcal{H}$. Then, we obtain the classical variational problem:

$$\text{Find } z \in C \text{ such that } \langle F(z), y - z\rangle \geq 0,\ \forall y \in C.$$

The set of solutions of this problem is denoted by $VI(F,C)$. In this case, the solution y_n of the minimization problem

$$argmin\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 :\ y \in C\},$$

can be expressed by

$$y_n = P_C(x_n - \lambda_n F(x_n)).$$

Let F be L-Lipschitz continuous on C. Then

$$f(x,y) + f(y,z) - f(x,z) = \langle F(x) - F(y), y - z\rangle,\ x, y, z \in C.$$

Therefore

$$|\langle F(x) - F(y), y - z\rangle| \leq L\|x - y\|\|y - z\| \leq \frac{L}{2}(\|x - y\|^2 + \|y - z\|^2),$$

hence f satisfies the Lipschitz-type continuous condition with $c_1 = c_2 = \frac{L}{2}$.

As a consequence of Theorem 3.1, we have the following strong convergence results for approximate computing of the common element of the set of common fixed points of a finite family of set-valued mappings and the solution set of the problem $VI(F,C)$.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let F be a function from C to $\mathcal{H}$ such that F is monotone and L-Lipschitz continuous on C. Let $T_i : C \to CB(C), (i = 1, 2, \ldots, m)$ be a finite family of set-valued mappings, each satisfying condition (E). Assume that $\Gamma = \bigcap_{i=1}^m F(T_i) \bigcap VI(F,C) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, \ldots, m)$ for each $p \in \Gamma$. Let h be a k-contraction of C into itself and $\mathcal{A}$ be a strongly positive bounded linear self-adjoint operator on $\mathcal{H}$ with coefficient $\overline{\gamma} < 1$, and $0 < \gamma < \frac{\overline{\gamma}}{k}$. Let $\{x_n\}$ be sequence generated by $x_0 \in C$ and by*

$$\begin{cases} w_n &= P_C(x_n - \lambda_n F(x_n)), \\ z_n &= P_C(x_n - \lambda_n F(w_n)), \\ y_{n,1} &= \alpha_{n,1} z_n + (1 - \alpha_{n,1}) u_{n,1}, \\ y_{n,2} &= \alpha_{n,2} z_n + (1 - \alpha_{n,2}) u_{n,2}, \\ &\vdots \\ y_{n,m} &= \alpha_{n,m} z_n + (1 - \alpha_{n,m}) u_{n,m}, \\ x_{n+1} &= \theta_n \gamma h(x_n) + (I - \theta_n \mathcal{A}) y_{n,m},\ \forall n \geq 0, \end{cases} \tag{4.1}$$

where $u_{n,1} \in T_1 z_n, u_{n,k} \in T_k y_{n,k-1}, (k = 2, \ldots, m)$. Let $\{\alpha_{n,i}\}, \{\lambda_n\}$ and $\{\theta_n\}$ satisfy

(i) $\{\theta_n\} \subset (0,1),\ \lim_{n\to\infty} \theta_n = 0,\ \sum_{n=1}^{\infty} \theta_n = \infty,$

(ii) $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$,

(iii) $\{\alpha_{n,i}\} \subset [c, d] \subset (0, 1)$ *for all* $1 \leq i \leq m$.

Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in \bigcap_{i=1}^m F(T_i) \bigcap Sol(f, C)$, which solves the variational inequality

$$\langle(\mathcal{A} - \gamma h)x^\star, x - x^\star\rangle \geq 0, \qquad \forall x \in \Gamma.$$

As a consequence of Theorem 3.2, we also have the following strong convergence results for computing the approximate common solution of $VI(F,C)$ and $F(T)$ for a set-valued mapping in real Hilbert space.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let F be a function from C to $\mathcal{H}$ such that F is monotone and L-Lipschitz continuous on C. Let $T : C \to K(C)$, be a set-valued mapping, such that P_T satisfies condition (E). Assume that $\Gamma = \bigcap F(T) \bigcap VI(F, C) \neq \emptyset$. Let h be a k-contraction of C into itself. Let $\{x_n\}$ be sequence generated by $x_0 \in C$ and by*

$$\begin{cases} w_n &= P_C(x_n - \lambda_n F(x_n)), \\ z_n &= P_C(x_n - \lambda_n F(w_n)), \\ y_n &= \alpha_n z_n + (1-\alpha_n)u_n, \\ x_{n+1} &= \theta_n h(x_n) + (1-\theta_n)y_n, \ \forall n \geq 0, \end{cases} \tag{4.2}$$

where $u_n \in P_T z_n$. Let $\{\alpha_n\}, \{\lambda_n\}$ and $\{\theta_n\}$ satisfy

(i) $\{\theta_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \theta_n = 0$, $\sum\limits_{n=1}^{\infty} \theta_n = \infty$,

(ii) $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$,

(iii) $\{\alpha_n\} \subset [c,d] \subset (0,1)$.

Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in \Gamma$ which solves the variational inequality

$$\langle (I-h)x^\star, x - x^\star \rangle \geq 0, \ \forall x \in \Gamma.$$

Remark 4.3. Theorems 3.1 and 4.1 generalize the result of Anh [3] and Anh, Kim and Muu [5], respectively, for a single valued nonexpansive mapping to a finite family of set-valued mappings satisfying condition (E). We also weaken or remove some control conditions on parameters.

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