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On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function

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Abstract

In this paper we establish new Hermite-Hadamard type inequalities involving fractional integrals with respect to another function. Such fractional integrals generalize the Riemann-Liouville fractional integrals and the Hadamard fractional integrals.

Keywords: Hermite-Hadamard inequality, fractional integral with respect to another function, Riemann-Liouville fractional integral, Hadamard fractional integral.
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1. Introduction

One of the most known inequalities for convex functions is the Hermite-Hadamard inequality [13]. It states that if $f : I \to \mathbb{R}$ is a convex function, where I is an interval of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

Many generalizations and extensions of the Hermite-Hadamard inequality exist in the literatures; see [1]-[12], [14, 15], [18]-[23] and references therein. Recently, several Hermite-Hadamard type inequalities were obtained for various classes of functions using fractional integrals; see [3, 5, 6, 14, 15, 22, 23] and references therein.

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Our aim in this paper is to establish new Hermite-Hadamard inequalities for convex functions involving fractional integrals with respect to another function. The obtained results generalize some existing results from the literature including those obtained in [23].

At first, let us recall some definitions and mathematical preliminaries that will be used through this paper.

Let $f : [a, b] \to \mathbb{R}$ be a given function, where $0 < a < b < \infty$.

Definition 1.1 ([16])**.** The left-sided Riemann-Liouville fractional integral $J^{\alpha}_{a^+}$ of order $\alpha > 0$ of f is defined by

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad x > a, \tag{1.1}$$

provided that the integral exists. The right-sided Riemann-Liouville fractional integral $J^{\alpha}_{b^-}$ of order $\alpha > 0$ of f is defined by

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau - x)^{\alpha-1} f(\tau)\, d\tau, \quad x < b, \tag{1.2}$$

provided that the integral exists.

Definition 1.2 ([17, 16])**.** The left-sided Hadamard fractional integral $\mathbf{J}^{\alpha}_{a^+}$ of order $\alpha > 0$ of f is defined by

$$\mathbf{J}^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{\tau}\right)^{\alpha-1} \frac{f(\tau)}{\tau}\, d\tau, \quad x > a, \tag{1.3}$$

provided that the integral exists. The right-sided Hadamard fractional integral $\mathbf{J}^{\alpha}_{b^-}$ of order $\alpha > 0$ of f is defined by

$$\mathbf{J}^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{\tau}{x}\right)^{\alpha-1} \frac{f(\tau)}{\tau}\, d\tau, \quad x < b, \tag{1.4}$$

provided that the integral exists.

Definition 1.3 ([16])**.** Let $g : [a, b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $g'(x)$ on (a, b). The left-sided fractional integral of f with respect to the function g on $[a, b]$ of order $\alpha > 0$ is defined by

$$I^{\alpha}_{a^+;g} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(\tau) f(\tau)}{[g(x) - g(\tau)]^{1-\alpha}}\, dt, \quad x > a, \tag{1.5}$$

provided that the integral exists. The right-sided fractional integral of f with respect to the function g on $[a, b]$ of order $\alpha > 0$ is defined by

$$I^{\alpha}_{b^-;g} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(\tau) f(\tau)}{[g(\tau) - g(x)]^{1-\alpha}}\, dt, \quad x < b, \tag{1.6}$$

provided that the integral exists.

Observe that for $g(x) = x$, the fractional integral (1.5) reduces to the left-sided Riemann-Liouville fractional integral (1.1), and the fractional integral (1.6) reduces to the right-sided Riemann-Liouville fractional integral (1.2). However, for $g(x) = \ln x$, the fractional integral (1.5) reduces to the left-sided Hadamard fractional integral (1.3), and the fractional integral (1.6) reduces to the right-sided Hadamard fractional integral (1.4).

Using the change of variable

$$s = \frac{\tau - a}{x - a},$$

we have

$$I^{\alpha}_{a^+;g} f(x) = \frac{(x-a)}{\Gamma(\alpha)} \int_0^1 \frac{g'(sx + (1-s)a) f(sx + (1-s)a)}{[g(x) - g(sx + (1-s)a)]^{1-\alpha}}\, ds, \quad x > a. \tag{1.7}$$

Using the change of variable

$$s = \frac{\tau - x}{b - x},$$

we have

$$I^{\alpha}_{b^-;g} f(x) = \frac{(b-x)}{\Gamma(\alpha)} \int_0^1 \frac{g'(sb + (1-s)x) f(sb + (1-s)x)}{[g(sb + (1-s)x) - g(x)]^{1-\alpha}}\, ds, \quad x < b. \tag{1.8}$$

2. Main results

Let $f : \mathring{I} \to \mathbb{R}$ be a given function, where $a, b \in \mathring{I}$ and $0 < a < b < \infty$. We suppose that $f \in L^\infty(a, b)$ in such a way that $I^{\alpha}_{a^+;g} f(x)$ and $I^{\alpha}_{b^-;g} f(x)$ are well defined. We define the functions

$$\tilde{f}(x) = f(a + b - x), \quad x \in [a, b]$$

and

$$F(x) = f(x) + \tilde{f}(x), \quad x \in [a, b].$$

We have the following result.

Theorem 2.1. *Let $\alpha > 0$. If f is a convex function on $[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{4[g(b) - g(a)]^\alpha} \left(I^{\alpha}_{a^+;g} F(b) + I^{\alpha}_{b^-;g} F(a)\right) \le \frac{f(a) + f(b)}{2}. \tag{2.1}$$

Proof. For $s \in [0, 1]$, let $u = as + (1-s)b$ and $v = (1-s)a + bs$. The convexity of f yields

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{u+v}{2}\right) \le \frac{1}{2} f(u) + \frac{1}{2} f(v),$$

that is,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2} f(as + (1-s)b) + \frac{1}{2} f((1-s)a + bs). \tag{2.2}$$

Multiplying both sides of (2.2) by

$$\frac{(b-a)}{\Gamma(\alpha)} \frac{g'(sb + (1-s)a)}{[g(b) - g(sb + (1-s)a)]^{1-\alpha}}$$

and integrating over $(0, 1)$ with respect to s, we get

$$\begin{aligned} &\frac{(b-a)}{\Gamma(\alpha)} f\left(\frac{a+b}{2}\right) \int_0^1 \frac{g'(sb + (1-s)a)}{[g(b) - g(sb + (1-s)a)]^{1-\alpha}}\, ds \\ &\quad \le \frac{1}{2} \frac{(b-a)}{\Gamma(\alpha)} \int_0^1 \frac{g'(sb + (1-s)a) f(as + (1-s)b)}{[g(b) - g(sb + (1-s)a)]^{1-\alpha}}\, ds \\ &\qquad + \frac{1}{2} \frac{(b-a)}{\Gamma(\alpha)} \int_0^1 \frac{g'(sb + (1-s)a) f((1-s)a + bs)}{[g(b) - g(sb + (1-s)a)]^{1-\alpha}}\, ds. \end{aligned}$$

Using (1.7), we get

$$\begin{aligned} &\int_0^1 \frac{g'(sb + (1-s)a)}{[g(b) - g(sb + (1-s)a)]^{1-\alpha}}\, ds = \frac{1}{\alpha(b-a)} [g(b) - g(a)]^\alpha, \\ &\frac{(b-a)}{\Gamma(\alpha)} \int_0^1 \frac{g'(sb + (1-s)a) f((1-s)a + bs)}{[g(b) - g(sb + (1-s)a)]^{1-\alpha}}\, ds = I^{\alpha}_{a^+;g} f(b), \end{aligned}$$

$$\frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)f(as+(1-s)b)}{[g(b)-g(sb+(1-s)a)]^{1-\alpha}}\,ds = I^{\alpha}_{a^+;g}\tilde{f}(b).$$

As consequence, we have

$$\frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \le \frac{I^{\alpha}_{a^+;g}F(b)}{2}. \tag{2.3}$$

Similarly, multiplying both sides of (2.2) by

$$\frac{(b-a)}{\Gamma(\alpha)}\frac{g'(sb+(1-s)a)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}},$$

and integrating over $(0,1)$ with respect to s, we get

$$\begin{aligned}\frac{(b-a)}{\Gamma(\alpha)} f\left(\frac{a+b}{2}\right)\int_0^1 \frac{g'(sb+(1-s)a)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}}\,ds \\ \le \frac{1}{2}\frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)f(as+(1-s)b)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}}\,ds \\ + \frac{1}{2}\frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)f((1-s)a+bs)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}}\,ds.\end{aligned}$$

Using (1.8), we get

$$\begin{aligned}\int_0^1 \frac{g'(sb+(1-s)a)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}}\,ds &= \frac{1}{\alpha(b-a)}[g(b)-g(a)]^{\alpha},\\ \frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)f((1-s)a+bs)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}}\,ds &= I^{\alpha}_{b^-;g}f(a),\\ \frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)f(as+(1-s)b)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}}\,ds &= I^{\alpha}_{b^-;g}\tilde{f}(a).\end{aligned}$$

As consequence, we have

$$\frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \le \frac{I^{\alpha}_{b^-;g}F(a)}{2}. \tag{2.4}$$

By adding the above inequalities (2.3) and (2.4), we get

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{4[g(b)-g(a)]^{\alpha}}\left(I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)\right),$$

and the first inequality is proved.

Since f is convex, for every $s\in[0,1]$, we have

$$f(as+(1-s)b)+f((1-s)a+bs) \le f(a)+f(b). \tag{2.5}$$

Multiplying both sides of (2.5) by

$$\frac{(b-a)}{\Gamma(\alpha)}\frac{g'(sb+(1-s)a)}{[g(b)-g(sb+(1-s)a)]^{1-\alpha}},$$

and integrating over $(0,1)$ with respect to s, we get

$$\begin{aligned}\frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)f(as+(1-s)b)}{[g(b)-g(sb+(1-s)a)]^{1-\alpha}}\,ds + \frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)f((1-s)a+bs)}{[g(b)-g(sb+(1-s)a)]^{1-\alpha}}\,ds \\ \le (f(a)+f(b))\frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)}{[g(b)-g(sb+(1-s)a)]^{1-\alpha}}\,ds,\end{aligned}$$

which yields

$$I^{\alpha}_{a^+;g}F(b) \le \frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\left(f(a)+f(b)\right). \tag{2.6}$$

Similarly, multiplying both sides of (2.5) by

$$\frac{(b-a)}{\Gamma(\alpha)}\frac{g'(sb+(1-s)a)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}},$$

and integrating over $(0,1)$ with respect to s, we get

$$\begin{aligned}\frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)f(as+(1-s)b)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}}\,ds + \frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)f((1-s)a+bs)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}}\,ds\\ \le (f(a)+f(b))\frac{(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{g'(sb+(1-s)a)}{[g(sb+(1-s)a)-g(a)]^{1-\alpha}}\,ds,\end{aligned}$$

which yields

$$I^{\alpha}_{b^-;g}F(a) \le \frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\left(f(a)+f(b)\right). \tag{2.7}$$

By adding the above inequalities (2.6) and (2.7), we get

$$\frac{\Gamma(\alpha+1)}{4[g(b)-g(a)]^{\alpha}}\left(I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)\right) \le \frac{f(a)+f(b)}{2},$$

and the second inequality is proved. □

Take $g(x)=x$ in (2.1), we obtain the following result proved in [23].

Corollary 2.2. *Let $\alpha>0$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right) \le \frac{f(a)+f(b)}{2}.$$

Take $g(x)=\ln x$ in (2.1), we obtain the following result involving the Hadamard fractional integral.

Corollary 2.3. *Let $\alpha>0$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{4\left(\ln\frac{b}{a}\right)^{\alpha}}\left(\mathbf{J}^{\alpha}_{a^+}F(b)+\mathbf{J}^{\alpha}_{b^-}F(a)\right) \le \frac{f(a)+f(b)}{2}.$$

For $\alpha>0$, let $\Xi_{\alpha,g}:[0,1]\to\mathbb{R}$ be the function defined by

$$\begin{aligned}\Xi_{\alpha,g}(t) =&[g(ta+(1-t)b)-g(a)]^{\alpha}-[g(bt+(1-t)a)-g(a)]^{\alpha}\\ &+[g(b)-g(bt+(1-t)a)]^{\alpha}-[g(b)-g(ta+(1-t)b)]^{\alpha}.\end{aligned}$$

Before stating and proving our next result, we need the following lemma.

Lemma 2.4. *Let $\alpha>0$. If $f\in C^1(\mathring{I})$, then*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{4[g(b)-g(a)]^{\alpha}}\left(I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)\right) = \frac{(b-a)}{4[g(b)-g(a)]^{\alpha}}\int_0^1 \Xi_{\alpha,g}(t)f'(ta+(1-t)b)\,dt. \tag{2.8}$$

Proof. Using an integration by parts, we obtain

$$I^{\alpha}_{a^+;g}F(b) = \frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}F(a)+\frac{(b-a)}{\Gamma(\alpha+1)}\int_0^1 [g(b)-g(bs+(1-s)a)]^{\alpha}F'(bs+(1-s)a)\,ds. \tag{2.9}$$

Similarly, we have

$$I^{\alpha}_{b^-;g}F(a) = \frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}F(b) - \frac{(b-a)}{\Gamma(\alpha+1)}\int_0^1 [g(bs+(1-s)a)-g(a)]^{\alpha}F'(bs+(1-s)a)\,ds. \tag{2.10}$$

Using (2.9) and (2.10), we obtain

$$\begin{aligned}\frac{4[g(b)-g(a)]^{\alpha}}{(b-a)}&\left(\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{4[g(b)-g(a)]^{\alpha}}[I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)]\right) \\ &= \int_0^1 \left([g(bs+(1-s)a)-g(a)]^{\alpha} - [g(b)-g(bs+(1-s)a)]^{\alpha}\right)F'(bs+(1-s)a)\,ds.\end{aligned} \tag{2.11}$$

On the other hand, we have

$$F'(bs+(1-s)a) = f'(bs+(1-s)a) - f'(sa+(1-s)b), \quad s\in[0,1].$$

Then, we obtain

$$\begin{aligned}\int_0^1 &[g(bs+(1-s)a)-g(a)]^{\alpha}F'(bs+(1-s)a)\,ds \\ &= \int_0^1 [g(ta+(1-t)b)-g(a)]^{\alpha}f'(ta+(1-t)b)\,dt \\ &\quad - \int_0^1 [g(bt+(1-t)a)-g(a)]^{\alpha}f'(ta+(1-t)b)\,dt\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\int_0^1 &[g(b)-g(bs+(1-s)a)]^{\alpha}F'(bs+(1-s)a)\,ds \\ &= \int_0^1 [g(b)-g(ta+(1-t)b)]^{\alpha}f'(ta+(1-t)b)\,dt \\ &\quad - \int_0^1 [g(b)-g(tb+(1-t)a)]^{\alpha}f'(ta+(1-t)b)\,dt.\end{aligned} \tag{2.13}$$

Finally, (2.8) follows form (2.11), (2.12) and (2.13). □

For $\alpha>0$, we introduce the following operator

$$\mathcal{L}^{\alpha}_g(x,y) = \int_a^{\frac{a+b}{2}} |x-u||g(y)-g(u)|^{\alpha}\,du - \int_{\frac{a+b}{2}}^b |x-u||g(y)-g(u)|^{\alpha}\,du, \quad x,y\in[a,b].$$

We have the following result.

Theorem 2.5. *Let $\alpha>0$. If $f\in C^1(\mathring{I})$ and $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{4[g(b)-g(a)]^{\alpha}}[I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)]\right| \le \frac{I^{\alpha}_g(a,b)}{4[g(b)-g(a)]^{\alpha}(b-a)}\left(|f'(a)|+|f'(b)|\right), \tag{2.14}$$

where

$$I^{\alpha}_g(a,b) = \mathcal{L}^{\alpha}_g(b,b)+\mathcal{L}^{\alpha}_g(a,b)-\mathcal{L}^{\alpha}_g(b,a)-\mathcal{L}^{\alpha}_g(a,a).$$

Proof. Using Lemma 2.4 and the convexity of $|f'|$, we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{4[g(b)-g(a)]^{\alpha}}[I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)]\right|$$

$$\leq \frac{(b-a)}{4[g(b)-g(a)]^\alpha} \int_0^1 |\Xi_{\alpha,g}(t)||f'(ta+(1-t)b)|\,dt$$
$$\leq \frac{(b-a)}{4[g(b)-g(a)]^\alpha} \left(|f'(a)| \int_0^1 t|\Xi_{\alpha,g}(t)|\,dt + |f'(b)| \int_0^1 (1-t)|\Xi_{\alpha,g}(t)|\,dt \right). \tag{2.15}$$

On the other hand,

$$\int_0^1 t|\Xi_{\alpha,g}(t)|\,dt = \frac{1}{(b-a)^2} \int_a^b |\varphi(u)|(b-u)\,du,$$

where

$$\varphi(u) = [g(u)-g(a)]^\alpha - [g(b+a-u)-g(a)]^\alpha + [g(b)-g(a+b-u)]^\alpha - [g(b)-g(u)]^\alpha, \quad u \in [a,b].$$

Observe that φ is a non-decreasing function on $[a,b]$. Moreover, we have

$$\varphi(a) = -2[g(b)-g(a)]^\alpha < 0$$

and

$$\varphi\left(\frac{a+b}{2}\right) = 0.$$

As consequence, we have

$$\begin{cases} \varphi(u) \leq 0 & \text{if} \quad a \leq u \leq \frac{a+b}{2}, \\ \varphi(u) > 0 & \text{if} \quad \frac{a+b}{2} < u \leq b. \end{cases}$$

Hence, we obtain

$$(b-a)^2 \int_0^1 t|\Xi_{\alpha,g}(t)|\,dt = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_a^{\frac{a+b}{2}} (b-u)[g(b)-g(u)]^\alpha\,du - \int_{\frac{a+b}{2}}^b (b-u)[g(b)-g(u)]^\alpha\,du,$$
$$I_2 = -\int_a^{\frac{a+b}{2}} (b-u)[g(u)-g(a)]^\alpha\,du + \int_{\frac{a+b}{2}}^b [g(u)-g(a)]^\alpha(b-u),$$
$$I_3 = \int_a^{\frac{a+b}{2}} [g(b+a-u)-g(a)]^\alpha(b-u)\,du - \int_{\frac{a+b}{2}}^b [g(b+a-u)-g(a)]^\alpha(b-u)\,du,$$
$$I_4 = -\int_a^{\frac{a+b}{2}} [g(b)-g(a+b-u)]^\alpha(b-u)\,du + \int_{\frac{a+b}{2}}^b [g(b)-g(a+b-u)]^\alpha(b-u)\,du.$$

Observe that

$$I_1 = \mathcal{L}_g^\alpha(b,b) \quad \text{and} \quad I_2 = -\mathcal{L}_g^\alpha(b,a).$$

On the other hand, using the change of variable $v = a+b-u$, we get

$$I_3 = -\mathcal{L}_g^\alpha(a,a) \quad \text{and} \quad I_4 = \mathcal{L}_g^\alpha(a,b).$$

Thus, we obtain

$$\int_0^1 t|\Xi_{\alpha,g}(t)|\,dt = \frac{\mathcal{L}_g^\alpha(b,b)+\mathcal{L}_g^\alpha(a,b)-\mathcal{L}_g^\alpha(b,a)-\mathcal{L}_g^\alpha(a,a)}{(b-a)^2}. \tag{2.16}$$

Similarly, we obtain

$$\int_0^1 (1-t)|\Xi_{\alpha,g}(t)|\,dt = \frac{\mathcal{L}_g^\alpha(b,b)+\mathcal{L}_g^\alpha(a,b)-\mathcal{L}_g^\alpha(b,a)-\mathcal{L}_g^\alpha(a,a)}{(b-a)^2}. \tag{2.17}$$

Finally, the desired result follows from (2.15), (2.16) and (2.17). □

Take $g(x) = x$ in (2.14), we obtain the following result proved in [23].

Corollary 2.6. *Let $\alpha > 0$. If $f \in C^1(\mathring{I})$ and $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)]\right| \le \frac{(b-a)}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)[f'(a)+f'(b)].$$

Take $g(x) = \ln x$ in (2.14), we obtain the following result.

Corollary 2.7. *Let $\alpha > 0$. If $f \in C^1(\mathring{I})$ and $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{4\left(\ln\frac{b}{a}\right)^\alpha}\left[\mathbf{J}^\alpha_{a^+}F(b) + \mathbf{J}^\alpha_{b^-}F(a)\right]\right| \le \frac{I^\alpha_{\ln}(a,b)}{4[g(b)-g(a)]^\alpha(b-a)}\left(|f'(a)|+|f'(b)|\right),$$

where

$$I^\alpha_{\ln}(a,b) = \mathcal{L}^\alpha_{\ln}(b,b) + \mathcal{L}^\alpha_{\ln}(a,b) - \mathcal{L}^\alpha_{\ln}(b,a) - \mathcal{L}^\alpha_{\ln}(a,a).$$

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