Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 1323–1333
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Strong convergence of hybrid algorithms for fixed point and bifunction equilibrium problems in reflexive Banach spaces

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Communicated by C. Park

Abstract

Fixed point and bifunction equilibrium problems are studied via hybrid algorithms. Strong convergence theorems are established in the framework of reflexive Banach spaces. The results presented in this paper improve the corresponding results announced by many authors recently.

Keywords: Asymptotically quasi-ϕ-nonexpansive mapping, equilibrium problem, fixed point, generalized projection.
2010 MSC: 47J20, 47J25.

1. Introduction and Preliminaries

Bifunction equilibrium problem [4], which was introduced two decades ago, have emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in industry, finance, optimization, social, pure and applied sciences; see [3], [11], [12], [20], [22], [23] and the references therein. Since bifunction equilibrium problem covers variational inequality problems, zero point problems, and variational inclusion problems, it has been investigated by many authors via fixed point algorithms; see, for example, [5, 6, 8, 9, 10], [13]-[17], [24]-[28], [30] and the references therein.

Let E be a real Banach space and let E^* be the dual space of E. Let C be a nonempty subset of E and let G be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem. Find $\bar{x} \in C$ such that

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Received 2015-09-01

$$G(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

We use $Sol(G)$ to denote the solution set of equilibrium problem (1.1). That is,

$$Sol(G) = \{x \in C : 0 \leq G(x, y) \quad \forall y \in C\}.$$

Given a mapping $A : C \to E^*$, let

$$G(x, y) := \langle y - x, Ax\rangle \quad \forall x, y \in C.$$

Then $\bar{x} \in Sol(G)$ if $\bar{x}$ is a solution of the following variational inequality. Find $\bar{x}$ such that

$$\langle y - \bar{x}, A\bar{x}\rangle \geq 0 \quad \forall y \in C. \tag{1.2}$$

In order to study the solution of problem (1.1), we assume that G satisfies the following conditions:

(C-1) $G(x, x) = 0 \ \ \forall x \in C$;

(C-2) $0 \geq G(x, y) + G(y, x) \ \ \forall x, y \in C$;

(C-3) $G(x, y) \geq \limsup_{t\downarrow 0} G(tz + (1 - t)x, y) \ \ \forall x, y, z \in C$;

(C-4) $y \mapsto G(x, y)$ is weakly lower semi-continuous and convex for each $x \in C$.

Recall that a Banach space E is said to be strictly convex iff $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\|x_n\| = \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|\frac{x_n + y_n}{2}\| = 1.$$

Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x\| - \|x + ty\|}{t}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth if and only if the above limit is attained uniformly for $x, y \in U_E$. It is known that if E is uniformly smooth if and only if E^* is uniformly convex.

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. Recall that a Banach space E has the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach spaces, then E has the Kadec-Klee property.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \|x\|^2 = \langle x, f^*\rangle = \|f^*\|^2\},$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing.

It is known if E is uniformly smooth, then J is uniformly norm-to-norm continuous on every bounded subset of E; if E is a smooth Banach space, then J is single-valued and demicontinuous, i.e., continuous from the strong topology of E to the weak star topology of E; if E is smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto.

Consider the functional defined by

$$\phi(x, y) = \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle \quad \forall x, y \in E.$$

Observe that, in a Hilbert space H, the equality is reduced to $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection P_C in Hilbert spaces.

Recall that generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y\in C} \phi(y, x).$$

Existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| + \|x\|)^2 \geq \phi(x, y) \geq (\|x\| - \|y\|)^2 \quad \forall x, y \in E, \tag{1.3}$$

and

$$\phi(x, y) - \phi(x, z) = \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \quad \forall x, y, z \in E. \tag{1.4}$$

Let $T : C \to C$ be a mapping. In this paper, we use $Fix(T)$ to denote the fixed point set of T. T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Recall that a point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{Fix}(T)$.

Recall that a mapping T is said to be relatively nonexpansive if

$$\widetilde{Fix}(T) = Fix(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x) \quad \forall x \in C,\ \forall p \in Fix(T).$$

Recall that a mapping T is said to be relatively asymptotically nonexpansive if

$$\widetilde{Fix}(T) = Fix(T) \neq \emptyset, \quad \phi(p, T^n x) \leq k_n\phi(p, x) \quad \forall x \in C,\ \forall p \in Fix(T), \forall n \geq 1,$$

where $\{k_n\} \subset [1, \infty)$ is a sequence such that $k_n \to 1$ as $n \to \infty$.

Remark 1.1. The class of relatively asymptotically nonexpansive mappings, which covers the class of relatively nonexpansive mappings, was first considered in [1]. See also [21] and the references therein.

Recall that a mapping T is said to be quasi-ϕ-nonexpansive if

$$Fix(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x) \quad \forall x \in C,\ \forall p \in Fix(T).$$

Recall that a mapping T is said to be asymptotically quasi-ϕ-nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$Fix(T) \neq \emptyset, \quad \phi(p, T^n x) \leq k_n\phi(p, x) \quad \forall x \in C,\ \forall p \in Fix(T),\ \forall n \geq 1.$$

Remark 1.2. The class of asymptotically quasi-ϕ-nonexpansive mappings, which covers the class of quasi-ϕ-nonexpansive mappings [18], was considered in Qin, Cho and Kang [19] and Agarwal and Qin [17]; see also Zhou, Gao and Tan [31].

Remark 1.3. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ- nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive do not require the restriction $Fix(T) = \widetilde{Fix}(T)$.

In order to our main results, we also need the following lemmas.

Lemma 1.4 ([4]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let G be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (C-1), (C-2), (C-3) and (C-4). Let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$rG(z, y) + \langle z - y, Jx - Jz\rangle \geq 0 \quad \forall y \in C.$$

Lemma 1.5 ([2]). *Let E be a reflexive, strictly convex, and smooth Banach space, C a nonempty, closed, and convex subset of E and $x \in E$. Then*

$$\phi(y, x) \geq \phi(y, \Pi_C x) + \phi(\Pi_C x, x) \quad \forall y \in C.$$

Lemma 1.6 ([2]). *Let C be a nonempty, closed, and convex subset of a smooth Banach space E and $x \in E$. Then*

$$\Pi_C x = x_0,$$

if and only if

$$\langle x_0 - y, Jx_0 - Jx\rangle \leq 0 \quad \forall y \in C.$$

Lemma 1.7 ([19]). *Let E be a uniformly convex and smooth Banach space and let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed asymptotically quasi-ϕ-nonexpansive mapping. Then $Fix(T)$ is a closed convex subset of C.*

Lemma 1.8 ([18]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let G be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (C-1), (C-2), (C-3) and (C-4). Let $r > 0$ and $x \in E$. Define a mapping $S_r : E \to C$ by*

$$S_r x = \{z \in C : rG(z, y) + \langle y - z, Jz - Jx\rangle \geq 0 \quad \forall y \in C\}.$$

Then the following conclusions hold:

(1) *S_r is single-valued;*

(2) *S_r is a firmly nonexpansive-type mapping, i.e. for all $x, y \in E$,*

$$\langle S_r x - S_r y, Jx - Jy\rangle \geq \langle S_r x - S_r y, JS_r x - JS_r y\rangle;$$

(3) *S_r is quasi-ϕ-nonexpansive;*

(4) *$\phi(q, x) \geq \phi(q, S_r x) + \phi(S_r x, x)$ $\forall q \in Fix(S_r)$;*

(5) *$Sol(G) = Fix(S_r)$ is convex and closed.*

2. Main results

Theorem 2.1. *Let E be a uniformly convex and smooth Banach space such that E^* has the Kadec-Klee property and let C be a nonempty closed and convex subset of E. Let Λ be an index set and let G_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (C-1), (C-2), (C-3) and (C-4). Let $T_i : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping for every $i \in \Lambda$. Assume that T_i is closed and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = \Pi_{C_1} x_0, \\ y_{(n,i)} = J^{-1}(\alpha_{(n,i)} Jx_1 + (1-\alpha_{(n,i)}) JT_i^n x_n), \\ u_{(n,i)} \in C_n \textit{ such that } r_{(n,i)} G_i(u_{(n,i)}, y) + \langle y - u_{(n,i)}, Ju_{(n,i)} - Jy_{(n,i)}\rangle \geq 0 \quad \forall y \in C_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(z, x_n) + \alpha_{(n,i)} D \geq \phi(z, u_{(n,i)})\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $D := \sup\{\phi(w, x_1) : p \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)\}$, $\{\alpha_{(n,i)}\}$ *is a real sequence in* $(0,1)$ *such that*

$\lim_{n\to\infty} \alpha_{(n,i)} = 0$ *and* $\{r_{(n,i)}\}$ *is a real sequence in* $[a_i, \infty)$, *where* $\{a_i\}$ *is a positive real number sequence for every* $i \in \Lambda$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)} x_1$.

Proof. Using Lemma 1.7 and Lemma 1.8, we find that $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)$ is convex and closed so that the generalization projection onto the set is well defined.

Next, we prove that C_n is convex and closed. To show C_n is convex and closed, it suffices to show that, for each fixed but arbitrary $i \in \Lambda$, $C_{(n,i)}$ is convex and closed. This can be proved by induction on n. It is obvious that $C_{(1,i)} = C$ is convex and closed. Assume that $C_{(m,i)}$ is convex and closed for some $m \geq 1$. Let For $z_1, z_2 \in C_{(m+1,i)}$, we see that $z_1, z_2 \in C_{(m,i)}$. It follows that $z = tz_1 + (1-t)z_2 \in C_{(m,i)}$, where $t \in (0,1)$. Notice that

$$\phi(z_1, x_n) + \alpha_{(m,i)} D \geq \phi(z_1, u_{(m,i)})$$

and

$$\phi(z_2, x_n) + \alpha_{(m,i)} D \geq \phi(z_2, u_{(m,i)}).$$

The above inequalities are equivalent to

$$\|x_m\|^2 + \alpha_{(m,i)} D \geq 2\langle z_1, Jx_m - Ju_{(m,i)}\rangle + \|u_{(m,i)}\|^2$$

and

$$\|x_m\|^2 + \alpha_{(m,i)} D \geq 2\langle z_2, Jx_m - Ju_{(m,i)}\rangle + \|u_{(m,i)}\|^2.$$

Multiplying t and $(1-t)$ on the both sides of the inequalities above, respectively yields that and

$$\|x_m\|^2 - \|u_{(m,i)}\|^2 + \alpha_{(m,i)} D \geq 2\langle z, Jx_m - Ju_{(m,i)}\rangle.$$

That is,

$$\phi(z, x_n) + \alpha_{(m,i)} D \geq \phi(z, u_{(m,i)}),$$

where $z \in C_{(m,i)}$. This finds that $C_{(m+1,i)}$ is convex and closed. We conclude that $C_{(n,i)}$ is convex and closed. This in turn implies that $C_n = \cap_{i\in\Lambda} C_{(n,i)}$ is convex and closed. This implies that $\Pi_{C_{n+1}} x_1$ is well defined.

Now, we are in a position to prove $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i) \subset C_n$. Note that

$$C = C_1 \supseteq \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)$$

is clear. Suppose that $C_{(m,i)} \supseteq \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)$ for some positive integer m. For any $w \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i) \subset C_{(m,i)}$, we see that

$$
\begin{aligned}
\phi(w,x_m)+\alpha_{(m,i)}D &\geq \alpha_{(m,i)}\phi(w,x_1)+(1-\alpha_{(m,i)})\phi(w,T_i^m x_m)\\
&= \|w\|^2-2\alpha_{(m,i)}\langle w,Jx_1\rangle-2(1-\alpha_{(m,i)})\langle w,JT_i^m x_m\rangle\\
&\quad+\alpha_{(m,i)}\|x_1\|^2+(1-\alpha_{(m,i)})\|T_i^m x_m\|^2\\
&\geq \|w\|^2-2\langle w,\alpha_{(m,i)}Jx_1+(1-\alpha_{(m,i)})JT_i^m x_m\rangle\\
&\quad+\|\alpha_{(m,i)}Jx_1+(1-\alpha_{(m,i)})JT_i^m x_m\|^2\\
&= \phi(w,J^{-1}(\alpha_{(m,i)}Jx_1+(1-\alpha_{(m,i)})JT_i^m x_m))\\
&= \phi(w,y_{(m,i)})\\
&\geq \phi(w,S_{r_{(m,i)}}y_{(m,i)})\\
&= \phi(w,u_{(m,i)}),
\end{aligned} \tag{2.1}
$$

where

$$D:=\sup\{\phi(w,x_1):w\in\cap_{i\in\Lambda}F(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)\}.$$

This shows that $w\in C_{(m+1,i)}$. This implies that $C_{(n,i)}\supseteq\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)$. Hence, $\cap_{i\in\Lambda}C_{(n,i)}\supseteq\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)$. This completes the proof that $C_n\supseteq\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)$. In the light of the construction $\Pi_{C_n}x_1=x_n$, we find from Lemma 1.6 that $\langle x_n-z,Jx_1-Jx_n\rangle\geq 0$ for any $z\in C_n$. Since

$$C_n\supseteq\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i),$$

we find that

$$0\geq\langle w-x_n,Jx_1-Jx_n\rangle,\quad\forall w\in\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i). \tag{2.2}$$

Using Lemma 1.5, one sees that

$$
\begin{aligned}
&\phi(\Pi_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)}x_1,x_1)\\
&\geq\phi(\Pi_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)}x_1,x_1)-\phi(\Pi_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)}x_1,x_n)\\
&\geq\phi(x_n,x_1).
\end{aligned}
$$

This implies that $\{\phi(x_n,x_1)\}$ is bounded. Hence, $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume that $x_n\rightharpoonup\bar{x}$. Since C_n is convex closed, we find $C_n\ni\bar{x}$. This implies that $\phi(\bar{x},x_1)\geq\phi(x_n,x_1)$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$
\begin{aligned}
\phi(\bar{x},x_1)&\geq\limsup_{n\to\infty}\phi(x_n,x_1)\\
&\geq\liminf_{n\to\infty}\phi(x_n,x_1)\\
&=\liminf_{n\to\infty}(\|x_n\|^2-2\langle x_n,Jx_1\rangle+\|x_1\|^2)\\
&\geq\|\bar{x}\|^2-2\langle\bar{x},Jx_1\rangle+\|x_1\|^2\\
&=\phi(\bar{x},x_1),
\end{aligned}
$$

which implies that $\lim_{n\to\infty}\phi(x_n,x_1)=\phi(\bar{x},x_1)$. Hence, we have $\lim_{n\to\infty}\|x_n\|=\|\bar{x}\|$. Using the Kadec-Klee property of E, one has $x_n\to\bar{x}$ as $n\to\infty$. Since $x_n=\Pi_{C_n}x_1$, and

$$x_{n+1}=\Pi_{C_{n+1}}x_1\in C_{n+1}\subset C_n,$$

one sees $\phi(x_{n+1},x_1)\geq\phi(x_n,x_1)$. This shows that $\{\phi(x_n,x_1)\}$ is nondecreasing. Since it is bounded, we find that $\lim_{n\to\infty}\phi(x_n,x_1)$ exists. It follows that

$$\phi(x_{n+1},x_1)-\phi(x_n,x_1)\geq\phi(x_{n+1},x_n).$$

This implies that

$$\lim_{n\to\infty}\phi(x_{n+1},x_n)=0. \tag{2.3}$$

In the light of $C_{n+1} \ni \Pi_{C_{n+1}} x_1 = x_{n+1}$, we find that

$$\phi(x_{n+1}, x_n) + \alpha_{(n,i)} D \geq \phi(x_{n+1}, u_{(n,i)}).$$

This implies from (2.3) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_{(n,i)}) = 0. \tag{2.4}$$

Using inequality (1.3), we see

$$\lim_{n\to\infty} (\|x_{n+1}\| - \|u_{(n,i)}\|) = 0.$$

This implies that

$$\lim_{n\to\infty} \|u_{(n,i)}\| = \|\bar{x}\|.$$

On the other hand, we have

$$\lim_{n\to\infty} \|u_{(n,i)}\| = \|\bar{x}\| = \lim_{n\to\infty} \|Ju_{(n,i)}\| = \|J\bar{x}\|. \tag{2.5}$$

This implies that $\{Ju_{(n,i)}\}$ is bounded. Since both E and E^* are reflexive, we may assume that $Ju_{(n,i)} \rightharpoonup u^{(*,i)} \in E^*$. Since E is reflexive, we see $J(E) = E^*$. This shows that there exists an element $u^i \in E$ such that $Ju^i = u^{(*,i)}$. It follows that

$$\phi(x_{n+1}, u_{(n,i)}) = \|Ju_{(n,i)}\|^2 + \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_{(n,i)}\rangle.$$

Therefore, one has

$$0 \geq \|\bar{x}\|^2 + \|u^{(*,i)}\|^2 - 2\langle \bar{x}, u^{(*,i)}\rangle = \|\bar{x}\|^2 + \|u^i\|^2 - 2\langle \bar{x}, Ju^i\rangle = \phi(\bar{x}, u^i) \geq 0.$$

That is, $\bar{x} = u^i$, which in turn implies that $u^{(*,i)} = J\bar{x}$. It follows that $Ju_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* has the Kadec-Klee property, we obtain from (2.5) that

$$\lim_{n\to\infty} Ju_{(n,i)} = J\bar{x}.$$

Since $J^{-1} : E^* \to E$ is demi-continuous and E has the Kadec-Klee property, we obtain $u_{(n,i)} \to \bar{x}$, as $n \to \infty$. Using (2.1) and (2.3), one has

$$\lim_{n\to\infty} \phi(x_{n+1}, y_{(n,i)}) = 0. \tag{2.6}$$

Using inequality (1.3), we see

$$\lim_{n\to\infty} (\|x_{n+1}\| - \|y_{(n,i)}\|) = 0.$$

This implies that

$$\lim_{n\to\infty} \|y_{(n,i)}\| = \|\bar{x}\|.$$

On the other hand, we have

$$\|\bar{x}\| = \|J\bar{x}\| = \lim_{n\to\infty} \|Jy_{(n,i)}\| = \lim_{n\to\infty} \|y_{(n,i)}\|. \tag{2.7}$$

This implies that $\{Jy_{(n,i)}\}$ is bounded. Since both E and E^* are reflexive, we may assume that $Jy_{(n,i)} \rightharpoonup y^{(*,i)} \in E^*$. Since E is reflexive, we see $E^* = J(E)$. This shows that there exists an element $y^i \in E$ such that $y^{(*,i)} = Jy^i$. It follows that

$$\|x_{n+1}\|^2 + \|Jy_{(n,i)}\|^2 - 2\langle x_{n+1}, Jy_{(n,i)}\rangle = \phi(x_{n+1}, y_{(n,i)}).$$

Therefore, one has

$$\begin{aligned}0 &\geq \|\bar{x}\|^2 + \|y^{(*,i)}\|^2 - 2\langle \bar{x}, y^{(*,i)}\rangle \\ &= \|\bar{x}\|^2 + \|y^i\|^2 - 2\langle \bar{x}, Jy^i\rangle \\ &= \phi(\bar{x}, y^i).\end{aligned}$$

That is, $\bar{x} = y^i$, which in turn implies that $y^{(*,i)} = J\bar{x}$. It follows that $Jy_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* has the Kadec-Klee property, we obtain from (2.7) that

$$\lim_{n\to\infty} Jy_{(n,i)} = J\bar{x}.$$

Since $J^{-1} : E^* \to E$ is demi-continuous and E has the Kadec-Klee property, we obtain $y_{(n,i)} \to \bar{x}$, as $n \to \infty$. In view of $S_{r_{(n,i)}} y_{(n,i)} = u_{(n,i)}$, we see that

$$\langle y - u_{(n,i)}, Ju_{(n,i)} - Jy_{(n,i)}\rangle + r_{(n,i)} G_i(u_{(n,i)}, y) \geq 0 \quad \forall y \in C_n.$$

It follows from (C-2) that

$$\|y - u_{(n,i)}\|\|Ju_{(n,i)} - Jy_{(n,i)}\| \geq r_{(n,i)} G_i(y, u_{(n,i)}) \quad \forall y \in C_n.$$

In view of (C-4), one has $G_i(y, \bar{x}) \leq 0$ $\forall y \in C_n$. For $0 < t_i < 1$ and $y \in C$, define

$$y_{(t,i)} = t_i y + (1 - t_i)\bar{x}.$$

It follows that $y_{(t,i)} \in C$, which yields that $G_i(y_{(t,i)}, \bar{x}) \leq 0$. It follows from the (C-1) and (C-4) that

$$\begin{aligned}t_i G_i(y_{(t,i)}, y) &\geq t_i G_i(y_{(t,i)}, y) + (1 - t_i) G_i(y_{(t,i)}, \bar{x}) \\ &\geq G_i(y_{(t,i)}, y_{(t,i)}) \\ &= 0.\end{aligned}$$

That is, $G_i(y_{(t,i)}, y) \geq 0$. Letting $t_i \downarrow 0$, we obtain from (C-3) that $0 \leq G_i(\bar{x}, y)$, $\forall y \in C$. This implies that $\bar{x} \in Sol(G_i)$ for every $i \in \Lambda$. This shows that $\bar{x} \in \cap_{i\in\Lambda} Sol(G_i)$.

Next, we prove $\bar{x} \in \cap_{i\in\Lambda} Fix(T_i)$. Using the condition imposed on $\{\alpha_{(n,i)}\}$, one has

$$\lim_{n\to\infty} \|Jy_{(n,i)} - JT_i^n x_n\| = 0.$$

Using the fact

$$\|J\bar{x} - JT_i^n x_n\| \leq \|Jy_{(n,i)} - J\bar{x}\| + \|Jy_{(n,i)} - JT_i^n x_n\|,$$

one has $JT_i^n x_n \to J\bar{x}$ as $n \to \infty$, for every $i \in \Lambda$. Since J^{-1} is demi-continuous, we have $T_i^n x_n \rightharpoonup \bar{x}$ for every $i \in \Lambda$. Since

$$\|J(T_i^n x_n) - J\bar{x}\| \geq |\|T_i^n x_n\| - \|\bar{x}\||,$$

one has $\|T_i^n x_n\| \to \|\bar{x}\|$, as $n \to \infty$, for every $i \in \Lambda$. Since E has the Kadec-Klee property, one obtains

$$\lim_{n\to\infty} \|T_i^n x_n - \bar{x}\| = 0.$$

On the other hand, we have

$$\|T_i^{n+1} x_n - \bar{x}\| \leq \|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - \bar{x}\|.$$

In view of the uniformly asymptotic regularity of T_i, one has

$$\lim_{n\to\infty} \|T_i^{n+1} x_n - \bar{x}\| = 0,$$

that is, $T_i T_i^n x_n - \bar{x} \to 0$ as $n \to \infty$. Since every T_i is closed, we find that $T_i \bar{x} = \bar{x}$ for every $i \in \Lambda$.

Finally, we prove $\bar{x} = \Pi_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)} x_1$. Letting $n \to \infty$ in (2.2), we see that

$$\langle \bar{x} - w, Jx_1 - J\bar{x}\rangle \geq 0, \quad \forall w \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i).$$

In view of Lemma 1.6, we find that that $\bar{x} = \Pi_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)} x_1$. This completes the proof. $\square$

Remark 2.2. The algorithm is efficient for an uncountable infinite family of operators and bifunctions. The space does not require the uniform smoothness. Theorem 2.1, which mainly improve the corresponding results in [7], [19] and [29], unify the corresponding results announced recently.

From Theorem 2.1, the following result is not hard to derive.

Corollary 2.3. *Let E be a uniformly convex and smooth Banach space such that E^* has the Kadec-Klee property and let C be a nonempty, convex and closed subset of E. Let G be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (C-1), (C-2), (C-3) and (C-4). Let $T : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping. Assume that T is closed and uniformly asymptotically regular on C and $Fix(T) \cap Sol(G)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}((1-\alpha_n)JT^n x_n + \alpha_n J x_1), \\ u_n \in C_n \text{ such that } r_n G(u_n, y) + \langle y - u_n, Ju_n - Jy_n \rangle \geq 0 \quad \forall y \in C_n, \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) + \alpha_n D \geq \phi(z, u_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $D := \sup\{\phi(w, x_1) : p \in Fix(T) \cap Sol(G)\}$, $\{\alpha_n\}$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, and $\{r_n\}$ is a real sequence in $[a, \infty)$, where a is a positive real number. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{Fix(T)\cap Sol(G)} x_1$.

In Hilbert spaces, Theorem 2.1 is reduced to the following.

Corollary 2.4. *Let E be a Hilbert space and let C be a nonempty closed and convex subset of E. Let Λ be an index set and let G_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (C-1), (C-2), (C-3) and (C-4). Let $T_i : C \to C$ be an asymptotically quasi-nonexpansive mapping for every $i \in \Lambda$. Assume that T_i is closed and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_{(1,i)} = C, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = P_{C_1} x_0, \\ y_{(n,i)} = \alpha_{(n,i)} x_1 + (1-\alpha_{(n,i)}) T_i^n x_n, \\ u_{(n,i)} \in C_n \text{ such that } r_{(n,i)} G_i(u_{(n,i)}, y) + \langle y - u_{(n,i)}, u_{(n,i)} - y_{(n,i)} \rangle \geq 0 \quad \forall y \in C_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|z - x_n\|^2 + \alpha_{(n,i)} D \geq \|z - u_{(n,i)}\|^2\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases}$$

where $D := \sup\{\|w - x_1\|^2 : p \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)\}$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_{(n,i)} = 0$ and $\{r_{(n,i)}\}$ is a real sequence in $[a_i, \infty)$, where $\{a_i\}$ is a positive real number sequence for every $i \in \Lambda$. Then the sequence $\{x_n\}$ converges strongly to $P_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)} x_1$.

Proof. In the framework of Hilbert space, the class of asymptotically quasi-ϕ-nonexpansive mappings is reduced to the class of asymptotically quasi-nonexpansive mappings and the $\phi(x, y) = \|x - y\|^2$. Using Theorem 2.1, we can conclude the desired result immediately. □

Acknowledgement

The authors are grateful to the reviewers for the useful suggestions which improve the contents of this article.

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