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Generalized Kudryashov method for nonlinear fractional double sinh–Poisson equation

Seyma Tuluce Demiray[a,*], Hasan Bulut[a]

[a] *Department of Mathematics, Firat University, 23119, Elazig, Turkey.*

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Abstract

Using the generalized Kudryashov method (GKM), we derive exact solutions of the nonlinear fractional double sinh-Poisson equation. We obtain novel dark soliton solutions. Some numerical simulations were done to see the behavior of these solutions.

Keywords: Nonlinear fractional double sinh-Poisson equation, generalized Kudryashov method, travelling wave transformation, dark soliton solution.
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1. Introduction

The foremost accomplishment of solutions of nonlinear fractional differential equations lies in the fact that the analysis of exact solutions of these equations have achieved in interpretation of some physical phenomena. In past years, many physical problems in almost all branches of sciences and engineering were with success modelled using the concept of fractional-order derivative. The variety of solutions of these equations, that are achieved by several mathematical methods, are highly important in several sciences such as elastic media, plasma physics, applied mathematics, computer engineering, meteorology, chemical kinematics and electromagnetic theory [13, 16, 18].

During recent years, many scholars have developed a lot of techniques to find solutions of nonlinear fractional differential equations such as extended trial equation method [4], variational iteration method [5], homotopy decomposition method [1], Crank–Nicholson method [7], local fractional variational iteration method [27], Sumudu transform method [22] and so on. In this study, GKM [6, 23, 24, 25] will be handled to find exact solutions of the nonlinear fractional double sinh-Poisson equation.

*Corresponding author
Email addresses: seymatuluce@gmail.com (Seyma Tuluce Demiray), hbulut@firat.edu.tr (Hasan Bulut)

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Onsager have considered a form of the sinh-Poisson equation in the late 1940s, though never published it [8]. In terms of published work, the sinh-Poisson equation has been introduced by Joyce and Montgomery [12]. The solutions of sinh-Poisson equation are relevant to the long-time states of turbulent two-dimensional Navier–Stokes flows. This equation arises from a continuum (or "mean field") limit of a very large number of interacting, ideal, parallel line vortices [17]. It defines a stream function configuration of a stationary two-dimensional Euler flow. This equation is closely related to some known integrable soliton models, and has a lot of exact solutions in theoretical studies and applications [9]. The sinh-Poisson vorticity model equation is privileged in that it is S-integrable. Moreover, it allows iterative solution-generating methods associated with invariance under Backlund transformations, and is also proper to the bilinear operator method [26]. The sinh-Poisson equation occurs in many significant areas, particularly as vorticity equation in classical hydrodynamics [2]. The occasion to investigate the sinh-Poisson equation arrives from the existing proofs that this equation controls the asymptotic states of ideal fluids, or, more generally, of two-dimensional systems that can be turned to the dynamics of point-like elements interacting by the potential which is the inverse of the Laplacean operator [11, 20]. This equation is also found when there are two kinds of elements such as positive and negative vorticity and they are of equal orders. Then, this equation governs the states with maximum entropy of the discrete statistical system at negative temperatures [12]. This equation is exactly integrable on periodic domains since it possesses a pair of Lax operators [21].

We consider the following nonlinear fractional double sinh-Poisson equation [3]

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \frac{\partial^{2\beta} u}{\partial y^{2\beta}} = \sinh u + \sinh 2u, \quad 0 < \alpha, \beta < 1, \tag{1.1}$$

where α, β are the parameters defining the orders of the fractional derivatives.

Our purpose in this paper is to seek exact solutions of nonlinear fractional double sinh-Poisson equation. In Section 2, we give the basic structure of GKM [6, 23, 24, 25]. In Section 3, as an application, we search for exact solutions of nonlinear fractional double sinh-Poisson equation by means of GKM.

2. Basic Structure of GKM

In the past years, some scientists have developed the Kudryashov method [14, 15, 19]. But, in this paper, we construct a generalized form of it.

We handle the following nonlinear partial differential equation of fractional order for a function u of two real variables, space x and time y:

$$P(u, D_x^{\alpha} u, D_y^{\beta} u, \ldots) = 0. \tag{2.1}$$

The general stages of the GKM are clarified as follows:

Step 1. Primarily, we must take the travelling wave solution of (2.1) in the form

$$u(x,y) = u(\eta), \quad \eta = \frac{kx^{\alpha}}{\Gamma[1+\alpha]} + \frac{cy^{\beta}}{\Gamma[1+\beta]}, \tag{2.2}$$

where k and c are arbitrary constants. By using the chain rule

$$D_x^{2\alpha} u = (\sigma_x)^2 \frac{d^2 u(\eta)}{d\eta^2} D_x^{2\alpha} \eta, \quad D_y^{2\beta} u = (\sigma_y)^2 \frac{d^2 u(\eta)}{d\eta^2} D_y^{2\beta} \eta, \tag{2.3}$$

where σ_x and σ_y are sigma indices [10]; without loss of generality, we can take $\sigma_x = s$ and $\sigma_y = r$, where s and r are constants. Substituting (2.2) and (2.3) into (2.1) yields the nonlinear ordinary differential equation

$$N(u, u', u'', u''', \cdots) = 0, \tag{2.4}$$

where the prime denotes differentiation wit respect to η.

Step 2. Assume that the exact solutions of (2.4) can be written in the form

$$u(\eta) = \frac{\sum_{i=0}^{N} a_i Q^i(\eta)}{\sum_{j=0}^{M} b_j Q^j(\eta)} = \frac{A\left[Q(\eta)\right]}{B\left[Q(\eta)\right]}, \tag{2.5}$$

where Q is $\frac{1}{1\pm e^{\eta}}$. We emphasize that the function Q is the solution of the equation

$$Q_\eta = Q' = Q^2 - Q. \tag{2.6}$$

Taking into account (2.5), we gain

$$u'(\eta) = \frac{A'Q'B - AB'Q'}{B^2} = Q'\left[\frac{A'B - AB'}{B^2}\right] = (Q^2 - Q)\left[\frac{A'B - AB'}{B^2}\right], \tag{2.7}$$

$$u''(\eta) = \frac{Q^2 - Q}{B^2}\left[(2Q-1)(A'B - AB') + \frac{Q^2 - Q}{B}\left[B(A''B - AB'') - 2B'A'B + 2A(B')^2\right]\right]. \tag{2.8}$$

Step 3. According to the structure of the recommended method, we assume that the solution of (2.4) can be written in the form

$$u(\eta) = \frac{a_0 + a_1 Q + a_2 Q^2 + \cdots + a_N Q^N + \cdots}{b_0 + b_1 Q + b_2 Q^2 + \cdots + b_M Q^M + \cdots}. \tag{2.9}$$

To compute the values M and N in (2.9) that are the pole orders for the general solution of (2.4), we do as in the classical Kudryashov method on balancing the highest-order nonlinear terms in (2.4) and we can constitute formulas for M and N. So we can attain some values of M and N.

Step 4. Replacing (2.5) into (2.4) yields a polynomial $R(Q)$ of Q. By equating the coefficients of $R(Q)$ to zero, we acquire a system of algebraic equations. Solving this system, we can define λ and the variable coefficients of $a_0,\ a_1,\ a_2, \ldots, a_N,\ b_0,\ b_1,\ b_2, \ldots, b_M$. In this way, we find the exact solutions to (2.4).

3. Application of GKM to Nonlinear Fractional Double sinh-Poisson Equation

In this section, we look for the exact solutions of nonlinear fractional double sinh-Poisson equation by using GKM.

Example 3.1. We get the travelling wave solutions of (1.1) and take the transformation $u(x,y) = u(\eta)$ with $\eta = \frac{kx^\alpha}{\Gamma[1+\alpha]} + \frac{cy^\beta}{\Gamma[1+\beta]}$ where k and c are constants. By using the chain rule [10] and by taking $\sigma_x = s$ and $\sigma_y = r$, we obtain

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \sigma_x \frac{\partial u}{\partial \eta}\frac{\partial^\alpha \eta}{\partial x^\alpha} = \sigma_x \frac{\partial u}{\partial \eta} k = sku', \tag{3.1}$$

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = \frac{\partial^\alpha}{\partial x^\alpha}\left(sk\frac{\partial u}{\partial \eta}\right) = sk\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial u}{\partial \eta}\right) = sk\left[\sigma_x \frac{\partial\left(\frac{\partial u}{\partial \eta}\right)}{\partial \eta}\frac{\partial^\alpha \eta}{\partial x^\alpha}\right] = k^2 s^2 u'', \tag{3.2}$$

$$\frac{\partial^\beta u}{\partial y^\beta} = \sigma_y \frac{\partial u}{\partial \eta}\frac{\partial^\beta \eta}{\partial y^\beta} = \sigma_y \frac{\partial u}{\partial \eta} c = rcu', \tag{3.3}$$

$$\frac{\partial^{2\beta} u}{\partial y^{2\beta}} = \frac{\partial^\beta}{\partial y^\beta}\left(rc\frac{\partial u}{\partial \eta}\right) = rc\frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial u}{\partial \eta}\right) = rc\left[\sigma_y \frac{\partial\left(\frac{\partial u}{\partial \eta}\right)}{\partial \eta}\frac{\partial^\beta \eta}{\partial y^\beta}\right] = r^2 c^2 u''. \tag{3.4}$$

Then, (1.1) turns to the following equation

$$(k^2 s^2 + r^2 c^2) u'' = \sinh u + \sinh 2u. \tag{3.5}$$

In order to implement GKM to (3.5), we take the transformations

$$\sinh u = \frac{e^u - e^{-u}}{2}, \quad \sinh 2u = \frac{e^{2u} - e^{-2u}}{2}, \quad v = e^u, \tag{3.6}$$

then we obtain the following relations

$$u'' = \frac{1}{v}v'' - \frac{1}{v^2}(v')^2, \quad \sinh u = \frac{v - v^{-1}}{2}, \quad \sinh 2u = \frac{v^2 - v^{-2}}{2}. \tag{3.7}$$

Putting (3.7) into (3.5), we get

$$2(k^2s^2 + r^2c^2)vv'' - 2(k^2s^2 + r^2c^2)(v')^2 - v^4 - v^3 + v + 1 = 0 \tag{3.8}$$

Setting (2.5) and (2.8) into (3.8) and balancing the highest order nonlinear terms of vv'' and v^4 (3.8), we obtain

$$2N - 2M + 2 = 4N - 4M \Longrightarrow N = M + 1. \tag{3.9}$$

Choosing $M = 1$ so that $N = 2$, we infer

$$u(\eta) = \frac{a_0 + a_1Q + a_2Q^2}{b_0 + b_1Q}, \tag{3.10}$$

$$u'(\eta) = (Q^2 - Q)\left[\frac{(a_1 + 2a_2Q)(b_0 + b_1Q) - b_1(a_0 + a_1Q + a_2Q^2)}{(b_0 + b_1Q)^2}\right], \tag{3.11}$$

$$\begin{aligned} u''(\eta) = &\frac{Q^2 - Q}{(b_0 + b_1Q)^2}(2Q - 1)\left[(a_1 + 2a_2Q)(b_0 + b_1Q) - b_1(a_0 + a_1Q + a_2Q^2)\right] \\ &+ \frac{(Q^2 - Q)^2}{(b_0 + b_1Q)^3}\left[2a_2(b_0 + b_1Q)^2 - 2b_1(a_1 + 2a_2Q)(b_0 + b_1Q) + 2b_1^2(a_0 + a_1Q + a_2Q^2)\right]. \end{aligned} \tag{3.12}$$

The exact solutions of (1.1) are found as follows.

Case 1

$$a_0 = \frac{1}{2}i(i + \sqrt{3})b_0, \quad a_1 = -\frac{1}{6}i\left[(-3i + \sqrt{3})a_2 + 6\sqrt{3}b_0\right], \quad b_1 = \frac{ia_2}{\sqrt{3}}, \quad c = -\frac{\sqrt{-\frac{3}{2} - k^2s^2}}{r}. \tag{3.13}$$

If we substitute (3.13) into (3.10), we find dark soliton solutions of (1.1):

$$u_1(x, y) = \ln\left[K + L(1 - \tanh(k_1x^\alpha + c_1y^\beta))\right], \tag{3.14}$$

$$u_2(x, y) = \ln\left[K + L(1 - \coth(k_1x^\alpha + c_1y^\beta))\right], \tag{3.15}$$

where $K = \frac{1}{2}i(i + \sqrt{3})$, $L = -\frac{i\sqrt{3}}{2}$, $k_1 = \frac{k}{2\Gamma(1+\alpha)}$ and $c_1 = -\frac{1}{2\Gamma(1+\beta)}\frac{\sqrt{-\frac{3}{2} - k^2s^2}}{r}$.

Case 2

$$a_0 = \frac{1}{6}i(3i + \sqrt{3})a_2, \quad a_1 = \frac{1}{6}(3 + i\sqrt{3})a_2, \quad b_0 = -\frac{ia_2}{\sqrt{3}}, \quad b_1 = -\frac{ia_2}{\sqrt{3}}, \quad c = -\frac{\sqrt{-\frac{3}{2} - k^2s^2}}{r}. \tag{3.16}$$

When we put (3.16) into (3.10), we acquire dark soliton solutions of (1.1):

$$u_3(x, y) = \ln\left[K + L(1 + \tanh(k_1x^\alpha + c_1y^\beta))\right], \tag{3.17}$$

$$u_4(x, y) = \ln\left[K + L(1 + \coth(k_1x^\alpha + c_1y^\beta))\right]. \tag{3.18}$$

Case 3

$$a_0 = \frac{1}{6}(3 + i\sqrt{3})a_2, \quad a_1 = \left[-1 - \frac{i}{\sqrt{3}}\right]a_2, \quad b_0 = -\frac{ia_2}{\sqrt{3}}, \quad b_1 = \frac{2ia_2}{\sqrt{3}}, \quad c = -\frac{1}{2}\frac{\sqrt{-\frac{3}{2} - 4k^2s^2}}{r}. \tag{3.19}$$

Figure 1: The graph of imaginary values of (3.20) is drawn for $k = 1$, $r = 0.9$, $s = 0.8$, $\alpha = 0.25$, $\beta = 0.75$, $-35 < x < 35$, $-1 < y < 1$ and the second graph shows imaginary values of (3.20) for $-35 < x < 35$, $y = 1$.

Figure 2: The graph of real values of (3.20) is drawn for $k = 1$, $r = 0.9$, $s = 0.8$, $\alpha = 0.25$, $\beta = 0.75$, $-35 < x < 35$, $-1 < y < 1$ and the second graph shows real values of (3.20) for $-35 < x < 35$, $y = 1$.

If we set (3.19) into (3.10), we procure dark soliton solutions of (1.1)

$$u_5(x,y) = \ln\left[K + P\left(\frac{\left(\frac{1}{2} - \frac{1}{2}\tanh(k_1 x^\alpha + c_2 y^\beta)\right)^2}{\tanh(k_1 x^\alpha + c_2 y^\beta)}\right)\right], \tag{3.20}$$

$$u_6(x,y) = \ln\left[K + P\left(\frac{\left(\frac{1}{2} - \frac{1}{2}\coth(k_1 x^\alpha + c_2 y^\beta)\right)^2}{\coth(k_1 x^\alpha + c_2 y^\beta)}\right)\right], \tag{3.21}$$

where $P = i\sqrt{3}$ and $c_2 = -\frac{1}{4\Gamma(1+\beta)}\frac{\sqrt{-\frac{3}{2}-4k^2s^2}}{r}$.

Case 4

$$a_0 = \frac{1}{4}(1+\sqrt{2})a_2, a_1 = -\frac{1}{2}(2+\sqrt{2})a_2, b_0 = -\frac{1}{4}(1+\sqrt{2})a_2, b_1 = \frac{a_2}{\sqrt{2}}, c = -\frac{\sqrt{1-k^2s^2}}{r}. \tag{3.22}$$

When we embed (3.22) into (3.10), we attain dark soliton solutions of (1.1)

$$u_7(x,y) = \ln\left[\frac{\tanh(k_1 x^\alpha + c_3 y^\beta)\left[\tanh(k_1 x^\alpha + c_3 y^\beta) - \sqrt{2}\right]}{-\sqrt{2}\tanh(k_1 x^\alpha + c_3 y^\beta) - 1}\right], \tag{3.23}$$

$$u_8(x,y) = \ln\left[\frac{\coth(k_1 x^\alpha + c_3 y^\beta)\left[\coth(k_1 x^\alpha + c_3 y^\beta) - \sqrt{2}\right]}{-\sqrt{2}\coth(k_1 x^\alpha + c_3 y^\beta) - 1}\right], \tag{3.24}$$

where $c_3 = -\frac{1}{2\Gamma(1+\beta)}\frac{\sqrt{1-k^2s^2}}{r}$.

Figure 3: The graph of imaginary values of (3.23) is drawn for $k = 2, r = 0.9, s = 0.8, \alpha = 0.75, \beta = 0.5, 0 < x < 25, 0 < y < 1$ and the second graph shows imaginary values of (3.23) for $0 < x < 25$, $y = 1$.

Figure 4: The graph of real values of (3.23) is drawn for $k = 2$, $r = 0.9$, $s = 0.8$, $\alpha = 0.75$, $\beta = 0.5$, $0 < x < 25$, $0 < y < 1$ and the second graph gives real values of (3.23) for $0 < x < 25$, $y = 1$.

Remark 3.2. By his method, Kudryashov has found exact solution of the equation considered as $u(\eta) = \sum_{i=0}^{N} a_i Q^i(\eta)$. But we construct a generalized form of his method and find an exact solution of equation (1.1) in the form (2.5). Necessary calculations are performed via Mathematica, Release 9. As far as we know, all the solutions of (1.1) that we reported here, are new and are not trackable in the former literature.

In Figures 1–2, we plot two- and three-dimensional graphics of imaginary and real values of (3.20), which illustrate the dynamics of solutions with appropriate parametric selections. Also in Figures 3–4, we draw two- and three-dimensional graphics of imaginary and real values of (3.23), which illustrate the dynamics of solutions with convenient parametric choices.

4. Conclusion

In this paper, we derive exact solutions of nonlinear fractional double sinh-Poisson equation by the help of GKM. Thus, we attain dark soliton solutions of this equation. According to these results, generalized Kudryashov method is influential and powerful in providing new exact solutions of nonlinear fractional differential equations. We have succeeded in constructing a proper algorithm to get exact solutions of this equation. We believe that the proposed method can also be implemented to other nonlinear fractional differential equations which arise in physical sciences.

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