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Quasilinearization method for nonlinear differential equations with causal operators

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Abstract

Employing quasilinearization technique coupled with the method of upper and lower solutions, we construct monotone sequences whose iterates are solutions to corresponding linear problems and show that the sequences converge uniformly and monotonically to the unique solution of the nonlinear problem with causal operator. Especially, instead of assuming convexity or concavity assumption on the nonlinear term that is demanded by the method of quasilinearization, we impose weaker conditions to be more useful in applications. The results obtained include several special cases and extend previous results.

Keywords: Quasilinearization, differential equation, causal operator, nonlinearity, initial value problem, quadratic convergence.
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1. Introduction

Causal differential equations (CDEs) or differential equations with causal operators, namely, non-anticipative or abstract Volterra operators, appear in applied sciences and many branches of engineering. Especially, they are very common equations for modeling problems in mechanical engineering, physical engineering, electric and electronics engineering, etc. [14, 22, 23, 24, 25]. Moreover, causality is a basic concept in physical sciences to describe the process of cause and effect in a particular situation.

In further development of nonlinear differential equations together with its importance in physical systems, much attention has been given to CDEs. They have been a very popular and considerable subject in applied mathematics because of the fact that the theory of CDEs has the powerful quality of unifying ordinary differential equations, differential equations with finite or infinite delay, integro-differential equations,

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integral equations, functional equations, and so on. All these equations are special cases of CDEs. The reader can consult [5, 6] for a monograph on this subject with more details. After Corduneanu's work [6], a lot of research has been done on CDEs. A systematic account of these developments and new extensions of CDEs in infinite dimensional spaces, fractional CDEs, set differential equations with causal operators and CDEs with retardation and anticipation, have been considered in detail by Lakshmikantham, founder of nonlinear federation, in [18].

On the other hand, linear and nonlinear initial value problems (IVPs) for differential equations appear in many areas of applied mathematics, physics and in variational problems of circuit, system and control theory. Since the initial value is commonly related to the time variable and every physical process has a starting time, IVPs are more commonly known (met, used, dealt, encountered) in physical systems. Moreover, it is known that various initial boundary value problems for ordinary and partial differential equations can be reduced to IVPs (see [9, 16]). This reduction allows us to study differential equations with operator coefficient in abstract spaces (see [1, 2, 15, 26] and the references therein). One of the most significant and important central point for a nonlinear IVP is to understand how nonlinearity affects the nature and characteristic of the solution. Even though questions of existence and uniqueness for linear and nonlinear boundary value problems (BVPs) appear more difficult than for IVPs, in fact, there is no specific general theory for the answer of this question. Nevertheless, there is a large cycle of works on the existence and uniqueness of the solutions of nonlinear IVPs. Monotone iterative technique (MIT) and the method of quasilinearization are well-known and the most common methods in the literature for the answer of the question.

As we know, for proving basic results of existence for nonlinear differential equations, the MIT is a powerful and flexible method providing a useful mechanism to construct monotone flows from corresponding linear equations [17] using the upper and lower solutions as initial iterations. Actually, it is shown that these flows converge monotonically to the extremal solutions of nonlinear equations. In other respects, the method of quasilinearization is a well-known technique to obtain approximate solutions of nonlinear differential equations with rapid convergence. The fundamental of the method of quasilinearization lies in the theory of dynamical programming. Indeed, the quasilinearization technique is a variant version of Newton's method. It can be used for both IVPs and BVPs. Besides, by the method of quasilinearization, one gets monotone schemes whose iterates converge uniformly and quadratically to the unique solution of the problem at hand [21, 29, 30]. Generally, this method is implemented to the problems with convex or concave nonlinearities. In view of its miscellaneous usage and applications, the quasilinearization approach is quite wondrous and easier for applications. Therefore, this method is effective for obtaining approximate solutions of nonlinear differential equations with finite or infinite delay, integral equations, functional equations, and so on.

A great deal of works on the method of quasilinearization spearheaded by Bellman [3, 4] and Kalaba [4] has been done in the literature. It provides a descent approach to obtain approximate solutions of nonlinear BVPs. Lakshmikantham and many coauthors have developed the method extensively and applied it to a wide range of problems. We refer the reader to the works by Lakshmikantham et al. [19, 20, 21] and the references therein.

To the best of our knowledge, in researches on causal differential equations, BVPs have mainly been considered as the main problem [7, 8, 10, 12, 13, 18, 27]. Moreover, MIT has been generally applied to the basic results concerning the existence problems (see [7, 10, 11, 12, 13, 28] and the references therein). For example, in [8], the notion of a causal operator has been extended to periodic BVPs

$$\begin{cases} x'(t) = (Qx)(t), \\ x(0) = x(2\pi), \end{cases}$$

and a MIT has been developed to obtain the existence of a solution in a closed set. In [12], Jankowski, a Polish mathematician on this subject, investigated the following nonlinear four-point BVPs for second order differential equation

$$\begin{cases} x''(t) = (Qx)(t), \ t \in J = [0, T], \\ g_1(x(0), x(\delta)) = 0, \ 0 < \delta < T, \\ g_2(x(T), x(\gamma)) = 0, \ 0 < \gamma < T, \end{cases} \tag{1.1}$$

where $g_1, g_2 \in C(R \times R,\ R)$. Sufficient conditions for the existence of solutions, using a monotone iterative method, were obtained. In [13], nonlinear two-point BVPs for first-order differential equations with causal operators Q of the form

$$\begin{cases} x'(t) = (Qx)(t),\ t \in J = [0, T], \\ g(x(0), x(T)) = 0 \end{cases}$$

were investigated by the same author and sufficient conditions for the existence of solutions were obtained by using MIT. Note that the above nonlinear BVP reduces to a periodic BVP for $g(x(0),\, x(T)) = x(0) - x(T)$ and an IVP for $g(x(0),\, x(T)) = x(0) - x_0$. In the same year, in [10], Geng considered the same problem and the existence of extremal solutions to problem (1.1) was established by utilizing MIT and the method of upper and lower solutions. After a couple of years, in [27], Wang and Tian considered the following problem

$$\begin{cases} x'(t) = (Qx)(t) + (Px)(t), t \in J = [0, T], \\ g(x(0), x(T)) = 0 \end{cases}$$

on the new concepts of upper and lower solutions by constructing monotone sequences and alternating sequences converging uniformly to the coupled minimal and maximal solutions to the problem.

Finally, the same authors of the paper [27] investigated nonlinear BVPs for difference equations with causal operators in [28] by using the method of upper and lower solutions coupled with the monotone iterative technique.

A large cycle of works exist on the basic results of linear and nonlinear BVPs for CDEs where MIT is used for the results of these problems. However, there are a few rare studies in which the method of quasilinearization is used for the basic results of IVPs by causal operators.

In [18], Lakshmikantham et al. considered the following causal differential equation

$$\begin{cases} x'(t) = (Qx)(t), t \in J = [0, T], \\ x(0) = x_0, \end{cases} \tag{1.2}$$

where the operator Q satisfies the convexity assumption. The method of quasilinearization is employed to get lower and upper bounds concurrently. Subsequently, they investigate the case when Q admits a decomposition into a difference of two convex or concave parts

$$\begin{cases} x'(t) = (Qx)(t) + (Px)(t), t \in J = [0, T], \\ x(0) = x_0, \end{cases}$$

with causal operators Q, P. The results are valid for initial value problems.

In [29], by choosing a weaker assumption with initial functions having different initial times, this technique is appropriately applied to equation (1.2).

The main goal of this study is to improve Lakshmikantham's study [18] for equation (1.2) containing general classes of functions. For this goal, we exert the quasilinearization technique coupled with upper and lower solutions to study CDEs for which particular and general results, including several special cases, are obtained. As a result, we find monotone sequences whose iterates are solutions to corresponding linear problems and show that the sequences converge uniformly and monotonically to the unique solution of the nonlinear problem with causal operators. Especially, instead of imposing the convexity or the concavity assumption on the operators involved, we assume weaker conditions. Furthermore, we demonstrate that these monotone sequences converge semi-quadratically.

2. Preliminaries

In this section, some basic definitions related to causal operator, upper and lower solutions, and some theorems about existence and uniqueness of CDEs are presented and they are necessary for the development of the paper.

Definition 2.1. An operator $S : E \to E$, $E = C([t_0, t_0 + T], R^n)$, is called causal (or non-anticipatory) if, for any $x, y \in E$ such that $x(s) = y(s)$, we have $(Sx)(s) = (Sy)(s)$ for $t_0 \le s < t_0 + T$.

Consider the following initial value problem (IVP) for CDE

$$\begin{cases} x'(t) = (Sx)(t), t \in J = [0, T], \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $S : E \to E$ is a continuous operator, $E = C(J, \mathbb{R})$, $J = [0, T]$.

Definition 2.2. Solutions $y,\ z \in C^1(J, \mathbb{R})$ are said to be lower and upper solutions to (2.1) respectively if they satisfy the following inequalities

$$y' \le (Sy)(t),\ y(0) \le x_0,$$

$$z' \ge (Sz)(t),\ z(0) \ge x_0,$$

for $t \in J$.

Definition 2.3. An operator $S : E \to E$, $E = C([t_0,\ t_0 + T],\ R^n)$ is said to be semi-nondecreasing at t for each x if

$$(Sx)(t_1) = (Sy)(t_1) \text{ and } (Sx)(t) \le (Sy)(t),\ t_0 \le t < t_1 < t_0 + T$$

for

$$x(t_1) = y(t_1),\ x(t) < y(t),\ t_0 \le t < t_1 < t_0 + T.$$

Next theorem is related to the uniqueness of the solution of (2.1) under the Lipschitz condition.

Theorem 2.4. *Assume that the operator S satisfies*

$$|(Sx)(t) - (Sy)(t)| \le L|x(t) - y(t)|,\ x, y \in \Omega,\ L > 0,$$

where $\Omega = \left\{ x, y \in E : \max\limits_{t_0 \le s \le t} |x(s) - y(s)| = |x(t) - y(t)| \right\}$. *Then, there exists a unique solution $x(t)$ to IVP* (2.1) *on J.*

For a proof of the above theorem, see [18].

If we know the existence of lower and upper solutions y, z such that $y(t) \le z(t),\ t \in J$ to IVP (2.1), then existence of a solution to (2.1) can be proved in the sector

$$\tilde{\Omega} = \{x \in E :\ y(t) \le x(t) \le z(t),\ t \in J\}.$$

Accordingly, we give an existence result in this special closed set $\tilde{\Omega}$ generated by lower and upper solutions.

Theorem 2.5. *Let $y,\ z \in C(J, \mathbb{R})$ be lower and upper solutions to* (2.1) *such that $y(t) \le z(t),\ t \in J$. Further, we assume that the operator S is bounded on $\tilde{\Omega}$. Then, there exists a solution $x(t)$ to* (2.1) *satisfying $y(t) \le x(t) \le z(t),\ t \in J$.*

For a detailed proof of the above theorem, see [18].

3. Main Theorem

Let the operator S in (2.1) admits a splitting into three parts as $P + Q + R$. In that case, problem (2.1) has the following form

$$\begin{cases} x'(t) = Px(t) + Qx(t) + Rx(t),\ t \in J = [0, T], \\ x(0) = x_0, \end{cases}$$

where $P, Q, R : E \to E$ are continuous operators. As we will state below, P satisfies a weaker condition than convexity, Q satisfies a weaker condition than concavity and R is Lipschitzian. We are now in a position to give the main result.

Theorem 3.1. *Suppose that the following hypotheses hold:*

H*$_1$*: $y_0, z_0 \in C^1(J,R),\ y_0(t) \le z_0(t)$ *on* J *and for* $t \in J$

$$\begin{aligned} y_0' &\le (Sy_0)(t),\ y_0(0) \le x_0, \\ z_0' &\ge (Sz_0)(t),\ z_0(0) \ge x_0 \end{aligned} \tag{3.1}$$

with $y_0(0) \le x_0 \le z_0(0)$ *where* $S = P + Q + R$.

H*$_2$*: *The Frechet derivatives* P_x, Q_x *exist and they are continuous and* $P_x(x)y$ *is semi-nondecreasing in* y *for each* x *and*

$$\begin{gathered} (Px)(t) \ge (Py)(t) + (P_x y)(x-y)(t),\ x \ge y \\ |(P_x x)(t) - (P_x y)(t)| \le L_1 |x(t) - y(t)| \ \text{ with } L_1 > 0,\ x, y \in \Omega. \end{gathered} \tag{3.2}$$

Moreover $Q_x(x)y$ *is semi-noninreasing in* y *for each* x *and*

$$\begin{gathered} (Qx)(t) \ge (Qy)(t) + (Q_x x)(x-y)(t),\ x \ge y \\ |(Q_x x)(t) - (Q_x y)(t)| \le L_2 |x(t) - y(t)| \ \text{ with } L_2 > 0,\ x, y \in \Omega \end{gathered} \tag{3.3}$$

***H*$_3$**: *The operator* R *satisfies the Lipschitz condition*

$$|(Rx)(t) - (Ry)(t)| \le K |x(t) - y(t)| \ \text{ with } K > 0,\ x, y \in \Omega.$$

Then, there exist monotone sequences $\{y_n\}$, $\{z_n\}$ *such that* $\lim_{n\to\infty} y_n = \rho$, $\lim_{n\to\infty} z_n = \eta$ *uniformly and monotonically on* J *to the unique solution* $\rho = \eta = x$ *to IVP* (2.1) *and the convergence is semi-quadratic i.e., there exist nonnegative constants* α, β, γ, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ *such that*

$$\max_{t\in J} |x - y_{n+1}| \le \alpha \max_{t\in J} |x - y_n|^2 + \beta \max_{t\in J} |z_n - x|^2 + \gamma \max_{t\in J} |x - y_n|$$

and

$$\max_{t\in J} |z_{n+1} - x| \le \bar{\alpha} \max_{t\in J} |x - y_n|^2 + \bar{\beta} \max_{t\in J} |z_n - x|^2 + \bar{\gamma} \max_{t\in J} |z_n - x|.$$

Proof. We consider the following linear IVPs with causal operators

$$\begin{cases} y_{n+1}'(t) = (Sy_n)(t) + [(P_x y_n)(t) + (Q_x z_n)(t) - K](y_{n+1} - y_n)(t), \\ y_{n+1}(0) = x_0 \end{cases} \tag{3.4}$$

$$\begin{cases} z_{n+1}'(t) = (Sz_n)(t) + [(P_x y_n)(t) + (Q_x z_n)(t) - K](z_{n+1} - z_n)(t). \\ z_{n+1}(0) = x_0. \end{cases} \tag{3.5}$$

Observe that each linear CDE with corresponding initial condition has unique solution. We intend to prove that

$$y_0 \le y_1 \le \cdots \le y_n \le z_n \le \cdots \le z_1 \le z_0, \text{ on } J. \tag{3.6}$$

First, we show that

$$y_0 \le y_1 \le z_1 \le z_0, \text{ on } J. \tag{3.7}$$

Set $m(t)= y_0 - y_1$ on J. Then, we get

$$\begin{aligned} m'(t) &= y_0'(t) - y_1'(t) \\ &\le (Sy_0)(t) - \{(Sy_0)(t) + [(P_x y_0)(t) + (Q_x z_0)(t) - K](y_1 - y_0)(t)\} \end{aligned}$$

$$= [(P_x y_0)(t) + (Q_x z_0)(t) - K]\, m(t)$$

and $m(0) \le 0$. It follows that $m(t) \le 0$ which implies $y_0 \le y_1$ on J.

Next, take $m(t) = y_1 - z_0$; then in view of (3.4) and H_1 on J, we write

$$\begin{aligned}
m'(t) &= y_1' - z_0' \\
&= (Sy_0)(t) + [(P_x y_0)(t) + (Q_x z_0)(t) - K]\,(y_1 - y_0)(t) - z_0' \\
&\le (Py_0)(t) + (Qy_0)(t) + (Ry_0)(t) \\
&\quad + [(P_x y_0)(t) + (Q_x z_0)(t) - K]\,(y_1 - y_0)(t) - (Sz_0)(t) \\
&= (Py_0)(t) + (Qy_0)(t) + (Ry_0)(t) \\
&\quad + [(P_x y_0)(t) + (Q_x z_0)(t) - K]\,(y_1 - y_0)(t) \\
&\quad - [(Pz_0)(t) + (Qz_0)(t) + (Rz_0)(t)] \\
&= [(Py_0)(t) - (Pz_0)(t)] + [(Qy_0)(t) - (Qz_0)(t)] \\
&\quad + [(Ry_0)(t) - (Rz_0)(t)] + [(P_x y_0)(t) + (Q_x z_0)(t) - K]\,(y_1 - y_0)(t).
\end{aligned}$$

At this point, taking into consideration H_2, H_3, and the fact that $y_0 \le z_0$, the following inequalities can be obtained:

$$\begin{aligned}
(Pz_0)(t) &\ge (Py_0)(t) + (P_x y_0)(z_0 - y_0)(t), \\
(P_x y_0)(y_0 - z_0)(t) &\ge (Py_0)(t) - (Pz_0)(t).
\end{aligned}$$

Similarly

$$\begin{aligned}
(Qz_0)(t) &\ge (Qy_0)(t) + (Q_x z_0)(z_0 - y_0)(t), \\
(Q_x z_0)(y_0 - z_0)(t) &\ge (Qy_0)(t) - (Qz_0)(t),
\end{aligned}$$

and

$$-K(z_0 - y_0) \le (Rz_0)(t) - (Ry_0)(t) \le K(z_0 - y_0).$$

Substituting these into the above inequality, we get

$$\begin{aligned}
m'(t) &\le (P_x y_0)(y_0 - z_0)(t) + (Q_x z_0)(y_0 - z_0)(t) + K(z_0 - y_0) \\
&\quad + [(P_x y_0)(t) + (Q_x z_0)(t) - K]\,(y_1 - y_0)(t) \\
&= [(P_x y_0)(t) + (Q_x z_0)(t) - K]\,(y_0 - z_0 + y_1 - y_0) \\
&= [(P_x y_0)(t) + (Q_x z_0)(t) - K]\,(y_1 - z_0)(t).
\end{aligned}$$

This implies that

$$m'(t) \le [(P_x y_0)(t) + (Q_x z_0)(t) - K]\, m(t) \text{ and } m(0) \le 0,$$

which yields $m(t) \le 0$. Thus, we achieve $y_0 \le y_1(t) \le z_0(t)$ on J. In a similar manner, one can obtain $y_0 \le z_1 \le z_0$ on J. We now prove that $y_1 \le z_1$ on J. For this aim, we put $m(t) = y_1 - z_1$ and note that $m(0) = 0$. Then, by (3.4) and (3.5)

$$\begin{aligned}
m'(t) &= y_1' - z_1' \\
&= (Py_0)(t) + (Qy_0)(t) + (Ry_0)(t) \\
&\quad + [(P_x y_0)(t) + (Q_x z_0)(t) - K]\,(y_1 - y_0)(t) \\
&\quad - (Pz_0)(t) - (Qz_0)(t) - (Rz_0)(t) \\
&\quad - [(P_x y_0)(t) + (Q_x z_0)(t) - K]\,(z_1 - z_0)(t) \\
&= [(P_x y_0)(t) + (Q_x z_0)(t) - K]\,(y_1 - y_0 + z_0 - z_1)(t)
\end{aligned}$$

$$
\begin{aligned}
&+ \left[(Py_0)(t) - (Pz_0)(t) \right] + \left[(Qy_0)(t) - (Qz_0)(t) \right] \\
&+ \left[(Ry_0)(t) - (Rz_0)(t) \right].
\end{aligned}
$$

From H_2, H_3, and the fact that $y_0 \le z_0$ on J, it follows that

$$
\begin{aligned}
m'(t) &\le \left[(P_x y_0)(t) + (Q_x z_0)(t) - K \right] (y_1 - y_0 + z_0 - z_1)(t) \\
&\quad - (P_x y_0)(z_0 - y_0)(t) - (Q_x z_0)(z_0 - y_0)(t) + K(z_0 - y_0) \\
&= \left[(P_x y_0)(t) + (Q_x z_0)(t) - K \right] (y_1 - y_0 + z_0 - z_1 + y_0 - z_0)(t) \\
&= \left[(P_x y_0)(t) + (Q_x z_0)(t) - K \right] (y_1 - z_1)(t),
\end{aligned}
$$

which shows that $m'(t) \le \left[(P_x y_0)(t) + (Q_x z_0)(t) - K \right] m(t)$ and $m(0) = 0$. Therefore, we reach $m(t) \le 0$, i.e., $y_1 \le z_1$ on J proving (3.7).

Now using the mathematical induction, assume that for an integer $k > 1$,

$$y_{k-1} \le y_k \le z_k \le z_{k-1} \text{ on } J.$$

We need to show that

$$y_k \le y_{k+1} \le z_{k+1} \le z_k \text{ on } J.$$

To do so, we take $m(t) = y_k - y_{k+1}$ and utilize H_2, H_3; we have

$$
\begin{aligned}
m'(t) &= y_k' - y_{k+1}' \\
&= (Sy_{k-1})(t) + \left[(P_x y_{k-1})(t) + (Q_x z_{k-1})(t) - K \right] (y_k - y_{k-1})(t) \\
&\quad - (Sy_k)(t) - \left[(P_x y_k)(t) + (Q_x z_k)(t) - K \right] (y_{k+1} - y_k)(t) \\
&= (Py_{k-1} - Py_k)(t) + (Qy_{k-1} - Qy_k)(t) + (Ry_{k-1} - Ry_k)(t) \\
&\quad + \left[(P_x y_{k-1})(t) + (Q_x z_{k-1})(t) - K \right] (y_k - y_{k-1})(t) \\
&\quad - \left[(P_x y_k)(t) + (Q_x z_k)(t) - K \right] (y_{k+1} - y_k)(t) \\
&\le (P_x y_{k-1})(y_{k-1} - y_k)(t) + (Q_x z_{k-1})(y_{k-1} - y_k)(t) \\
&\quad + K(y_{k-1} - y_k) \\
&\quad + (P_x y_{k-1})(y_k - y_{k-1})(t) + (Q_x z_{k-1})(y_k - y_{k-1})(t) \\
&\quad + \left[(P_x y_k)(t) + (Q_x z_k)(t) \right] (y_k - y_{k+1})(t) \\
&\quad + K(y_{k-1} - y_k + y_{k+1} - y_k) \\
&= \left[(P_x y_k)(t) + (Q_x z_k)(t) - K \right] (y_k - y_{k+1})(t),
\end{aligned}
$$

where we used the semi-nondecreasing and semi-nonincreasing property of $P_x(x)$ and $Q_x(x)y$, respectively. Thus, one attains $m'(t) \le \left[(P_x y_k)(t) + (Q_x z_k)(t) - K \right] m(t)$, $m(0) = 0$ which yields $y_k \le y_{k+1}$ on J. Analogously, it can be shown that $z_{k+1} \le z_k$ on J. To prove $y_{k+1} \le z_{k+1}$, let $m(t) = y_{k+1} - z_{k+1}$; then, by employing H_2, H_3 and in view of $y_k \le z_k$, we get

$$
\begin{aligned}
m'(t) &= y_{k+1}' - z_{k+1}' \\
&= (Sy_k)(t) + \left[(P_x y_k)(t) + (Q_x z_k)(t) - K \right] (y_{k+1} - y_k)(t) \\
&\quad - (Sz_k)(t) - \left[(P_x y_k)(t) + (Q_x z_k)(t) - K \right] (z_{k+1} - z_k)(t) \\
&\le (P_x y_k)(y_k - z_k)(t) + (Q_x z_k)(y_k - z_k)(t) - K(y_k - z_k) \\
&\quad + \left[(P_x y_k)(t) + (Q_x z_k)(t) - K \right] (y_{k+1} - y_k - z_{k+1} + z_k)(t) \\
&\le \left[(P_x y_k)(t) + (Q_x z_k)(t) - K \right] (y_{k+1} - z_{k+1}).
\end{aligned}
$$

Owing to $m'(t) \le \left[(P_x y_k)(t) + (Q_x z_k)(t) - K \right] m(t)$ and $m(0) = 0$, we find $m(t) \le 0$, i.e., $y_{k+1} \le z_{k+1}$ on J.

Obviously, the constructed sequences $\{y_n\}$, $\{z_n\}$ are equicontinuous and uniformly bounded. Therefore, employing the Ascoli-Arzela theorem, we find subsequences $\{y_{n_k}\}$ and $\{z_{n_k}\}$ converging uniformly to the functions ρ and η, respectively. However, since the sequences $\{y_n\}$, $\{z_n\}$ are monotonic, we infer that the whole sequences converge uniformly and monotonically to ρ and η on J, respectively when $n \to \infty$.

Using the corresponding Volterra integral equations of (3.4), (3.5), and passing to limit as $n \to \infty$, we conclude that ρ and η are solutions to (2.1). Since the operator S satisfies the Lipschitz condition, a unique solution exists, that is, $\rho = x = \eta$ on J.

It remains to demonstrate the semi-quadratic convergence. For this aim, we set $p_{n+1} = x - y_{n+1}$, $r_{n+1} = z_{n+1} - x$. Note that $p_{n+1}(0) = 0$, $r_{n+1}(0) = 0$. Now consider the following relations

$$\begin{aligned} p'_{n+1} &= x' - y'_{n+1} \\ &= (Sx)(t) - (Sy_n)(t) - [(P_x y_n)(t) + (Q_x z_n)(t) - K](y_{n+1} - y_n)(t) \\ &= [(P_x\xi) + (Q_x\sigma) + K](x - y_n)(t) \\ &\quad + [(P_x y_n)(t) + (Q_x z_n)(t) - K](y_n - y_{n+1})(t), \end{aligned}$$

where $y_n \le \xi, \sigma \le x$. Due to monotone-nondecreasing property of P_x and monotone-nonincreasing property of Q_x, it follows that

$$\begin{aligned} p'_{n+1} &\le [(P_x x)(t) + (Q_x y_n)(t) + K] p_n(t) \\ &\quad + [(P_x y_n)(t) + (Q_x z_n)(t) - K](p_{n+1} - p_n)(t) \\ &= [(P_x x)(t) - (P_x y_n)(t)] p_n(t) \\ &\quad + [(Q_x y_n)(t) - (Q_x z_n)(t)] p_n(t) + 2K p_n(t) \\ &\quad + [(P_x y_n)(t) + (Q_x z_n)(t) - K] p_{n+1}(t) \\ &\le L_1 p_n^2 + L_2 (p_n + r_n) p_n + 2K p_n \\ &\quad + [(P_x y_n)(t) + (Q_x z_n)(t) - K] p_{n+1}(t) \\ &\le \left(L_1 + \frac{3}{2}L_2\right) p_n^2 + \frac{1}{2} L_2 r_n^2 + 2K p_n \\ &\quad + [(P_x y_n)(t) + (Q_x z_n)(t) - K] p_{n+1} \end{aligned}$$

by the inequalities given in H_2, H_3. Moreover,

$$p'_{n+1} \le M p_{n+1} + \left(L_1 + \frac{3}{2}L_2\right) p_n^2 + \frac{1}{2} L_2 r_n^2 + 2K p_n,$$

where $M = M_1 + M_2 - K$ and $|P_x(x)| \le M_1$ and $|Q_x(x)| \le M_2$. Now, Gronwall's inequality implies

$$0 < p_{n+1} \le \int_0^t e^{M(t-s)} \left[\left(L_1 + \frac{3}{2}L_2\right) p_n^2(s) + \frac{1}{2} L_2 r_n^2(s) + 2K p_n(s)\right] ds.$$

Let $\alpha = \frac{e^{MT}}{M}\left(L_1 + \frac{3}{2}L_2\right)$, $\beta = \frac{e^{MT}}{2M} L_2$, $\gamma = \frac{2e^{MT}}{M} K$. Then, we reach the desired result for $t \in J$,

$$\max_J |x - y_{n+1}| \le \alpha \max_J |x - y_n|^2 + \beta \max_J |z_n - x|^2 + \gamma \max_J |x - y_n|,$$

which shows semi-quadratic convergence.

In a similar way, by using similar computation, we arrive at

$$\begin{aligned} \max_J |z_{n+1} - x| &\le \frac{e^{MT}}{M} \left[\frac{1}{2} L_2 \max_J |x - y_n|^2 + \left(L_1 + \frac{3}{2}L_2\right) \max_J |z_n - x|^2 \right. \\ &\quad \left. + 2K \max_J |z_n - x| \right] \\ &= \bar{\alpha} \max_J |x - y_n|^2 + \bar{\beta} \max_J |z_n - x|^2 + \bar{\gamma} \max_J |z_n - x|, \end{aligned}$$

where $\bar{\alpha} = \frac{e^{MT}}{2M} L_1$, $\bar{\beta} = \frac{e^{MT}}{M}\left(\frac{3}{2}L_1 + L_2\right)$, $\bar{\gamma} = \frac{2e^{MT}}{M} K$ which completes the proof. □

Remark 3.2. Let $(Px)(t) + (Qx)(t) \equiv 0$ on J; then we have the monotone method and the convergence is linear.

Remark 3.3. Let $(Qx)(t) + (Rx)(t) \equiv 0$ on J; then Theorem 3.1 reduces to Theorem (3.6.1) in [18], and the convergence is quadratic.

Remark 3.4. Let $(Rx)(t) \equiv 0$ on J; then Theorem 3.1 transforms into Theorem (3.7.1) in [18], and the convergence is quadratic.

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