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A new kind of generalized fuzzy integrals

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Abstract

Fuzzy integral is an important tool to study fuzzy differential equations. Under normal circumstances, there are two basic limitations: limited of integral interval and boundedness of integrand. However, in practical problems, it is difficult to calculate when integral interval is not common interval. Then fuzzy integral on infinite interval is taken into consideration. In this paper, definition of a kind of generalized Liu integral is given. Moreover, properties and theorems of this kind of generalized fuzzy integral are obtained.

Keywords: Fuzzy variable, fuzzy process, Liu process, generalized fuzzy integral.
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1. Introduction

In real world, there exist many fuzzy phenomena. The uncertainty of fuzzy phenomenon is a basic type of subjective uncertainty which is characterized by membership function given by experts. To describe a set without definite boundary, fuzzy set was initiated by Zadeh [20] in 1965, and a possibility measure was presented by Zadeh [21] in 1978. However, possibility measure has no self duality. Then credibility measure, a self-duality measure was introduced by Liu and Liu [11] in 2002. A sufficient and necessary condition for credibility measure was given by Li and Liu [6] in 2006. Credibility theory, founded by Liu [7] in 2004 and refined by Liu [9] in 2007, is a branch of mathematics for studying the behavior of fuzzy phenomena. A survey of credibility theory can be found in Liu [8], and interested reader may consult the book [9].

There are many types of fuzzy integrals in literatures, such as Choquet fuzzy integral and Sugeno fuzzy integral (see[5], [15] and [16]). However, these fuzzy integrals are all integrals with respect to variable,

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which have any relationship with fuzzy process. To describe dynamic fuzzy phenomena, a fuzzy process (Liu process), a differential formula (Liu formula) and a fuzzy integral (Liu integral) were introduced by Liu [10] in 2008. Here the fuzzy integral is the integral of fuzzy process with respect to Liu process.

As for Liu process, some researches concerning have been done. You, Huo and Wang [17] extended Liu process, Liu integral and Liu formula to the case of multi-dimensional. Complex Liu process was studied by Qin and Wen[14]. Dai [2] and Dai [3] proposed Lipschitz continuity and reflection principle of Liu process. Some properties of Liu integral were studied by You and Wang [18]. You, Wang and Huo [19] discussed existence and uniqueness theorems for some special fuzzy differential equations. Chen and Qin [1] studied a new existence and uniqueness theorem for fuzzy differential equations, which is a general case. Liu process has also been applied to stock model and fuzzy finance. A basic stock model was proposed by Liu [10], which is called Liu's stock model. Since then, fuzzy calculus was widely used in finance. Assumed that stock price is modeled by geometric Liu process, Qin and Li [13] first deduced option pricing formula for European option. Most results concerning fuzzy finance were studied by Gao [4], Peng [12]. Fuzzy process is also used in control fields by Zhu [22].

The purpose of this paper is to discuss generalized fuzzy integral based on credibility theory. The structure of this paper is as follows: In Section 2 of this paper, some concepts and results of Liu integral will be given as preliminaries. The definitions and properties of infinite Liu integral will be discussed in Section 3. In the end, a brief summary is given in Section 4.

2. Preliminaries

In the setting of credibility theory, let T be an index set, Θ an empty set, $\mathcal{P}$ the power set of Θ and Cr a credibility measure. Then $(\Theta, \mathcal{P}, \mathrm{Cr})$ is called a credibility space. A fuzzy process $X_t(\theta)$ is defined as a function from $T \times (\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers, where t is time and θ is a point in credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. In other words, $X_t*(\theta)$ is a fuzzy variable for each t^*; $X_t(\theta^*)$ is a function of t for any given $\theta^* \in \Theta$, such a function is called a sample path of $X_t(\theta)$. For simplicity, we use the symbol X_t to replace $X_t(\theta)$ in the following sections.

A fuzzy process X_t is called continuous if the sample paths of X_t are all continuous functions of t for almost all $\theta \in \Theta$.

Definition 2.1 ([10]). A fuzzy process C_t is said to be a Liu process if

(i) $C_0 = 0$,

(ii) C_t has stationary and independent increments,

(iii) every increment $C_{t+s} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$, whose membership function is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \ -\infty < x < +\infty.$$

The Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

Definition 2.2. (Liu integral, [10]) Let X_t be a fuzzy process and let C_t be a standard Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\triangle = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then the Liu integral of X_t with respect to C_t is defined as follows,

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\triangle \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limitation exists almost surely and is a fuzzy variable. In this case, X_t is called Liu integrable.

Theorem 2.3 (Liu Formula, [10]). *Let C_t be a standard Liu process and let $h(t,x)$ be a continuously differentiable function. If fuzzy process X_t is given by $\mathrm{d}X_t = u_t\mathrm{d}t + v_t\mathrm{d}C_t$, where u_t, v_t are absolutely integrable fuzzy process and Liu integrable fuzzy process, respectively. Define $Y_t = h(t, X_t)$. Then*

$$\mathrm{d}Y_t = \frac{\partial h}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial h}{\partial x}(t, X_t)\mathrm{d}X_t,$$

which is called Liu formula.

Theorem 2.4 ([18]). *Let $a < k < b$. If fuzzy process X_t is Liu integrable on any closed interval $[a,k]$ and $[k,b]$, then X_t is Liu integral for closed interval $[a,b]$, and*

$$\int_a^b X_t\mathrm{d}C_t = \int_a^k X_t\mathrm{d}C_t + \int_k^b X_t\mathrm{d}C_t.$$

Theorem 2.5 ([18]). *Let fuzzy process X_t and Y_t be Liu integrable on closed interval $[a,b]$. Then*

$$\int_a^b (k_1X_t + k_2Y_t)\mathrm{d}C_t = k_1\int_a^b X_t\mathrm{d}C_t + k_2\int_a^b X_t\mathrm{d}C_t$$

for any real numbers k_1 and k_2.

3. Infinite Liu Integral

This section aims to give the definition of infinite Liu integral and discuss some properties of infinite Liu integral.

Definition 3.1. Let X_t be a fuzzy process and let C_t be a standard Liu process. Suppose X_t is defined on interval $[a,+\infty)$ and integrable on any finite closed interval $[a,u]$ with respect to C_t. If the limitation

$$\lim_{u\to+\infty}\int_a^u X_t\mathrm{d}C_t = J(\theta)$$

exists almost surely and is a fuzzy variable, then the limitation $J(\theta)$ is called infinite Liu integral of fuzzy process X_t on interval $[a,+\infty)$ (For short, infinite Liu integral). Denote

$$\int_a^{+\infty} X_t\mathrm{d}C_t = J(\theta).$$

In this case, $\int_a^{+\infty} X_t\mathrm{d}C_t$ is called convergent almost surely to $J(\theta)$.

On the contrary, if the limitation $\lim\limits_{u\to+\infty}\int_a^u X_t\mathrm{d}C_t$ does not exist, $\int_a^{+\infty} X_t\mathrm{d}C_t$ is called divergent.

Example 3.2. Let C_t be a standard Liu process and $C_t \to +\infty$ when $t \to +\infty$. Discuss the convergence of infinite Liu integral $\int_0^{+\infty}\exp(-C_t)\mathrm{d}C_t$.

By using Liu formula, we have

$$\mathrm{d}\exp(-C_t) = \exp(-C_t)\mathrm{d}(-C_t),$$

that is

$$\begin{aligned}\int_0^{+\infty}\exp(-C_t)\mathrm{d}C_t &= -\int_0^{+\infty}\exp(-C_t)\mathrm{d}(-C_t)\\ &= -\int_0^{+\infty}\mathrm{d}\exp(-C_t) = -\lim_{u\to+\infty}\exp(-C_u) + 1 = 1.\end{aligned}$$

Thus infinite Liu integral $\int_0^{+\infty}\exp(-C_t)\mathrm{d}C_t$ is convergent almost surely.

Example 3.3. Let C_t be a standard Liu process. If $t \to +\infty$, $C_t \to +\infty$, discuss the convergence of the following infinite Liu integral $\int_0^{+\infty} \frac{1}{1+C_t^2} \mathrm{d}C_t$.

It follows from Liu formula that

$$\mathrm{d} \arctan C_t = \frac{1}{1+C_t^2} \mathrm{d}C_t,$$

then

$$\int_0^{+\infty} \frac{1}{1+C_t^2} \mathrm{d}C_t = \int_0^{+\infty} \mathrm{d} \arctan C_t = \lim_{u \to +\infty} \arctan C_u = \frac{\pi}{2},$$

thus $\int_0^{+\infty} \frac{1}{1+C_t^2} \mathrm{d}C_t$ is convergent almost surely.

The definition shows that the convergence or divergence of infinite Liu integral $\int_a^{+\infty} X_t \mathrm{d}C_t$ is determined by the existence of limitation of Liu integral $\lim\limits_{u \to \infty} \int_a^u X_t \mathrm{d}C_t$.

Next, some properties of infinite Liu integral will be derived.

Theorem 3.4. *If infinite Liu integral $\int_a^{+\infty} X_t \mathrm{d}C_t$ is convergent almost surely, then there exists a fuzzy event A with $\mathrm{Cr}\{A\} = 1$ and a real number $G \geq a$ such that for every $\varepsilon(\theta) > 0$, we have*

$$\Big| \int_a^{u_2} X_t \mathrm{d}C_t - \int_a^{u_1} X_t \mathrm{d}C_t \Big| = \Big| \int_{u_1}^{u_2} X_t \mathrm{d}C_t \Big| < \varepsilon(\theta),$$

if $u_1, u_2 > G$, for each $\theta \in A$.

Proof. Since $\int_a^{+\infty} X_t \mathrm{d}C_t$ is convergent almost surely, denoting the limitation by $J(\theta)$, we know there exists a fuzzy event A with $\mathrm{Cr}\{A\} = 1$ such that $\lim\limits_{G \to \infty} \int_a^G X_t \mathrm{d}C_t = J(\theta)$ for each $\theta \in A$.

Fix $\varepsilon(\theta) > 0$, by the definition of limitation, there exists $G \geq a$ such that

$$\Big| \int_a^{u_1} X_t \mathrm{d}C_t - J(\theta) \Big| < \varepsilon(\theta), \ \Big| \int_a^{u_2} X_t \mathrm{d}C_t - J(\theta) \Big| < \varepsilon(\theta),$$

when $u_1 > G, u_2 > G$, for each $\theta \in A$.

According to Theorem 2.4, we have

$$\begin{aligned} \Big| \int_{u_1}^{u_2} X_t \mathrm{d}C_t \Big| &= \Big| \int_a^{u_2} X_t \mathrm{d}C_t - \int_a^{u_1} X_t \mathrm{d}C_t \Big| \\ &= \Big| \int_a^{u_1} X_t \mathrm{d}C_t - J(\theta) - \int_a^{u_2} X_t \mathrm{d}C_t + J(\theta) \Big| \\ &\leq \Big| \int_a^{u_1} X_t \mathrm{d}C_t - J(\theta) \Big| + \Big| \int_a^{u_2} X_t \mathrm{d}C_t - J(\theta) \Big| \\ &< 2\varepsilon(\theta). \end{aligned}$$

The theorem is proved. □

Theorem 3.5. *Let C_t be a standard Liu process. If infinite Liu integral $\int_a^{+\infty} X_t \mathrm{d}C_t$ and $\int_a^{+\infty} Y_t \mathrm{d}C_t$ are both convergent almost surely, then infinite Liu integral $\int_a^{+\infty} (k_1 X_t + k_2 Y_t) \mathrm{d}C_t$ is convergent, and*

$$\int_a^{+\infty} (k_1 X_t + k_2 Y_t) \mathrm{d}C_t = k_1 \int_a^{+\infty} X_t \mathrm{d}C_t + k_2 \int_a^{+\infty} Y_t \mathrm{d}C_t$$

for any constant k_1 and k_2.

Proof. Since infinite Liu integrals $\int_a^{+\infty} X_t \mathrm{d}C_t$ and $\int_a^{+\infty} Y_t \mathrm{d}C_t$ are both convergent almost surely, then $\lim_{u\to\infty} \int_a^u X_t \mathrm{d}C_t$ and $\lim_{u\to\infty} \int_a^u Y_t \mathrm{d}C_t$ exist. Let

$$\lim_{u\to\infty} \int_a^u X_t \mathrm{d}C_t = J(\theta), \ \lim_{u\to\infty} \int_a^u X_t \mathrm{d}C_t = K(\theta).$$

It follows from Theorem 2.5 that

$$\begin{aligned}\int_a^{+\infty} (k_1 X_t + k_2 Y_t) \mathrm{d}C_t &= \lim_{u\to\infty} \int_u^{+\infty} k_1 X_t \mathrm{d}C_t + \lim_{u\to\infty} k_2 \int_u^{+\infty} Y_t \mathrm{d}C_t \\ &= k_1 \lim_{u\to\infty} \int_u^{+\infty} X_t \mathrm{d}C_t + k_2 \lim_{u\to\infty} \int_u^{+\infty} Y_t \mathrm{d}C_t \\ &= k_1 J(\theta) + k_2 K(\theta).\end{aligned}$$

Hence

$$\int_a^{+\infty} (k_1 X_t + k_2 Y_t) \mathrm{d}C_t = k_1 \int_a^{+\infty} X_t \mathrm{d}C_t + k_2 \int_a^{+\infty} Y_t \mathrm{d}C_t.$$

The theorem is proved. □

Theorem 3.6. *Let X_t be Liu integrable fuzzy process on any finite closed interval $[a, u]$, and $a < b$. Then infinite Liu integral $\int_a^{+\infty} X_t \mathrm{d}C_t$ and $\int_b^{+\infty} X_t \mathrm{d}C_t$ are convergent or divergent at the same time and*

$$\int_a^{+\infty} X_t \mathrm{d}C_t = \int_a^b X_t \mathrm{d}C_t + \int_b^{+\infty} X_t \mathrm{d}C_t.$$

Proof. It follows from the definition of infinite Liu integral and Theorem 2.4. that

$$\begin{aligned}\int_a^{+\infty} X_t \mathrm{d}C_t &= \lim_{c\to+\infty} \int_a^c X_t \mathrm{d}C_t \\ &= \lim_{c\to+\infty} (\int_a^b X_t \mathrm{d}C_t + \int_b^c X_t \mathrm{d}C_t) \\ &= \int_a^b X_t \mathrm{d}C_t + \lim_{c\to+\infty} \int_b^c X_t \mathrm{d}C_t \\ &= \int_a^b X_t \mathrm{d}C_t + \int_b^{+\infty} X_t \mathrm{d}C_t.\end{aligned}$$

The theorem is proved. □

Theorem 3.7. *Let C_t be a standard Liu process and $F(t)$ be an absolutely continuous function. If $\lim_{t\to+\infty} F(t)$ and $\lim_{t\to+\infty} C_t$ exist, then*

$$\int_0^{+\infty} F(t) \mathrm{d}C_t = \lim_{t\to+\infty} F(t) C_t - \int_0^{+\infty} C_t \mathrm{d}F(t).$$

Proof. Taking $h(t, C_t) = F(t)\mathrm{d}C_t$, it follows from Liu Formula that

$$\mathrm{d}(F(t)C_t) = C_t \mathrm{d}F(t) + F(t)\mathrm{d}C_t.$$

Thus

$$\lim_{t\to+\infty} F(t) C_t = \int_0^{+\infty} \mathrm{d}(F(t)C_t) = \int_0^{+\infty} C_t \mathrm{d}F(t) + \int_0^{+\infty} F(t) \mathrm{d}C_t,$$

that is

$$\int_0^{+\infty} F(t) \mathrm{d}C_t = \lim_{t\to+\infty} F(t) C_t - \int_0^{+\infty} C_t \mathrm{d}F(t).$$

The theorem is proved. □

4. Conclusions

This paper was mainly to extend Liu integral to a kind of generalized Liu integral, that is Liu integral on infinite interval. The results of this paper can be summarized as follows: (a) the definition of infinite Liu integral was presented; (b) some properties of infinite Liu integral were given, which include linear properties, the additivity of integral interval, the formula of integration by parts and etc..

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