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Conjugacy between trapezoid maps

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Abstract

Trapezoid maps are a kind of continuous and piecewise linear maps with a flat top. By the conjugacy relationship, we present a complete classification for four families of trapezoid maps. Firstly, using an extension method, we construct all homeomorphic solutions of conjugacy equation $\varphi \circ f = g \circ \varphi$ for some non-monotone continuous maps f and g. Secondly, using an iterative construction method and an extension method, we construct respectively all topological conjugacies for four families of trapezoid maps. Finally, all construction algorithms are implemented in MATLAB, and three examples are illustrated to construct topological conjugacies and a topological semi-conjugacy.

Keywords: Trapezoid map, topological conjugacy, topological classification, conjugacy equation.
2010 MSC: 37E05, 37C15.

1. Introduction

An outstanding problem of iteration theory and ergodic theory [4] is to decide whether two maps $f : I \to I$ and $g : J \to J$ are topologically conjugate, i.e., whether there exists a homeomorphism $\varphi : I \to J$ such that $\varphi \circ f = g \circ \varphi$. Such a homeomorphism φ is called a *conjugacy*. If there exists a continuous, monotone (not necessarily strictly monotone) and onto map φ such that $\varphi \circ f = g \circ \varphi$, then we say that f is topologically semi-conjugate to g, and φ is a *semi-conjugacy*. If f, g are topological conjugate, write in symbols $f \sim g$, otherwise, $f \nsim g$.

Topological conjugation is an equivalence relation which is useful in the topological classification of dynamical systems.

The aim of this paper is to give a complete classification of four families of trapezoid maps, respectively denoted by $\mathcal{M}_i$, $i = 1, 2, 3, 4$, of forms:

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$$t_{a,b}(x)=\begin{cases}\frac{x}{a}, & x\in[0,a],\\ 1, & x\in(a,b],\\ \frac{1-x}{1-b}, & x\in(b,1],\end{cases}\qquad \tilde{t}_{a,b}(x)=\begin{cases}\frac{bx}{a}, & x\in[0,a],\\ b, & x\in(a,b],\\ \frac{b(1-x)}{1-b}, & x\in(b,1],\end{cases}$$

$$\bar{t}_{a,b,c}(x)=\begin{cases}\frac{bx}{a}, & x\in[0,a],\\ b, & x\in(a,c],\\ \frac{b(1-x)}{1-c}, & x\in(c,1],\end{cases}\qquad \hat{t}_{a,b,c}(x)=\begin{cases}\frac{ax}{b}, & x\in[0,b],\\ a, & x\in[b,c],\\ \frac{a(1-x)}{1-c}, & x\in[c,1],\end{cases}$$

where a,b,c are three fixed parameters with $0<a<b<c<1$, see Figs. 1, 2, 3, 4.

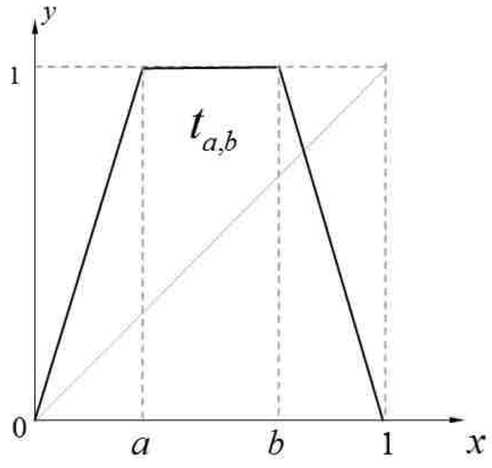

Figure 1: $t_{a,b}$

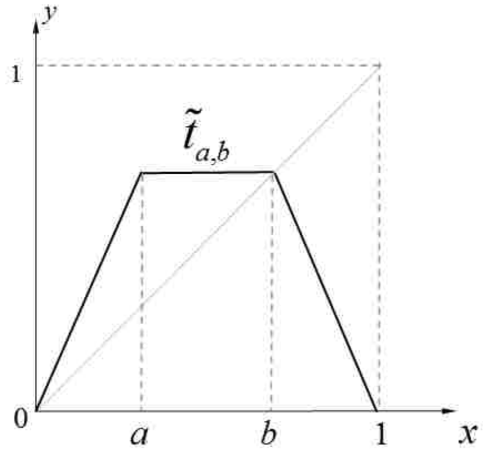

Figure 2: $\tilde{t}_{a,b}$

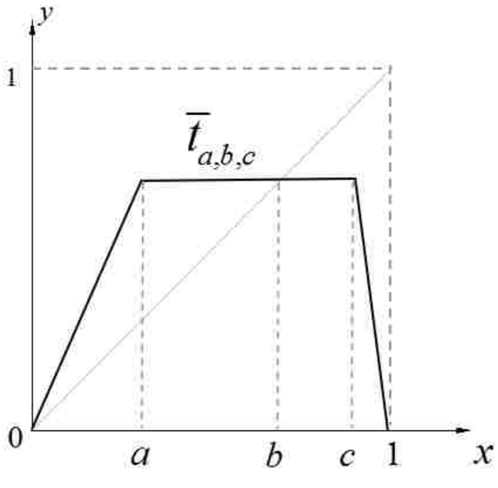

Figure 3: $\bar{t}_{a,b,c}$

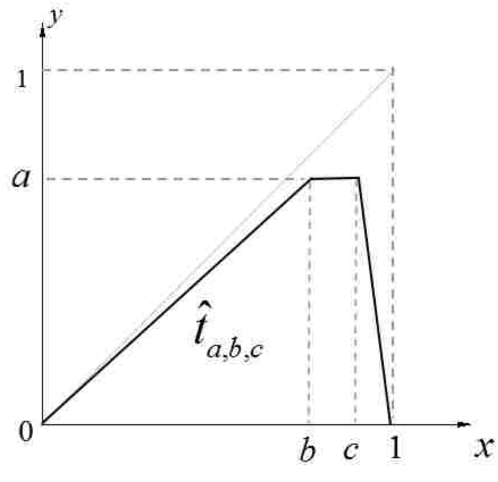

Figure 4: $\hat{t}_{a,b,c}$

Schweizer and Sklar in [8] investigated the family $\mathcal{M}_1$ of trapezoid maps. They showed that any two maps in $\mathcal{M}_1$ are topologically conjugate, Their results were generalized to Markov maps [2], and to combination of multiple trapezoids, which is the so-called weakly multimodal maps [7, 9]. In this paper, we construct all homeomorphic solutions of conjugacy equation $\varphi\circ f=g\circ\varphi$ for some non-monotone continuous maps f and g. And we shall prove that: (i) $f\sim g$ if $f,g\in\mathcal{M}_i$, $i=1,2,3,4$; (ii)$f\nsim g$ if $f\in\mathcal{M}_i$ and $g\in\mathcal{M}_j$, $i\neq j$. Meanwhile, we show that $f^n\in\mathcal{M}_3$ for $n\geq 2$ if $f\in\mathcal{M}_2$, and $f\in\mathcal{M}_3$ is topologically semi-conjugate to $g\in\mathcal{M}_2$. Using an iterative construction method and an extension method, we construct respectively all topological conjugacies for four families of trapezoid maps. Finally, all construction algorithms are implemented in MATLAB, and three examples are illustrated to construct topological conjugacies and a topological semi-conjugacy.

2. Preliminaries

2.1. Dynamics of trapezoid maps

Each trapezoid map in $\mathcal{M}_1$ is a chaotic map, which produces various interesting behaviors. Many authors investigated their delta-pseudo orbit shadowing [5], symbolic dynamics [6] and monotonicity properties of kneading sequences [3].

In this subsection, we first study the dynamics of other three families of trapezoid maps $\mathcal{M}_i$, $i = 2, 3, 4$.

Lemma 2.1. *Assume that $\tilde{t}_{a_1,b_1} \in \mathcal{M}_2$. Then*

(i) *$\tilde{t}_{a_1,b_1}$ has only two fixed points 0 and b_1;*

(ii) *for any $x_0 \in (0, b_1) \cup (b_1, 1]$, the sequence $\{\tilde{t}^n_{a_1,b_1}(x_0)\}_{n=1}^\infty$ converges to b_1;*

(iii) *$\tilde{t}^n_{a_1,b_1} = \bar{t}_{a_n,b_n,c_n}$ for $n \geq 2$ where*

$$a_n = \frac{a_1^{n+1}}{b_1^n}, \quad b_n = b_1, \quad c_n = \frac{b_1^n - a_1^n + a_1^n b_1}{b_1^n}.$$

Proof. One immediately verifies that the results of (i) and (ii) by direct computation. Now we prove (iii) by induction.

We first check the case $n = 2$. Put f_1, f_2, f_3 be respectively the restrictions of $\tilde{t}_{a_1,b_1}$ to the subintervals $[0, a_1]$, $[a_1, b_1]$ and $[b_1, 1]$. One can see that

(1) $\tilde{t}^2_{a_1,b_1}(x) = f_1^2(x) \in [0, a_1]$ for $x \in \left[0, \frac{a_1^3}{b_1^2}\right]$;

(2) $\tilde{t}^2_{a_1,b_1}(x) = \tilde{t}_{a_1,b_1}(f_1(x)) = b_1$ for $x \in \left[\frac{a_1^3}{b_1^2}, a_1\right]$;

(3) $\tilde{t}^2_{a_1,b_1}(x) = f_2^2(b_1) = b_1$ for $x \in [a_1, b_1]$;

(4) $\tilde{t}^2_{a_1,b_1}(x) = \tilde{t}_{a_1,b_1}(f_3(x))$ for $x \in \left[b_1, \frac{b_1^2 - a_1^2 + a_1^2 b}{b_1^2}\right]$;

(5) $\tilde{t}^2_{a_1,b_1}(x) = f_1(f_3(x)) \in [0, a_1]$ for $x \in \left[\frac{b_1^2 - a_1^2 + a_1^2 b}{b_1^2}, 1\right]$.

Thus $\tilde{t}^2_{a_1,b_1} = \bar{t}_{a_2,b_2,c_2}$. Assume that $\tilde{t}^k_{a_1,b_1} = \bar{t}_{a_k,b_k,c_k}$ for some positive integer $k \geq 2$. When $n = k + 1$, we have for $x \in [0, 1]$

$$\tilde{t}^{n+1}_{a_1,b_1}(x) = \tilde{t}_{a_1,b_1}(\tilde{t}^n_{a_1,b_1}(x)) = \tilde{t}_{a_1,b_1}(\bar{t}_{a_n,b_n,c_n}(x)) = \bar{t}_{a_{n+1},b_{n+1},c_{n+1}}(x).$$

Therefore the result of (iii) follows. $\square$

Proofs of the following two lemmas are easily supplied by a similar method in proving Lemma 2.1.

Lemma 2.2. *Assume that $\bar{t}_{a_1,b_1,c_1} \in \mathcal{M}_3$. Then*

(i) *$\bar{t}_{a_1,b_1,c_1}$ has only two fixed points 0 and b_1;*

(ii) *for any $x_0 \in (0, b_1) \cup (b_1, 1]$, the sequence $\{\bar{t}^n_{a_1,b_1,c_1}(x_0)\}_{n=1}^\infty$ converges to b_1;*

(iii) *$\bar{t}^n_{a_1,b_1,c_1} = \bar{t}_{a_n,b_n,c_n}$ for $n \geq 2$ where $a_n = \frac{a_1^{n+1}}{b_1^n}$, $b_n = b_1$, $c_n = 1 - \frac{(1-c_1)c_1 a_1^n}{b_1^{n+1}}$.*

Lemma 2.3. *Assume that* $\hat{t}_{a_1,b_1,c_1} \in \mathcal{M}_4$. *Then*

(i) $\hat{t}_{a_1,b_1,c_1}$ *has only one fixed point* 0;

(ii) *for any* $x_0 \in (0,1]$, *the sequence* $\{\hat{t}^n_{a_1,b_1,c_1}(x_0)\}_{n=1}^{\infty}$ *is monotonically decreasing, and converges to zero;*

(iii) $\hat{t}^n_{a_1,b_1,c_1} = \hat{t}_{a_n,b_n,c_n}$ *for* $n \geq 2$ *where* $a_n = \frac{a_1^{n+1}}{b_1^n}$, $b_n = b_1$, $c_n = c_1$.

2.2. Homeomorphic solutions of conjugacy equation

Lemma 2.4. *Let* f_1, g_1 *be respectively the restrictions of* $\tilde{t}_{a_1,b_1}$ *and* $\tilde{t}_{a_2,b_2}$ *(or* $\bar{t}_{a_1,b_1,c_1}$ *and* $\bar{t}_{a_2,b_2,c_2}$*) to the subintervals* $[0,a_1]$ *and* $[0,a_2]$. *Then all homeomorphic solutions of conjugacy equation* $\varphi_1 \circ f_1 = g_1 \circ \varphi_1$ *are given by* $\varphi_1 : [0,a_1] \to [0,a_2]$

$$\varphi_1(x) = \begin{cases} \phi_0(x), & x \in [f_1^{-1}(a_1), a_1], \\ g^{-i} \circ \phi_0 \circ f^i(x), & x \in [(f^{-i-1}(a_1), f^{-i}(a_1)],\ i = 1,2,\ldots, \\ 0, & x = 0, \end{cases} \tag{2.1}$$

where ϕ_0 *is any strictly increasing continuous map from* $[f_1^{-1}(a_1), a_1]$ *onto* $[g_1^{-1}(a_2), a_2]$.

Proof. Firstly, we prove that any homeomorphic solution of $\varphi_1 \circ f_1 = g_1 \circ \varphi_1$ is strictly increasing. Since $f(x) > x$, $\forall x \in (0,a_1]$, we can see that $f^{-1}(x) < x$. Similarly, $g^{-1}(x) < x$. Suppose that φ is a homeomorphic solution of $\varphi_1 \circ f_1 = g_1 \circ \varphi_1$. Assume that $\varphi(x_0) = y_0$ for some $x_0 \in [0,a_1]$ and $y_0 \in [0,b_1]$. We have $\varphi(f^{-1}(x_0)) = g^{-1}(y_0)$. Since $f^{-1}(x_0) < x_0$ and $g^{-1}(y_0) < x_0$, we see that φ is strictly increasing. Consequently, $\varphi(a_1) = b_1$ and $\varphi(f^{-1}(a_1)) = g^{-1}(b_1)$.

Secondly, one can easily check that the function $\varphi_1(x)$ in (2.1) is a homeomorphic solution of $\varphi_1 \circ f_1 = g_1 \circ \varphi_1$ and every homeomorphic solution can be obtained in this manner. □

The analogous statements hold for $\hat{t}_{a_1,b_1,c_1}$.

Lemma 2.5. *Let* f_1, g_1 *be respectively the restrictions of* $\hat{t}_{a_1,b_1,c_1}$ *and* $\hat{t}_{a_2,b_2,c_2}$ *to the subintervals* $[0,a_1]$ *and* $[0,a_2]$. *Then all homeomorphic solutions of conjugacy equation* $\varphi_1 \circ f_1 = g_1 \circ \varphi_1$ *are given by* $\varphi_1 : [0,b_1] \to [0,b_2]$

$$\varphi_1(x) = \begin{cases} \phi_0(x), & x \in [f_1(b_1), b_1], \\ g^{i} \circ \phi_0 \circ f^{-i}(x), & x \in [(f^{i+1}(b_1), f^{i}(b_1)],\ i = 1,2,\ldots, \\ 0, & x = 0, \end{cases}$$

where ϕ_0 *is any strictly increasing continuous map from* $[f_1(b_1), b_1]$ *onto* $[g_1(b_2), b_2]$.

Lemma 2.6. *Let* f_2 *and* g_2 *be respectively the restrictions of* $\tilde{t}_{a_1,b_1}$ *and* $\tilde{t}_{a_2,b_2}$ *(or* $\bar{t}_{a_1,b_1,c_1}$ *and* $\bar{t}_{a_2,b_2,c_2}$*) to the subintervals* $[0,b_1]$ *and* $[0,b_2]$. *Then all homeomorphic solutions of conjugacy equation* $\varphi_2 \circ f_2 = g_2 \circ \varphi_2$ *are given by* $\varphi_2 : [0,b_1] \to [0,b_2]$

$$\varphi_2(x) = \begin{cases} \varphi_1(x), & x \in [0,a_1], \\ h_1(x), & x \in [a_1,b_1], \end{cases} \tag{2.2}$$

where φ_1 *is determined by* Lemma 2.4 *and* $h_1 : [a_1,b_1] \to [a_2,b_2]$ *is any strictly increasing continuous map.*

Proof. Note that both $f_2 : [0,b_1] \to [0,b_1]$ and $g_2 : [0,b_2] \to [0,b_2]$ are self-maps.

Firstly, we prove by contradiction that any homeomorphic solution of $\varphi_2 \circ f_2 = g_2 \circ \varphi_2$ is strictly increasing. Assume that φ is a strictly decreasing homeomorphic solution. Thus there exists a subinterval $[x_0,1] \subset [a_1,1]$ such that $\varphi([x_0,1]) \subset [0,a_2]$. For some $x \in (x_0,1)$, we have $\varphi \circ f_2(x) = \varphi(b_1) = 0$. However $g_2 \circ \varphi(x) > 0$ due to $\varphi(x) > 0$. This contradicts the fact $\varphi \circ f_2(x) = g_2 \circ \varphi(x)$. Therefore φ is a strictly increasing homeomorphism with $\varphi(b_1) = b_2$.

Secondly, we prove by contradiction that $\varphi(a_1) = a_2$. For any $x \in [a_1, 1]$, we have $g_2 \circ \varphi(x) = \varphi \circ f_2(x) = \varphi(b_1) = b_2$. Then $\varphi(x) \in [a_2, b_2]$. Assume that $\varphi(a_1) = y_0 > a_2$. Then there exists a point $x_0 < a_1$ such that $\varphi(x_0) = a_2$. Thus $\varphi \circ f_2(x_0) = g_2 \circ \varphi(x_0) = g_2(a_2) = b_2$ while $f_2(x_0) < b_1$, which contradicts $\varphi(b_1) = b_2$.

Finally, one can easily check that $\varphi_2(x)$ in (2.2) is a homeomorphic solution of $\varphi_1 \circ f_1 = g_1 \circ \varphi_1$ and every homeomorphic solution can be obtained in this manner. □

3. Classification of trapezoid maps

According to [7, the proof of Theorem 2.1] (cf. [2]), we have the following result for $\mathcal{M}_1$.

Theorem 3.1. *Any two trapezoid maps t_{a_1,b_1} and t_{a_2,b_2} in $\mathcal{M}_1$ are topologically conjugate. Further, any conjugacy from t_{a_1,b_1} to t_{a_2,b_2} is the limit of the functional sequence*

$$\varphi_{n+1}(x) = \begin{cases} g_1^{-1} \circ \varphi_n \circ f_1(x), & x \in [1, a_1), \\ h(x), & x \in [a_1, b_1], \\ g_2^{-1} \circ \varphi_n \circ f_2(x), & x \in (b_1, 1], \end{cases} \tag{3.1}$$

where $g_1 = t_{a_2,b_2}|_{[0,a_2]}$, $f_1 = t_{a_1,b_1}|_{[0,a_1]}$, $g_2 = t_{a_2,b_2}|_{[b_2,1]}$, $f_2 = t_{a_1,b_1}|_{[b_1,1]}$, and h is any order-preserving homeomorphism from $[a_1, b_1]$ onto $[a_2, b_2]$.

When $a_2 = b_2$, the trapezoid map t_{a_2,b_2} degenerates to a skew tent map t_{a_2,a_2}. According to [7, Corollary 3.1] (cf. [2]), we have the following result:

Proposition 3.2. *Any trapezoid map t_{a_1,b_1} in $\mathcal{M}_1$ is topologically semi-conjugate to a skew tent map t_{a_2,a_2}. Further, there exists a unique semi-conjugacy from t_{a_1,b_1} to t_{a_2,a_2}, which is given by the limit of the functional sequence*

$$\varphi_{n+1}(x) = \begin{cases} g_1^{-1} \circ \varphi_n \circ f_1(x), & x \in [1, a_1), \\ a_2, & x \in [a_1, b_1], \\ g_2^{-1} \circ \varphi_n \circ f_2(x), & x \in (b_1, 1], \end{cases} \tag{3.2}$$

where $g_1 = t_{a_2,a_2}|_{[0,a_2]}$, $f_1 = t_{a_1,b_1}|_{[0,a_1]}$, $g_2 = t_{a_2,a_2}|_{[a_2,1]}$, $f_2 = t_{a_1,b_1}|_{[b_1,1]}$.

Theorem 3.3. *Any two trapezoid maps $\tilde{t}_{a_1,b_1}$ and $\tilde{t}_{a_2,b_2}$ in $\mathcal{M}_2$ are topologically conjugate. Further, all conjugacies are given by*

$$\varphi(x) = \begin{cases} \varphi_2(x), & x \in [0, b_1], \\ g_3^{-1} \circ \varphi_2 \circ f_3(x), & x \in (b_1, 1], \end{cases} \tag{3.3}$$

where φ_2 is determined by Lemma 2.6, *$f_3 = \tilde{t}_{a_1,b_1}|_{[b_1,1]}$ and $g_3 = \tilde{t}_{a_2,b_2}|_{[b_2,1]}$.*

Proof. One can immediately check that $\varphi(x)$ given in (3.3) is a homeomorphic solution of $\varphi \circ \tilde{t}_{a_1,b_1} = \tilde{t}_{a_2,b_2} \circ \varphi$ and every homeomorphic solution can be obtained in this manner. □

Theorem 3.4. *Any two trapezoid maps $\bar{t}_{a_1,b_1,c_1}$ and $\bar{t}_{a_2,b_2,c_2}$ in $\mathcal{M}_3$ are topologically conjugate. Further, all conjugacies are given by*

$$\varphi(x) = \begin{cases} \varphi_2(x), & x \in [0, b_1], \\ h_2(x), & x \in [b_1, c_1], \\ g_4^{-1} \circ \varphi_2 \circ f_4(x), & x \in [c_1, 1], \end{cases} \tag{3.4}$$

where φ_2 is determined by Lemma 2.6, *$h_2 : [b_1, c_1] \to [b_2, c_2]$ is any strictly increasing continuous map, $f_4 = \bar{t}_{a_1,b_1,c_1}|_{[c_1,1]}$ and $g_4 = \bar{t}_{a_2,b_2,c_2}|_{[c_2,1]}$.*

Proof. Firstly, suppose φ is a conjugacy from $\bar{t}_{a_1,b_1,c_1}$ to $\bar{t}_{a_2,b_2,c_2}$. Let f_2 and g_2 be respectively the restrictions of $\bar{t}_{a_1,b_1,c_1}$ and $\bar{t}_{a_2,b_2,c_2}$ to the subintervals $[0, b_1]$ and $[0, b_2]$. By Lemma 2.6, the restriction of φ to $[0, b_1]$ is φ_2. Consequently, φ is a strictly increasing homeomorphism with $\varphi(b_1) = b_2$.

Secondly, we prove by contradiction that $\varphi(c_1) = c_2$. For any $x \in [b_1, c_1]$, we have $\bar{t}_{a_2,b_2,c_2} \circ \varphi(x) = \varphi \circ \bar{t}_{a_1,b_1,c_1}(x) = \varphi(b_1) = b_2$. Then $\varphi(x) \in [b_2, c_2]$. Assume that $\varphi(c_1) = y_0 < c_2$. Then there exists a point $x_0 > c_1$ such that $\varphi(x_0) = c_2$. Thus $\varphi \circ \bar{t}_{a_1,b_1,c_1}(x_0) = \bar{t}_{a_2,b_2,c_2} \circ \varphi(x_0) = \bar{t}_{a_2,b_2,c_2}(c_2) = b_2$ while $\bar{t}_{a_1,b_1,c_1}(x_0) < b_1$, which contradicts $\varphi(b_1) = b_2$.

Finally, one can immediately check that $\varphi(x)$ in (3.4) is a homeomorphic solution of $\varphi \circ \bar{t}_{a_1,b_1,c_1} = \bar{t}_{a_2,b_2,c_2} \circ \varphi$ and every homeomorphic solution can be obtained in this manner. □

With the similar argument as the proof of Theorem 3.4, we have the following result for $\mathcal{M}_4$.

Theorem 3.5. *Any two trapezoid maps $\hat{t}_{a_1,b_1,c_1}$ and $\hat{t}_{a_2,b_2,c_2}$ in $\mathcal{M}_4$ are topologically conjugate. Further, all conjugacies are given by*

$$\varphi(x) = \begin{cases} \varphi_1(x), & x \in [0, b_1], \\ h_2(x), & x \in [b_1, c_1], \\ g_3^{-1} \circ \varphi_1 \circ f_3(x), & x \in [c_1, 1], \end{cases}$$

where φ_1 is determined by Lemma 2.5*, $h_2 : [b_1, c_1] \to [b_2, c_2]$ is any strictly increasing continuous map, $f_3 = \hat{t}_{a_1,b_1,c_1}|_{[c_1,1]}$ and $g_3 = \hat{t}_{a_2,b_2,c_2}|_{[c_2,1]}$.*

The following result can be found in [1, Theorem 12.4.2].

Lemma 3.6. *If f is topologically conjugate to g via a homeomorphism h, then h maps every periodic of f to a periodic point of g with the same period.*

Theorem 3.7. *Suppose $f \in \mathcal{M}_i$ and $g \in \mathcal{M}_j$ for $i \neq j$. Then $f \nsim g$.*

Proof. Since any map in $\mathcal{M}_1$ has a four-periodic point, according to Lemmas 2.1, 2.2, 2.3 and 3.6, any map in $\mathcal{M}_1$ is not topologically conjugate to any map in $\mathcal{M}_i, i = 2, 3, 4$.

By Lemmas 2.1, 2.2 and 2.3, any map in $\mathcal{M}_4$ has only one fixed point while any map in $\mathcal{M}_2$ or $\mathcal{M}_3$ has two fixed points. It follows from Lemma 3.6 that any map in $\mathcal{M}_4$ is not topologically conjugate to any map in $\mathcal{M}_i, i = 2, 3$.

Suppose $\bar{t}_{a_1,b_1,c_1} \in \mathcal{M}_3$ and $\tilde{t}_{a_2,b_2} \in \mathcal{M}_2$. We prove by contradiction that $f \nsim g$. Assume that $f \sim g$ via a homeomorphism φ. It follows from Lemma 2.6 that $\varphi(b_1) = b_2$. Let g_3 be the restrictions of $\tilde{t}_{a_2,b_2}$ to the subintervals $[b_2, 1]$. For $\forall x \in [b_1, c_1]$, we have $\varphi(x) \in [b_2, 1]$ and

$$\tilde{t}_{a_2,b_2} \circ \varphi(x) = \varphi \circ \bar{t}_{a_1,b_1,c_1}(x) = \varphi(b_1) = b_2.$$

Thus,

$$\varphi(x) = g_3^{-1}(b_2) = constant, \quad \forall x \in [b_1, c_1].$$

This is a contradiction. □

Proposition 3.8. *Suppose that $\tilde{t}_{a_2,b_2} \in \mathcal{M}_2$ and $\bar{t}_{a_1,b_1,c_1} \in \mathcal{M}_3$. Then $\bar{t}_{a_1,b_1,c_1}$ is topologically semi-conjugate to $\tilde{t}_{a_2,b_2}$.*

Proof. Let f_4 and g_3 be respectively the restrictions of $\bar{t}_{a_1,b_1,c_1}$ and $\tilde{t}_{a_2,b_2}$ to the subintervals $[c_1, 1]$ and $[b_2, 1]$. Define $\varphi : [0, 1] \to [0, 1]$ as

$$\varphi(x) = \begin{cases} \varphi_2(x), & x \in [0, b_1], \\ b_2, & x \in [b_1, c_1], \\ g_3^{-1} \circ \varphi_2 \circ f_4(x), & x \in [c_1, 1], \end{cases} \tag{3.5}$$

where φ_2 is determined by Lemma 2.6. One can immediately check that $\varphi(x)$ in (3.5) is a continuous, increasing and surjective solution of $\varphi \circ \bar{t}_{a_1,b_1,c_1} = \bar{t}_{a_2,b_2,c_2} \circ \varphi$. Therefore, $\bar{t}_{a_1,b_1,c_1}$ is topologically semi-conjugate to $\tilde{t}_{a_2,b_2}$. □

4. Examples

In this section, we end the paper with the following three examples.

Example 4.1. Let $f := t_{\frac{2}{9},\frac{7}{9}}$ and $g := t_{\frac{3}{8},\frac{5}{8}}$. By Theorem 3.1, choosing $h(x) = \frac{9}{20}x + \frac{11}{40}$. we can obtain a piecewise linear conjuacy from f to g, see Fig. 5.

By Corollary 3.2, we can obtain the unique semi-conjugacy from the isosceles trapezoid map $t_{\frac{1}{3},\frac{2}{3}}$ to the tent map $t_{\frac{1}{2},\frac{1}{2}}$ is just the Cantor function (cf. [7]), see Fig. 6.

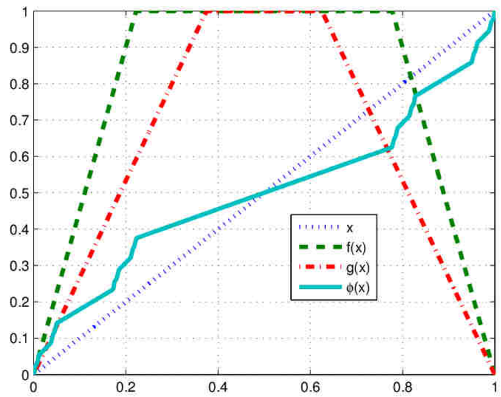

Figure 5: $f := t_{\frac{2}{9},\frac{7}{9}}$ and $g := t_{\frac{3}{8},\frac{5}{8}}$.

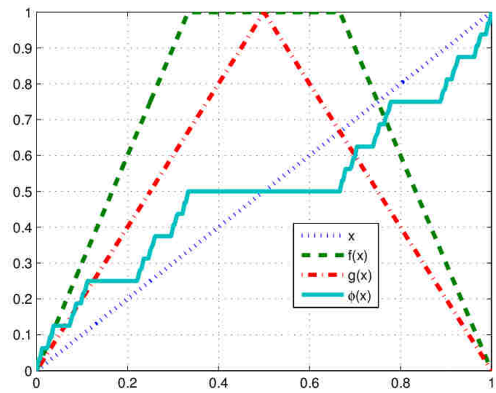

Figure 6: $f := t_{\frac{1}{3},\frac{1}{3}}$ and $g := t_{\frac{1}{2},\frac{1}{2}}$.

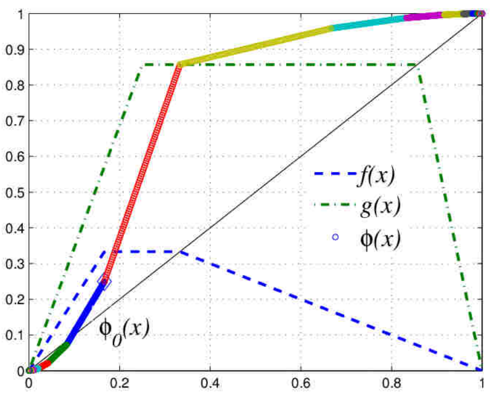

Figure 7: A piecewise linear conjuacy.

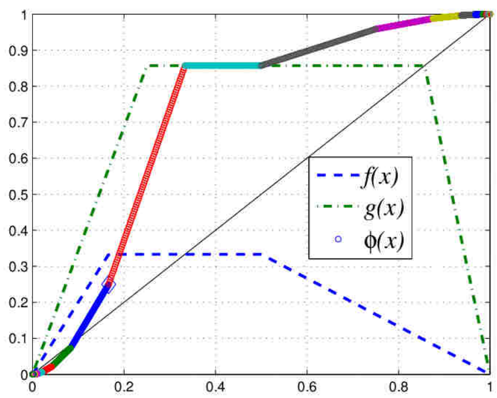

Figure 8: A piecewise linear semi-conjuacy.

Example 4.2. Let $f := \tilde{t}_{\frac{1}{6},\frac{1}{3}}$ and $g := \tilde{t}_{\frac{1}{4},\frac{6}{7}}$. By Theorem 3.3, we can obtain a piecewise linear conjuacy from f to g, see Fig. 7.

Example 4.3. Let $f := \bar{t}_{\frac{1}{6},\frac{1}{3},\frac{1}{2}}$ and $g := \tilde{t}_{\frac{1}{2},\frac{6}{7}}$. By Theorem 3.8, we can obtain a piecewise linear semi-conjuacy from f to g, see Fig. 8.

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