Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 827–835
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Existence and uniqueness of mild and classical solutions to fractional order Hadamard-type Cauchy problem

Qutaibeh Katatbeh[a,*], Ahmad Al-Omari[b]

[a] *Department of Mathematics and Statistics, Faculty of Science and Arts, Jordan University of Science and Technology, Irbid-Jordan 22110.*

[b] *Al al-Bayt University, Faculty of Sciences, Department of Mathematics P.O. Box 130095, Mafraq 25113, Jordan.*

Communicated by R. Saadati

Abstract

We consider the existence and uniqueness of a mild and classical solution to impulsive nonlocal conditions fractional-order Hadamard-type Cauchy problem. The results are obtained by means of fixed point methods. Finally, we illustrate our results by an example of fractional-order Hadamard-type Cauchy problem.

Keywords: Hadamard fractional derivative, integral boundary conditions, fixed point theorems, impulsive equations.
2010 MSC: 34A08, 34N05, 34A12.

1. Introduction and preliminaries

In recent studies, the theory of fractional differential equations and inclusions has been into the focus of many of them. This is due to its extensive applications in numerous branches of applied sciences such as, physics, economics, and engineering sciences [2, 3, 5, 8, 10, 12, 13]. Fractional differential equations and their applications make it possible to find appropriate models for describing real-world problems which cannot be described by using classical integral-order differential equations. Some recent contributions to the subject can be found in [1] and references therein. It has been noticed that most of the work on the topic is based

*Corresponding author
Email addresses: qutaibeh@just.edu.jo (Qutaibeh Katatbeh), omarimutah1@yahoo.com (Ahmad Al-Omari)

Received 2015-08-17

on Riemann–Liouville and Caputo-type fractional differential equations. However, there are other kinds of fractional derivatives that appears side by side with Riemann–Liouville and Caputo derivatives. The fractional derivative is due to Hadamard who introduced it in 1892 [7], but which differs from the preceding ones, in the sense that the kernel of the integral (in the definition of the Hadamard derivative) contains a logarithmic function of arbitrary exponent. The details and properties of the Hadamard fractional derivative and integral can be found in [4, 6, 9]. In this paper, we study a boundary value problem of Hadamard-type fractional differential inclusions.

Our aim is to study the existence and uniqueness of the mild and classical solutions to impulsive nonlocal-conditions fractional Cauchy problem on Hadamard-type fractional differential inclusions. Throughout the paper we shall use the notation:

$$\Omega = \{(t,s) : 1 \leq s \leq t \leq T\},$$
$$M = \sup\{\|S(t)\|\,,\ t \in [1,T]\},$$

and

$$\mathbb{X} = C([1,T],\mathbb{R}).$$

$$D^\alpha u(t) + A\,u(t) = f(t,u(t),u(b_1(t)),\dots,u(b_r(t))) + \int_1^t f_1(t,s,u(s))\,ds + \int_1^{1+\epsilon} f_2(t,s,u(s))\,ds, \tag{1.1}$$
$$\Delta u|_{t_k} = I_k(u(t_k^-)),\ k = 1,2,\dots m.$$
$$u(1) + g(u) = u_0,$$

where $t \in J \setminus \{t_1,t_2,\dots,t_m\} \subseteq [1,T] = J$ and $1 < t_1 < \cdots < t_m < T$ where $0 < \alpha \leq 1$. The operator $-A$ generates an analytic compact semigroup $(S(t))_{t\geq 0}$ of uniformly bounded linear operators on a Banach space $\mathbb{R}$. D^α is a Hadamard-type fractional derivative operator and $f : C([1,T]\times\mathbb{R}^{r+1} \to \mathbb{R}$, $f_i : \Omega\times\mathbb{R} \to \mathbb{R}$ $(i = 1,2)$, $g : \mathbb{X} \to \mathbb{R}$, $b_i : J \to J$ $(i = 1,\dots,r)$ are function satisfying some assumptions, and $u_0 \in \mathbb{R}$ and $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$, represent the right and left limits of $u(t)$ at $t = t_k$; I_k $(k = 1,2,\dots,m)$ are functions to be specified later.

Definition 1.1 ([9])**.** The Hadamard derivative of fractional order α for a function $g : [1,\infty) \to \mathbb{R}$ is defined by

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{g(s)}{s}ds,$$

$\alpha \in (n-1,n)$, $n = [\alpha]+1$, where $[\alpha]$ denotes the integer part of the real number α and $\log(.) = \log_e(.)$.

Definition 1.2 ([9])**.** The Hadamard fractional integral of order α for a function $g : [1,\infty) \to \mathbb{R}$ is defined as

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds$$

for $\alpha > 0$, provided the integral exists.

Let us recall the generalized Gronwall inequality, which can be found in [11],

Lemma 1.3. *Suppose $\alpha > 0$, $a(t)$ and $u(t)$ are nonnegative function and locally integrable on $1 \leq t < T$ and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $1 \leq t < T$, and $g(t) \leq M$ (constant). If*

$$u(t) \leq a(t) + g(t)\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} u(s)\,\frac{ds}{s} \qquad \text{on } 1 \leq t < T,$$

then

$$u(t) \leq a(t) + \int_1^t \left[\sum_{n=1}^{\infty}\frac{[g(t)\Gamma(\alpha)]^n}{\Gamma(n\,\alpha)}\left(\log\frac{t}{s}\right)^{n\,\alpha-1} a(s)\right]\frac{ds}{s}.$$

Definition 1.4. A function $u \in C([1,T],\mathbb{R})$ satisfying the integral equation

$$\begin{aligned} u(t) &= S(t-1)u_0 - S(t-1)g(u) \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^t S(t-s)\left(\log\frac{t}{s}\right)^{\alpha-1}\Big[f(s,u(s),u(b_1(s)),\ldots,u(b_r(s))) \\ &+ \int_1^s f_1(s,\tau,u(\tau))\,d\tau + \int_1^{1+\epsilon} f_2(s,\tau,u(\tau))\,d\tau\Big]\frac{ds}{s} \\ &+ \sum_{k=1}^m S(t-t_k)I_k(u(t_k^-)), \qquad t\in J \end{aligned}$$

is said to be a mild solution to the integrodifferential evolution impulsive nonlocal Cauchy problem (1.1) with fractional-order Hadamard-type.

2. Main result

Theorem 2.1. *The integrodifferential evolution impulsive nonlocal Cauchy problem* (1.1) *has a unique mild solution u, if the following conditions are satisfied:*

(i) $f : J\times\mathbb{R}^{m+1}\to\mathbb{R}$ *is a continuous function with respect to the first variable in J,* $f_i:\Omega\times\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ *are continuous functions with respect to the first and second variables in Ω;* $g: X\to\mathbb{R}$, *and* $b_i : J\to J$ $(i=1,\ldots,m)$ *are continuous functions on J and there exist positive constants L, L_i* $(i=1,2)$ *and K such that:*

$$\|f(s,z_0,z_1,\ldots,z_m) - f(s,\widetilde{z}_0,\widetilde{z}_1,\ldots,\widetilde{z}_m)\| \le L\left(\sum_{i=0}^m \|z_i-\widetilde{z}_i\|\right) \tag{2.1}$$

for $s\in J$, $z_i,\widetilde{z}_i\in\mathbb{R}$ $(i=1,2,\ldots,m)$,

$$\|f_i(s,\tau,z)-f_i(s,\tau,\widetilde{z})\| \le L_i(\|z-\widetilde{z}\|)\ \ (i=1,2) \tag{2.2}$$

for $(s,\tau)\in\Omega$, $z,\widetilde{z}\in\mathbb{R}$ *and*

$$\|g(u)-g(\widetilde{u})\| \le K\|u-\widetilde{u}\|_\infty \ \textit{for}\ u,\widetilde{u}\in X. \tag{2.3}$$

(ii)

$$\big[(mL+K)\Gamma(\alpha+1)\log^{-\alpha}T + L(m+1) + L_1T + L_2T\big]\frac{M\log^\alpha T}{\Gamma(\alpha+1)} < 1.$$

(iii) The functions $I_k:\mathbb{R}\to\mathbb{R}$ *are continuous and there exists a* ρ_1 *such that* $\|I_k(x)\|\le\rho_1$ *for all* $x\in\mathbb{R}$ *and* $k=1,\ldots,m$ *and* $u_0\in\mathbb{R}$.

Proof. Introduce the operator $F:\mathbb{X}\to\mathbb{X}$ by

$$\begin{aligned} (F\omega)(t) &= S(t-1)u_0 - S(t-1)g(\omega) \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^t S(t-s)(\log\frac{t}{s})^{\alpha-1}\Big[f(s,\omega(s),\omega(b_1(s)),\ldots,\omega(b_r(s))) \\ &+ \int_1^s f_1(s,\tau,\omega(\tau))\,d\tau + \int_1^{1+\epsilon} f_2(s,\tau,\omega(\tau))\,d\tau\Big]\frac{ds}{s} \\ &+ \sum_{k=1}^m S(t-t_k)I_k(\omega(t_k^-)), \qquad t\in J. \end{aligned} \tag{2.4}$$

We have

$$
\begin{aligned}
||(F\omega)(t)-(F\hat{\omega})(t)|| &\leq ||S(t-1)||||g(\omega)-g(\hat{\omega})|| \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_1^t S(t-s)\left(\log\frac{t}{s}\right)^{\alpha-1}\Big[f(s,\omega(s),\omega(b_1(s)),\ldots,\hat{\omega}(b_r(s))) \\
&\quad-f(s,\hat{\omega}(s),\hat{\omega}(b_1(s)),\ldots,\hat{\omega}(b_r(s)))\Big]\frac{ds}{s} \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_1^t S(t-s)\left(\log\frac{t}{s}\right)^{\alpha-1}\left[\int_1^s ||f_1(s,\tau,\omega(\tau))-f_1(s,\tau,\hat{\omega}(\tau))||\,d\tau\right]\frac{ds}{s} \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_1^t S(t-s)\left(\log\frac{t}{s}\right)^{\alpha-1}\left[\int_1^{1+\epsilon} ||f_2(s,\tau,\omega(\tau))-f_2(s,\tau,\hat{\omega}(\tau))||\,d\tau\right]\frac{ds}{s} \\
&\quad+\sum_{k=1}^{m}||S(t-t_k)||\left[||I_k(\omega(t_k^-))-I_k(\hat{\omega}(t_k^-))||\right] \\
&\leq MK||\omega-\hat{\omega}||_\infty+\frac{ML}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{(\alpha-1)}\left[||\omega(s)-\hat{\omega}(s)||+\sum_{i=1}^{m}||\omega(b_i(s))-\hat{\omega}(b_i(s))||\right]\frac{ds}{s} \\
&\quad+\frac{ML_1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left[\int_1^s ||\omega(\tau)-\hat{\omega}(\tau)||\,d\tau\right]\frac{ds}{s} \\
&\quad+\frac{ML_2}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left[\int_1^{1+\epsilon} ||\omega(\tau)-\hat{\omega}(\tau)||\,d\tau\right]\frac{ds}{s} \\
&\quad+MmC||\omega-\hat{\omega}||_\infty \\
&\leq MK||\omega-\hat{\omega}||_\infty+\frac{ML(m+1)||\omega-\hat{\omega}||_\infty}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{(\alpha-1)}\frac{ds}{s} \\
&\quad+\frac{ML_1T||\omega-\hat{\omega}||_\infty}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&\quad+\frac{ML_2T||\omega-\hat{\omega}||_\infty}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}+MmC||\omega-\hat{\omega}||_\infty \\
&\leq MK||\omega-\hat{\omega}||_\infty+\frac{ML(m+1)||\omega-\hat{\omega}||_\infty}{\Gamma(\alpha)}\frac{\log^\alpha T}{\alpha} \\
&\quad+\frac{ML_1T||\omega-\hat{\omega}||_\infty}{\Gamma(\alpha)}\frac{\log^\alpha T}{\alpha}+\frac{ML_2T||\omega-\hat{\omega}||_\infty}{\Gamma(\alpha)}\frac{\log^\alpha T}{\alpha}+MmC||\omega-\hat{\omega}||_\infty \\
&\leq\left[(mL+K)\Gamma(\alpha+1)\log^{-\alpha}T+L(m+1)+L_1T+L_2T\right]\frac{M\log^\alpha T}{\Gamma(\alpha+1)}||\omega-\hat{\omega}||_\infty.
\end{aligned}
$$

Using the assumption, we get

$$\left[(mL+K)\Gamma(\alpha+1)\log^{-\alpha}T+L(m+1)+L_1T+L_2T\right]\frac{M\log^\alpha T}{\Gamma(\alpha+1)}<1.$$

This implies that

$$||(F\omega)(t)-(F\hat{\omega})(t)||\leq||\omega-\hat{\omega}||_\infty$$

and hence F is a contraction on $\mathbb{X}$. □

The following theorem gives the conditions for the existence of a unique classical solution for fractional implicit nonlocal conditions with fractional order Hadamard-type.

Theorem 2.2. *Assume that:*

(i) $\mathbb{R}$ *is a reflexive Banach space and* $u_0 \in \mathbb{R}$.

(ii) $f : J \times \mathbb{R}^{m+1} \to \mathbb{R}$ $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$, *are continuous function with respect to the second variable in* J; $g : X \to \mathbb{R}$, $b_i : J \to J$ $(i = 1, \dots, m)$ *are continuous function on* J *and there exist positive constants* C, C_i $(i = 1, 2)$ *and* K *such that:*

$$\|f(s, z_0, z_1, \dots, z_m) - f(\widetilde{s}, \widetilde{z}_0, \widetilde{z}_1, \dots, \widetilde{z}_m)\| \leq C \left(|s - \widetilde{s}| + \sum_{i=0}^{m} \|z_i - \widetilde{z}_i\| \right) \tag{2.5}$$

for $s, \widetilde{s} \in J$, $z_i, \widetilde{z}_i \in \mathbb{R}$ $(i = 1, 2, \dots, m)$,

$$\|f_i(s, \tau, z) - f_i(\widetilde{s}, \tau, \widetilde{z})\| \leq C_i \left(|s - \widetilde{s}| + \|z - \widetilde{z}\| \right) \;\; (i = 1, 2) \tag{2.6}$$

for (s, τ), $(\widetilde{s}, \tau) \in \Omega$, $z, \widetilde{z} \in \mathbb{R}$ *and*

$$\|g(u) - g(\widetilde{u})\| \leq K \|u - \widetilde{u}\|_\infty \text{ for } u, \widetilde{u} \in X. \tag{2.7}$$

(iii) $\left[(mC + K)\Gamma(\alpha + 1) \log^{-\alpha} T + C(m + 1) + T(C_1 + C_2)\right] \dfrac{M \log^\alpha T}{\Gamma(\alpha + 1)} < 1.$

(iv) *The functions* $I_k : \mathbb{R} \to \mathbb{R}$ *are continuous and there exists a* ρ_1 *such that* $\|I_k(x)\| \leq \rho_1$ *for all* $x \in \mathbb{R}$ *and* $k = 1, \dots, m$.

Then the integrodifferential evolution impulsive nonlocal Cauchy problem (1.1) *has a unique mild solution* u. *Moreover, if* $u_0 \in D(A)$, $g(u) \in D(A)$, *and if there is a positive constant* κ *such that*

$$\|u(b_i(s)) - u(b_i(\widetilde{s}))\| \leq \kappa \|u(s) - u(\widetilde{s})\| \text{ for } s, \widetilde{s} \in J \quad (i = 1, \dots, m), \tag{2.8}$$

then u *is a unique classical solution to problem* (1.1).

Proof. Since all the assumptions of Theorem 2.1 are satisfied, it is easy to see that the presented problem possesses a unique mild solution u. Now, we shall show that u is a classical solution to the problem. Introducing the notation

$$N := \max_{s \in J} ||f(s, u(s), u(b_1(s), \dots, u(b_m(s)))||$$
$$N_i := \max_{(s,\tau) \in \Gamma} ||f_i(s, \tau, u(\tau))||, \quad (i = 1, 2)$$

we my write

$$\begin{aligned} u(t + \theta) - u(t) = {} & (S(t + \theta - 1)u_0 - S(t + \theta - 1)g(u)) - (S(t - 1)u_0 - S(t - 1)g(u)) \\ & + \frac{1}{\Gamma(\alpha)} \int_1^{t+\theta} S(t + \theta - s) \left(\log \frac{t + \theta}{s} \right)^{\alpha - 1} \Big[f(s, u(s), u(b_1(s)), \dots, u(b_r(s))) \Big] \\ & + \int_1^s f_1(s, \tau, u(\tau)) \, d\tau + \int_1^{1+\epsilon} f_2(s, \tau, u(\tau)) \, d\tau \Big] \frac{ds}{s} \\ & - \frac{1}{\Gamma(\alpha)} \int_1^t S(t - s) \left(\log \frac{t}{s} \right)^{\alpha - 1} \Big[f(s, u(s), u(b_1(s)), \dots, u(b_r(s))) \Big] \\ & + \int_1^s f_1(s, \tau, u(\tau)) \, d\tau + \int_1^{1+\epsilon} f_2(s, \tau, u(\tau)) \, d\tau \Big] \frac{ds}{s} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{m} S(t+\theta-t_k)I_k(u(t_k^-)) - \sum_{k=1}^{m} S(t-t_k)I_k(u(t_k^-))\\
&= S(t-1)\left(S(\theta)-I\right)u_0 - S(t-1)\left(S(\theta)-I\right)g(u)\\
&+\frac{1}{\Gamma(\alpha)}\int_1^{1+\theta} S(t+\theta-s)\left(\log\frac{t+\theta}{s}\right)^{\alpha-1}\Big[\,f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\Big]\\
&+\int_1^{s} f_1(s,\tau,u(\tau))\,d\tau + \int_1^{1+\epsilon} f_2(s,\tau,u(\tau))\,d\tau\Big]\,\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_1^{t} S(t-s)\left(\log\frac{t}{s}\right)^{\alpha-1}\Big[\,f(s+\theta,u(s+\theta),u(b_1(s+\theta)),\ldots,u(b_r(s+\theta)))\\
&-\ f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\Big]\,\frac{ds}{s}\\
&-\frac{1}{\Gamma(\alpha)}\int_1^{t} S(t-s)\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\int_1^{s}\left(f_1(s+\theta,\tau,u(\tau))-f_1(s,\tau,u(\tau))\right)d\tau\right)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_1^{t} S(t-s)\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\int_s^{s+\theta}\left(f_1(s+\theta,\tau,u(\tau))\right)d\tau\right)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_1^{t} S(t-s)\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\int_1^{1+\epsilon}\left(f_2(s+\theta,\tau,u(\tau))-f_2(s,\tau,u(\tau))\right)d\tau\right)\frac{ds}{s}\\
&+\sum_{k=1}^{m} S(\theta)I_k(u(t_k^-));
\end{aligned}
$$

therefore,

$$
\begin{aligned}
||u(t+\theta)-u(t)|| \le{}& \theta M||Au_0|| + \theta M||Ag(u)||\\
&+\frac{(M(N+TN_1+TN_2))}{\Gamma(\alpha)}\int_1^{1+\theta}\left(\log\frac{t+\theta}{s}\right)^{\alpha-1}\frac{ds}{s} + \frac{(MC\theta)}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\
&+\frac{(MC)}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left(|\theta| + \|u(s+\theta)-u(s)\| + \sum_{i=1}^{m}\|u(b_i(s+\theta))-u(b_i(s))\|\right)\frac{ds}{s}\\
&+\frac{(MC_1\theta T)}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\
&+\frac{(MN_1\theta)}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} + \frac{(MC_2\theta T)}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} + Mm\rho_1.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
||u(t+\theta)-u(t)|| \le{}& \theta M||Au_0|| + \theta M||Ag(u)|| + Mm\rho_1\\
&+\frac{(M(N+TN_1+TN_2))}{\Gamma(\alpha)}\int_1^{1+\theta}\left(\log\frac{t+\theta}{s}\right)^{\alpha-1}\frac{ds}{s}\\
&+\frac{(MC(1+m\kappa))}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\|u(s+\theta)-u(s)\|\,\frac{ds}{s}\\
&+\frac{(2MC\theta + MC_1\theta T + MN_1\theta + MC_2\theta T)}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s},\\
||u(t+\theta)-u(t)|| \le{}& \theta M||Au_0|| + \theta M||Ag(u)|| + Mm\rho_1\\
&+\frac{T^*(M(N+TN_1+TN_2))}{\Gamma(\alpha)}
\end{aligned}
$$

$$+\frac{(MC(1+m\kappa))}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\|u(s+\theta)-u(s)\|\frac{ds}{s}$$
$$+\frac{\log^\alpha T(2MC\theta+MC_1\theta T+MN_1\theta+MC_2\theta T)}{\Gamma(\alpha)},$$

where

$$T^*:=Max\left\{\int_1^{1+\theta}\left(\log\frac{t+\theta}{s}\right)^{\alpha-1}\frac{ds}{s},\ t\in[1,T]\right\}$$
$$||u(t+\theta)-u(t)||\le C_*(t)\ +C_{**}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\|u(s+\theta)-u(s)\|\frac{ds}{s}.$$

By the generalized Gronwall inequality from [11], it follows that

$$||u(t+\theta)-u(t)||\le C_*(t)+\int_1^t\left[\sum_{n=1}^{\infty}\frac{[C_{**}\Gamma(\alpha)]^n}{\Gamma(n\,\alpha)}\left(\log\frac{t}{s}\right)^{n\,\alpha-1}C_*(s)\right]\frac{ds}{s}$$
$$||u(t+\theta)-u(t)||\le C_*(t)+\sum_{n=1}^{\infty}\frac{[C_{**}\Gamma(\alpha)]^n}{\Gamma(n\,\alpha+1)}(\log t)^{n\,\alpha}C_*$$
$$||u(t+\theta)-u(t)||\le C_*(t)+C_*(t)\mathbb{E}_\alpha(C_{**}\Gamma(\alpha)\log^\alpha t),$$

where $\mathbb{E}_\alpha$ is the Mittag-Leffler function defined by $\mathbb{E}_\alpha(t)=\sum\limits_{k=0}^{\infty}\dfrac{t^k}{\Gamma(k\alpha+1)}$ for $t\in[1,T)$, $\theta>0$ and $t+\theta\in(1,T]$. Hence u is Lipschitz continuous on J. The Lipschitz continuity of u on J combined with the Lipschitz continuity of f on $J\times\mathbb{R}^{m+1}$ and $f_i\quad(i=1,2)$ in J imply that the function

$$J\ni\ t\to f(t,u(t),u(b_1(t)),\ldots,u(b_r(t)))+\int_1^t f_1(t,s,u(s))\,ds+\int_1^{1+\epsilon}f_2(t,s,u(s))\,ds$$

is Lipschitz continuous on J. This property of f, together with assumptions and by Theorem 2.1 implies that u is a unique classical solution to the present problem. □

3. Example

In this section we provide an example to illustrate our main result. Consider the following impulsive nonlocal fractional differential equation

$$D^\alpha u(t)+\frac{1}{100}e^{-t}|u(t)|=\frac{e^{-t}|u(t)|}{(9+e^t)(1+|u(t)|)}+\int_1^t\frac{e^{t-s}}{5+|u(s)|}ds+\int_1^2\frac{e^s|u(s)|}{(t+5)^2(1+|u(s)|)}ds$$
$$t\in[1,2],\qquad t\neq\frac{3}{2},\qquad 0<\alpha\le 1.$$
$$\Delta u|_{t=\frac{3}{2}}=\frac{|u(\frac{3}{2}^-)|}{8+|u(\frac{3}{2}^-)|}$$
$$u(1)+\frac{|u(t)|}{3+|u(t)|}=u_0(t)$$
$$f(t,u(t),u(b(t)))=\frac{e^{-t}|u(t)|}{(9+e^t)(1+|u(t)|)},\quad f_1(t,s,u(s))=\frac{e^{t-s}}{5+|u(s)|},$$

$$f_2(t,s,u(s)) = \frac{e^s|u(s)|}{(t+5)^2(1+|u(s)|)} \quad \text{and} \quad I_k(u) = \frac{u(t)}{8+u(t)}.$$

We have

$$\begin{aligned}
\|f(t,u(t),u(b(t))) - f(t,v(t),v(b(t)))\| &= \frac{e^{-t}}{9+e^t}\left|\frac{u}{1+u} - \frac{v}{1+v}\right| \\
&= \frac{e^{-t}|u-v|}{(9+e^t)(1+u)(1+v)} \\
&\le \frac{e^{-t}}{9+e^t}|u-v| \\
&\le \frac{1}{10}|u-v|.
\end{aligned}$$

Therefore, the first condition in Theorem 2.2 holds with $C = \frac{1}{10}$. We also have

$$\begin{aligned}
\|f_1(t,s,u(s)) - f_1(t,s,v(s))\| &= e^{t-s}\left|\frac{1}{5+u} - \frac{1}{5+v}\right| \\
&= \frac{e^{t-s}|u-v|}{(5+u)(5+v)} \\
&\le \frac{e}{25}|u-v|.
\end{aligned}$$

$$\begin{aligned}
\|f_2(t,s,u(s)) - f_2(t,s,v(s))\| &= \frac{e^s}{(t+5)^2}\left|\frac{u(s)}{1+u(s)} - \frac{v(s)}{1+v(s)}\right| \\
&= \frac{e^s|u-v|}{(t+5)^2(1+u)(1+v)} \\
&\le \frac{e}{25}|u-v|.
\end{aligned}$$

Hence the second condition in Theorem 2.2 holds with $C_i = \frac{e}{25}$ $(i = 1, 2)$. Moreover, we have

$$\|g(u) - g(v)\| = \left|\frac{u}{3+u} - \frac{v}{3+v}\right| = \frac{3|u-v|}{(3+u)(3+v)} \le \frac{1}{3}|u-v|,$$

and

$$\|I_k(u) - I_k(v)\| = \left|\frac{u}{8+u} - \frac{v}{8+v}\right| = \frac{8|u-v|}{(8+u)(8+v)} \le \frac{1}{8}|u-v|,$$

and

$$\|I_k(u)\| = |\frac{u}{u+8}| \le \frac{1}{8}.$$

Hence the condition (iv) in Theorem 2.2 holds with $\rho_1 = \frac{1}{8}$. Let $t \in [1,2]$. We shall check whether the condition

$$\left[(mC+K)\Gamma(\alpha+1)(\log 2)^{-\alpha} + C(m+1) + 2(C_1+C_2)\right]\frac{M(\log 2)^\alpha}{\Gamma(\alpha+1)} < 1$$

is satisfied with $m = 1$ and we can take $M = 1$. Indeed

$$\left[(mC+K)\Gamma(\alpha+1)(\log 2)^{-\alpha} + C(m+1) + 2(C_1+C_2)\right]\frac{M(\log 2)^\alpha}{\Gamma(\alpha+1)} < 1$$

if and only if $\Gamma(\alpha+1) > \dfrac{44}{49}$, which is satisfied for an $\alpha \in (0,1]$, such as $\alpha = 0.2$ or $\alpha = 0.8$. Then by Theorem 2.2, the problem (1.1) has a unique solution in $[1,2]$.

Acknowledgement

The authors thank Jordan University of Science and Technology and Al-Bayt University for their support.

References

[1] B. Ahmad, S. K. Ntouyas, *On Hadamard fractional integro-differential boundary value problems*, J. Appl. Math. Comput., **47** (2015), 119–131. 1

[2] B. Ahmad, S. K. Ntouyas, A. Alsaedi, *New results for boundary value problems of Hadamard-type fractional differential inclusions and integral boundary conditions*, Bound. Value Probl., **2013** (2013), 14 pages. 1

[3] A. Anguraja, M. L. Maheswari, *Existence of solutions for fractional impulsive neutral functional infinite delay integrodifferential equations with nonlocal conditions*, J. Nonlinear Sci. Appl., **5** (2012), 271–280. 1

[4] P. L. Butzer, A. A. Kilbas, J. J. Trujillo, *Compositions of Hadamard-type fractional integration operators and the semigroup property*, J. Math. Anal. Appl., **269** (2002), 387–400. 1

[5] P. L. Butzer, A. A. Kilbas, J. J. Trujillo, *Fractional calculus in the Mellin setting and Hadamard-type fractional integrals*, J. Math. Anal. Appl., **269** (2002), 1–27. 1

[6] P. L. Butzer, A. A. Kilbas, J. J. Trujillo, *Mellin transform analysis and integration by parts for Hadamard-type fractional integrals*, J. Math. Anal. Appl., **270** (2002), 1–15. 1

[7] J. Hadamard, *Essai sur l'étude des fonctions donnés par leur développement de Taylor*, J. Math. Pures Appl., **8** (1892), 101–186. 1

[8] A. A. Kilbas, *Hadamard-type fractional calculus*, J. Korean Math. Soc., **38** (2001), 1191–1204. 1

[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, Elsevier Science B. V., Amsterdam, (2006). 1, 1.1, 1.2

[10] A. A. Kilbas, J. J. Trujillo, *Hadamard-type integrals as G-transforms*, Integral Transforms Spec. Funct., **14** (2003), 413–427. 1

[11] S.-Y. Lin, *Generalized Gronwall inequalities and their applications to fractional differential equations*, J. Inequal. Appl., **2013** (2013), 9 pages. 1, 2

[12] J. A. Nanwarea, D. B. Dhaigude, *Existence and uniqueness of solutions of differential equations of fractional order with integral boundary conditions*, J. Nonlinear Sci. Appl., **7** (2014), 246–254. 1

[13] T. Qiu, Z. Bai, *Positive solutions for boundary of nonlinear fractional differential equation*, J. Nonlinear Sci. Appl., **1** (2008), 123–131. 1