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Pedal curves of fronts in the sphere

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Abstract

Notions of the pedal curves of regular curves are classical topics. T. Nishimura [T. Nishimura, Demonstratio Math., **43** (2010), 447–459] has done some work associated with the singularities of pedal curves of regular curves. But if the curve has singular points, we can not define the Frenet frame at these singular points. We also can not use the Frenet frame to define and study the pedal curve of the original curve. In this paper, we consider the differential geometry of pedal curves of singular curves in the sphere. We define the pedal curve of a front and give properties of such pedal curve by using a moving frame along a front. At last, we give the classification of singularities of the pedal curves of fronts.

Keywords: Pedal curve, front, singularity, Legendre curve.
2010 MSC: 51B20, 53B50, 53A35.

1. Introduction

The notions of pedal curves of regular curves in Euclidean plane or 3-space are classical topics in differential geometry. As well known, for a plane curve $\boldsymbol{\gamma}$ and a given fixed pedal point P, the pedal curve of $\boldsymbol{\gamma}$ is the locus of points X so that the line PX is perpendicular to a tangent T to the curve passing through the point X. In [4], T. Nishimura gives the concept and the classification of the singularities of pedal curves of regular curves in the unit sphere. Unfortunately, if the curve is not regular at a point, then we can not define the pedal curve at this point as the classical way. In [2] , T. Fukunaga and M. Takahashi firstly define frontals (or fronts) in Euclidean plane and Legendrian curves (or Legendrian immersions) in the unit tangent bundle of $\mathbb{R}^2$. The differential geometric properties of the frontal is studied in [3]. The most important difference between a regular curve and a frontal is that the frontal might exist singular points. A key tool for studying of the frontal is the so called moving frame defined in the unit tangent bundle. By

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using the moving frame, we can give a new definition of pedal curve of the front. We remark that this new definition on the pedal curve is consistent with the classical one when the curve is a regular curve.

On the other hand, singularity theory, which is a direct descendant of differential calculus has a great deal of interest to speak about geometry, equation, physics, astronomy and other disciplines. In general, the current theory always does not allow for singularities, however, it is unavoidable in some real life circumstances. Thus, we apparently need to understand the ontology of singularities if we want to research the nature of space and time in the actual universe. For this reason, it has been studied extensively by both physicists and geometers. There are several articles concerning singularities of pedal curves [4, 5]. In those papers, all the pedal curves are produced by regular curves. However, to the best of the authors knowledge, no literature exists regarding the singularities of pedal curves of fronts in the sphere. Thus the current study hopes to serve such a need and it is inspired by the work T. Nishimura [4] and M. Takahashi [6]. In this paper, we introduce the notions of frontals (or fronts), Legendre curves (or Legendre immersions) and the pedal curves of fronts in the sphere etc. The main result of this paper is Theorem 3.3 which gives the classification of singularities of pedal curves of fronts in S^2.

The rest of this paper is organized as follows. Firstly, we introduce the moving frame of Legendre curves (frontals) in S^2 that will be useful to the study of pedal curves of fronts in S^2. Then, we give the definition of pedal curves of fronts in S^2. We remark that this new definition on the pedal curve is consistent with the classical one when the curve is a regular curve and give the classification of singularities of pedal curves of fronts in S^2. Finally, we also give an example of pedal curve of front.

2. The frontals in the sphere

In this section, we consider the differential geometry of smooth curves in Euclidean 2-sphere. If the curve has singular points, we can not define the orthonormal Frenet frame at these singular points. We also can not use the Frenet-Serret type formula to study the properties of the original curve. In order to overcome this difficulty, we take advantage of the way developed by T. Fukunaga and M. Takahashi in [2] instead of the classical way. We give the detailed descriptions about this way as follows. For more detailed descriptions see [6].

We consider the differential geometry of curves in Euclidean 2-sphere. Let $\boldsymbol{\gamma} : I \to S^2$ be a smooth curve. We say that $\boldsymbol{\gamma}$ is a frontal in S^2, if there exists a smooth mapping $\nu : I \to S^2$ such that the pair $(\boldsymbol{\gamma}, \nu) : I \to \Delta \subset S^2 \times S^2$ satisfies $(\boldsymbol{\gamma}(t), \nu(t))^*\theta = 0$ for all $t \in I$. Here

$$\Delta = \{(\boldsymbol{v}, \boldsymbol{w}) \in S^2 \times S^2 \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$$

is a 3-dimensional manifold and θ is a canonical contact 1-form on Δ. The condition

$$(\boldsymbol{\gamma}(t), \nu(t))^*\theta = 0$$

is equivalent to

$$\langle \dot{\boldsymbol{\gamma}}(t), \nu(t) \rangle = 0 \ for \ all \ t \in I,$$

and we call $(\boldsymbol{\gamma}, \nu) : I \to \Delta$ the *Legendre curve*. We consider the canonical contact structure on the unit spherical bundle $T_1S^2 = S^2 \times S^2$ over S^2. If $(\boldsymbol{\gamma}, \nu)$ is a Legendre curve, then $(\boldsymbol{\gamma}, \nu)$ is an integral curve with respect to the contact structure [1]. Moreover, if $(\boldsymbol{\gamma}, \nu) : I \to \Delta$ is an immersion, namely, $(\dot{\boldsymbol{\gamma}}(t), \dot{\nu}(t)) \neq 0$ and $(\boldsymbol{\gamma}(t), \nu(t))^*\theta = 0$ for each $t \in I$. Then we call $\boldsymbol{\gamma}$ the front in S^2 and $(\boldsymbol{\gamma}, \nu) : I \to \Delta$ the Legendre immersion.

Let $(\boldsymbol{\gamma}, \nu) : I \to \Delta$ be a Legendre curve. If $\boldsymbol{\gamma}$ is singular at a point t_0 in S^2, then we can not define the Frenet-Serret formula at this point. By the definition of the Legendre curve, however, the ν is always well defined even if at a singular point of $\boldsymbol{\gamma}$. Let $\boldsymbol{\mu}(t) = \boldsymbol{\gamma} \wedge \nu(t)$. Then $\boldsymbol{\mu}(t) \in S^2$, $\boldsymbol{\gamma}(t) \cdot \boldsymbol{\mu}(t) = 0$ and $\nu(t) \cdot \boldsymbol{\mu}(t) = 0$. We have a moving frame $\{\boldsymbol{\gamma}(t), \nu(t), \boldsymbol{\mu}(t)\}$ which is called the Legendre Frenet frame along $\boldsymbol{\gamma}(t)$. By the standard arguments, we have the following Legendre Frenet-Serret type formula [6] :

$$\begin{pmatrix} \boldsymbol{\gamma}'(t) \\ \nu'(t) \\ \boldsymbol{\mu}'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m(t) \\ 0 & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}(t) \\ \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{2.1}$$

where $m(t) = \langle \dot{\boldsymbol{\gamma}}(t), \boldsymbol{\mu}(t)\rangle$ and $n(t) = \langle \dot{\nu}(t), \boldsymbol{\mu}(t)\rangle$. We call the pair (m, n) the Legendre curvature of Legendre curve $(\boldsymbol{\gamma}, \nu) : I \to \Delta \subset S^2 \times S^2$. We remark that if $(\boldsymbol{\gamma}, \nu) : I \to \Delta \subset S^2 \times S^2$ is a Legendre curve (Legendre immersion) with the Legendre curvature $(-m, n)$, then $(\boldsymbol{\gamma}, -\nu)$ is a Legendre curve (Legendre immersion). Also $(-\boldsymbol{\gamma}, \nu)$ is a Legendre curve (Legendre immersion) with the Legendre curvature $(m, -n)$. Moreover $(\nu, \boldsymbol{\gamma})$ is a Legendre curve (Legendre immersion) with the Legendre curvature $(-n, -m)$.

Let I and $\tilde{I}$ be intervals. A smooth function $u : \tilde{I} \to I$ is a (positive) change of parameter when u is surjective and has a positive derivative at every point. It follows that u is a diffeomorphism.

Let $(\boldsymbol{\gamma}, \nu) : I \to \Delta \subset S^2 \times S^2$ and $(\tilde{\boldsymbol{\gamma}}, \tilde{\nu}) : I \to \Delta \subset S^2 \times S^2$ be Legendre curves whose curvatures are (m, n) and $(\tilde{m}, \tilde{n})$ respectively. Suppose that $(\boldsymbol{\gamma}, \nu)$ and $(\tilde{\boldsymbol{\gamma}}, \tilde{\nu})$ are parametrically equivalent via the change of parameter $t : \tilde{I} \to I$, that is, $(\tilde{\boldsymbol{\gamma}}(u), \tilde{\nu}(u)) = (\boldsymbol{\gamma}(t(u)), \nu(t(u)))$ for all $u \in \tilde{I}$. By differentiation, we have

$$\tilde{m}(u) = m(t(u))\dot{t}(u), \;\; \tilde{n}(u) = n(t(u))\dot{t}(u). \tag{2.2}$$

Hence the curvature is dependent on the parametrization.

3. The pedal curves of fronts in the sphere

In this section, We first recall the concepts of pedal curves of regular curves in the sphere [4]. Let P be a point in $S^2 - \{\pm \boldsymbol{n}(t) \mid t \in I\}$, the pedal curve $Pe_{\boldsymbol{\gamma},P} : I \to S^2$ of a regular curve $\boldsymbol{\gamma} : I \to S^2$ is given by

$$Pe_{\boldsymbol{\gamma},P}(t) = \frac{1}{\sqrt{1 - (P \cdot \boldsymbol{n}(t))^2}}(P - (P \cdot \boldsymbol{n}(t))\boldsymbol{n}(t)), \tag{3.1}$$

where, $\boldsymbol{n}(t)$ is unit normal vector of the regular curve $\boldsymbol{\gamma}$.

We assume that $(\boldsymbol{\gamma}, \nu) : I \to \Delta \subset S^2 \times S^2$ is a Legendre immersion, that is, $(m(t), n(t)) \neq (0, 0)$ for all $t \in I$. We define a pedal curve of the front and give properties of the pedal curve in the sphere.

Definition 1. Let P be any point in $S^2 - \{\pm \nu(t) \mid t \in I\}$, we define a pedal curve $\mathcal{P}e_{\boldsymbol{\gamma},P} : I \to S^2$ of the front $\boldsymbol{\gamma}$ by

$$\mathcal{P}e_{\boldsymbol{\gamma},P}(t) = \frac{1}{\sqrt{1 - (P \cdot \nu(t))^2}}(P - (P \cdot \nu(t))\nu(t)). \tag{3.2}$$

Proposition 2. Let $\boldsymbol{\gamma} : I \to S^2$ be a regular curve and P be any point in $S^2 - \{\pm \nu(t) \mid t \in I\}$. Then the pedal curve of the regular curve and the pedal curve of the front are coincide.

Proof. Let $\boldsymbol{\gamma} : I \to S^2$ be a regular curve and P be any point in $S^2 - \{\pm \nu(t) \mid t \in I\}$. Without loss of generality, if we take $\nu(t) = \boldsymbol{n}(t)$, then $(\boldsymbol{\gamma}, \boldsymbol{n})$ is a Legendre immersion, and by the definition of pedal curve of the regular curve (3.1), we have

$$\begin{aligned} Pe_{\boldsymbol{\gamma},P}(t) =& \frac{1}{\sqrt{1 - (P \cdot \boldsymbol{n}(t))^2}}(P - (P \cdot \boldsymbol{n}(t))\boldsymbol{n}(t)) \\ =& \frac{1}{\sqrt{1 - (P \cdot \nu(t))^2}}(P - (P \cdot \nu(t))\nu(t)) = \mathcal{P}e_{\boldsymbol{\gamma},P}(t). \end{aligned}$$

□

Proposition 3. Suppose that $(\boldsymbol{\gamma}, \nu) : I \to \Delta \subset S^2 \times S^2$ is a Legendre immersion with Legendre curvature (m, n) and P be any point in $S^2 - \{\pm \nu(t) \mid t \in I\}$. Then the pedal curve $\mathcal{P}e_{\boldsymbol{\gamma},P}$ of $\boldsymbol{\gamma}$ is independent of the parametrization of $(\boldsymbol{\gamma}, \nu)$.

Proof. Let $(\boldsymbol{\gamma}, \nu) : I \to \Delta \subset S^2 \times S^2$ and $(\tilde{\boldsymbol{\gamma}}, \tilde{\nu}) : \tilde{I} \to \Delta \subset S^2 \times S^2$ be parametrically equivalent via the change of parameter $t : \tilde{I} \to I$. By the assumption, we have $(\tilde{\boldsymbol{\gamma}}(u), \tilde{\nu}(u)) = (\boldsymbol{\gamma}(t(u)), \nu(t(u)))$, then we have

$$\begin{aligned}\mathcal{P}e_{\tilde{\boldsymbol{\gamma}},P}(u) =& \frac{1}{\sqrt{1-(P\cdot\tilde{\nu}(u))^2}}(P-(P\cdot\tilde{\nu}(u))\tilde{\nu}(u))\\ =& \frac{1}{\sqrt{1-(P\cdot\nu(t(u)))^2}}(P-(P\cdot\nu(t(u)))\nu(t(u))) = \mathcal{P}e_{\boldsymbol{\gamma},P}(t(u)).\end{aligned}$$

□

Lemma 3.1. *Let $(\boldsymbol{\gamma}, \nu) : I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with Legendre curvature (m, n) and P be any point in $S^2 - \{\pm\nu(t) \mid t \in I\}$. $\mathcal{P}e_{\boldsymbol{\gamma},P}$ is the pedal curve of $\boldsymbol{\gamma}$, then $\mathcal{P}'e_{\boldsymbol{\gamma},P}(t) = 0$ if and only if $n(t) = 0$ or $P = \boldsymbol{\gamma}(t)$.*

Proof. By differentiating $\mathcal{P}e_{\boldsymbol{\gamma},P}$ and using (2.2), we have the following:

$$\begin{aligned}\mathcal{P}'e_{\boldsymbol{\gamma},P}(t) =& n(t)\frac{(P\cdot\nu(t))(P\cdot\boldsymbol{\mu}(t))}{(1-(P\cdot\nu(t))^2)^{\frac{3}{2}}}\left((P\cdot\boldsymbol{\gamma}(t))\boldsymbol{\gamma}(t)+(P\cdot\boldsymbol{\mu}(t))\boldsymbol{\mu}(t)\right)\\ &- n(t)\frac{1}{(1-(P\cdot\nu(t))^2)^{\frac{1}{2}}}\left((P\cdot\boldsymbol{\mu}(t))\nu(t)-(P\cdot\nu(t))\boldsymbol{\mu}(t)\right).\end{aligned}$$

Since $\{\boldsymbol{\gamma}(t), \nu(t), \boldsymbol{\mu}(t)\}$ is an orthogonal frame, we see that $\mathcal{P}'e_{\boldsymbol{\gamma},P}(t) = 0$ if and only if $n(t) = 0$ or $P = \boldsymbol{\gamma}(t)$. □

Let P be a point of $S^2 - \{\pm\nu(t) \mid t \in I\}$. We consider the following C^∞ map :

$$\psi_p : S^2 - \{\pm P\} \to S^2,$$

$$\psi_P(\boldsymbol{x}) = \frac{1}{\sqrt{1-(P\cdot\boldsymbol{x})^2}}(P-(P\cdot\boldsymbol{x})\boldsymbol{x}).$$

We see that the image $\psi_P(S^2 - \{\pm P\})$ is inside the open hemisphere centered at P. Let this open hemisphere, the set $\pi(S^2 - \{\pm P\})$ be denoted by $\mathbf{X}_P, \mathbf{B}_P$ respectively, where $\pi : S^2 \to P^2(\mathbb{R})$ is the canonical projection. Note that $\mathbf{X}_P$ is C^∞ diffeomorphic to the 2-dimensional open disc $\{(x, y) \mid x^2 + y^2 < 1\}$ and $\mathbf{B}_P$ is C^∞ diffeomorphic to the open Möbius band. Since $\psi_P(\boldsymbol{x}) = \psi_P(-\boldsymbol{x})$, ψ_P induces the map $\tilde{\psi}_P : \mathbf{B}_P \to \mathbf{X}_P$. Then, $\mathcal{P}e_{\boldsymbol{\gamma},P}(t)$ is factored into three maps in the following way:

$$\mathcal{P}e_{\boldsymbol{\gamma},P}(t) = \tilde{\psi}_P \circ \pi \circ \nu(t).$$

We let B be the set

$$\{(x_1, x_2) \times [\xi_1 : \xi_2] \in \mathbb{R}^2 \times P^1(\mathbb{R}) \mid x_1\xi_2 = x_2\xi_1\}$$

and let $p : B \to \mathbb{R}^2$ be the blow up of $\mathbb{R}^2$ centered at the origin, then we have,

Lemma 3.2. *Let P be a point of $S^2 - \{\pm\nu(t) \mid t \in I\}$. Then, there exist C^∞ diffeomorphisms $h_s : B_P \to B$ and $h_t : X_P \to \mathbb{R}^2$ such that the equality $h_t \circ \tilde{\psi}_p \equiv p \circ h_s$ is satisfied.*

Proof. It is reasonable to call $\tilde{\psi}_p$ a map of blow up type. First, by a suitable rotation of S^2 if necessary, we may assume that $P = (0, 0, 1)$. We put

$$\begin{aligned}U_1 &= \{(x_1, x_2) \times [\xi_1 : \xi_2] \in \mathbb{R}^2 \times P^1(\mathbb{R}) \mid x_1\xi_2 = x_2\xi_1, \xi_1 \neq 0\},\\ U_2 &= \{(x_1, x_2) \times [\xi_1 : \xi_2] \in \mathbb{R}^2 \times P^1(\mathbb{R}) \mid x_1\xi_2 = x_2\xi_1, \xi_2 \neq 0\}\end{aligned}$$

and

$$\begin{aligned}U_{p,1} &= \{\pi(x_1, x_2, x_3) \mid x_1 \neq 0\},\\ U_{p,2} &= \{\pi(x_1, x_2, x_3) \mid x_2 \neq 0\}.\end{aligned}$$

Furthermore, we put

$$\varphi_1 : U_1 \to \mathbb{R}^2,\ (x_1, x_2) \times [\xi_1 : \xi_2] \mapsto (u_1, u_2) = \left(x_1, \frac{\xi_2}{\xi_1}\right),$$
$$\varphi_2 : U_2 \to \mathbb{R}^2,\ (x_1, x_2) \times [\xi_1 : \xi_2] \mapsto (u_1', u_2') = \left(\frac{\xi_1}{\xi_2}, x_2\right)$$

and

$$\varphi_{P,1}(\pi(x_1, x_2, x_3)) = \left(-\tan(\lambda)x_1, \frac{x_2}{x_1}\right),$$
$$\varphi_{P,2}(\pi(x_1, x_2, x_3)) = \left(\frac{x_1}{x_2}, -\tan(\lambda)x_2\right),$$

where $\lambda = \sin^{-1}(x_3)$ $(-\frac{\pi}{2} < \lambda < \frac{\pi}{2})$. Then, the following equality holds

$$\varphi_{P,2} \circ \varphi_{P,1}^{-1} \equiv \varphi_2 \circ \varphi_1^{-1},$$

where $\{(U_1, \varphi_1),\ (U_2, \varphi_2)\}$ is the standard atlas for the blowing up $P : \mathbf{B} \to \mathbb{R}^2$. Thus, the set

$$\{(U_{P,1}, \varphi_{P,1}),\ (U_{P,2}, \varphi_{P,2})\}$$

can be an atlas for $\pi(S^2 - \{P\})$.

Next, we express our map $\tilde{\psi}_P$ by using Euclidean coordinates (u_1, u_2). Since we have assumed $P = (0, 0, 1)$, for $\boldsymbol{x} = (x_1, x_2, \sin(\lambda))$ we have

$$\frac{1}{\sqrt{1 - (P \cdot \boldsymbol{x})^2}}(P - (P \cdot \boldsymbol{x})\boldsymbol{x}) = (-\tan(\lambda)x_1, -\tan(\lambda)x_2, \cos(\lambda))$$

and therefore

$$q \circ \tilde{\psi}_P \circ \varphi_{P,1}^{-1}(u_1, u_2) = (u_1, u_1 u_2),$$
$$q \circ \tilde{\psi}_P \circ \varphi_{P,2}^{-1}(u_1', u_2') = (u_1' u_2', u_2'),$$

where $q : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is the canonical projection.

Since this expression is completely the same as that of the blow up by using the standard coordinate system (U_1, φ_1), (U_2, φ_2) and the restriction $q|_{X_P} : X_P \to q(X_p)$ is a C^∞ diffeomorphism, we see that Lemma 3.2 is proved for $\tilde{\psi}_{P|U_{P,1}}$, $\tilde{\psi}_{P|U_{P,2}}$ and $P|U_1$, $P|U_2$. Thus, in order to finish the proof of Lemma 3.2 it suffices to show that the equality

$$\varphi_1^{-1} \circ \varphi_{P,1}(\pi(x_1, x_2, x_3)) = \varphi_2^{-1} \circ \varphi_{P,2}(\pi(x_1, x_2, x_3))$$

holds for any $\pi(x_1, x_2, x_3) \in U_{P,1} \cap U_{P,2}$. This holds since we have already checked that the patching relations for our $\{(U_{P,i}, \varphi_{P,i})\}_{1 \le i \le 2}$ are completely the same as for the standard atlas of B. □

Then we give our main result.

Theorem 3.3. *Let $(\boldsymbol{\gamma}, \nu) : I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with Legendre curvature (m, n), and P be a point of $S^2 - \{\pm\nu(t_0)\}$. Then the followings hold.*

(1) *If $P \in S^2 - \{\pm\nu(t_0)\} - \{\pm\boldsymbol{\gamma}(t_0)\}$, then the map-germ $\mathcal{P}e_{\boldsymbol{\gamma},P} : (I, t_0) \to (S^2, \mathcal{P}e_{\boldsymbol{\gamma},P}(t_0))$ is smooth, that is to say, it is C^∞ right-left equivalent to the map-germ $(\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ given by $\sigma \mapsto (\sigma, 0)$.*

(2) *If $P \in \{\pm\boldsymbol{\gamma}(t_0)\}$, then the map-germ $\mathcal{P}e_{\boldsymbol{\gamma},P} : (I, t_0) \to (S^2, \mathcal{P}e_{\boldsymbol{\gamma},P}(t_0))$ is C^∞ right-left equivalent to the map-germ $(\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ given by $\sigma \mapsto (\sigma^2, \sigma^3)$.*

Proof. Since $\left\{S^2 - \{\pm\nu(t_0)\} - \{\pm\boldsymbol{\gamma}(t_0)\},\ \{\pm\boldsymbol{\gamma}(t_0)\}\right\}$ gives a stratification of $S^2 - \{\pm\nu(t_0)\}$, "if parts" of Theorem 3.3 follows from "only if parts" of 1, 2 of Theorem 3.3. Thus, we show only "only if parts" in the following.

[Proof of "only if part" of 1] By Lemma 3.2, $\mathcal{P}'e_{\boldsymbol{\gamma},P}(t_0) \neq 0$ in this case. Thus, the map-germ $\mathcal{P}e_{\boldsymbol{\gamma},P}(t_0)$ is non-singular.

[Proof of "only if part" of 2] By a suitable rotation of S^2 if necessary we may assume that $P = (0,0,1) \in \mathbb{R}^3$. Then, since $P \in \{\pm\boldsymbol{\gamma}(t_0)\}$, therefore, by Lemma 3.2 let ε_1 be the set of all C^∞ function-germs with one variable $(\mathbb{R},0) \to \mathbb{R}$, m_1 be its subset consisting of all function-germs with zero constant terms. Then, $m_1^2\varepsilon_1$ is a finitely generated ε_1-module. We put $f(t) = t^2$ and apply the Malgrange preparation theorem (for instance, see [7]) to $m_1^2\varepsilon_1$ and f. Then we see that for any function-germ $g \in m_1^2\varepsilon_1$ there exists a certain C^∞ function-germ ψ such that

$$g(t) = \psi(t^2, t^3).$$

Thus, for our map-germ $\mathcal{P}e_{\boldsymbol{\gamma},P}(t) : (I, t_0) \to (S^2, \mathcal{P}e_{\boldsymbol{\gamma},P}(t_0))$ there exists a germ of C^∞ diffeomorphism $h_t : (S^2, \mathcal{P}e_{\boldsymbol{\gamma},P}(t_0)) \to (\mathbb{R}^2, 0)$ such that

$$h_t \circ \mathcal{P}e_{\boldsymbol{\gamma},P}(t) = \left((t-t_0)^2, (t-t_0)^3\right).$$

□

4. Examples

In this section, we give an example of the pedal curve of front.

Example 4.1. Let $\boldsymbol{\gamma} : I \to S^2$ be a curve,

$$\boldsymbol{\gamma}(t) = \left(\frac{3}{4}\cos(t) - \frac{1}{4}\cos(3t), \frac{3}{4}\sin(t) - \frac{1}{4}\sin(3t), \frac{\sqrt{3}}{2}\cos(t)\right).$$

We get

$$\dot{\boldsymbol{\gamma}}(t) = \left(-\frac{3}{4}\sin(t) + \frac{3}{4}\sin(3t), \frac{3}{4}\cos(t) - \frac{3}{4}\cos(3t), -\frac{\sqrt{3}}{2}\sin(t)\right).$$

So $\boldsymbol{\gamma}$ is singular at $t = 0$. Take

$$\nu(t) = \left(\frac{3}{4}\sin(t) + \frac{1}{4}\sin(3t), -\frac{3}{4}\cos(t) - \frac{1}{4}\cos(3t), -\frac{\sqrt{3}}{2}\sin(t)\right).$$

We have $\langle\boldsymbol{\gamma}(t), \nu(t)\rangle = 0$ and $\langle\dot{\boldsymbol{\gamma}}(t), \nu(t)\rangle = 0$. Hence $(\boldsymbol{\gamma}, \nu)$ is a Legendre curve.

If $P = \left(\frac{3\sqrt{3}}{8}, \frac{1}{8}, \frac{3}{4}\right)$, then $P = \boldsymbol{\gamma}\left(\frac{\pi}{6}\right) \in \boldsymbol{\gamma}(t)$. Thus,

$$\mathcal{P}e_{\boldsymbol{\gamma},P} = \left(\frac{X}{K}, \frac{Y}{K}, \frac{Z}{K}\right),$$

where

$$X = \frac{3\sqrt{3}}{8} + \left(\frac{3}{32}\sqrt{3}\sin(t) - \frac{3}{32}\sqrt{3}\sin(3t) + \frac{3}{32}\cos(t) + \frac{1}{32}\cos(3t)\right)\left(\frac{3}{4}\sin(t) + \frac{1}{4}\sin(3t)\right),$$
$$Y = \frac{1}{8} + \left(-\frac{3}{4}\cos(t) - \frac{1}{4}\cos(3t)\right),$$

$$Z = \frac{3}{4} - \sqrt{3}\sin(t)\left(\frac{3}{64}\sqrt{3}\sin(t) - \frac{3}{64}\sqrt{3}\sin(3t) + \frac{3}{64}\cos(t) + \frac{1}{64}\cos(3t)\right),$$
$$K = \sqrt{1 - \left(-\frac{3}{32}\sqrt{3}\sin(t) + \frac{3}{32}\sqrt{3}\sin(3t) - \frac{3}{32}\cos(t) - \frac{1}{32}\cos(3t)\right)^2}.$$

According to the Theorem 3.3, we have the map-germ $\mathcal{P}e_{\gamma,P} : (I, \frac{\pi}{6}) \to (S^2, \mathcal{P}e_{\gamma,P}(\frac{\pi}{6}))$ is C^∞ right-left equivalent to the map-germ $(\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ given by $\sigma \mapsto (\sigma^2, \sigma^3)$. See Figure 1.

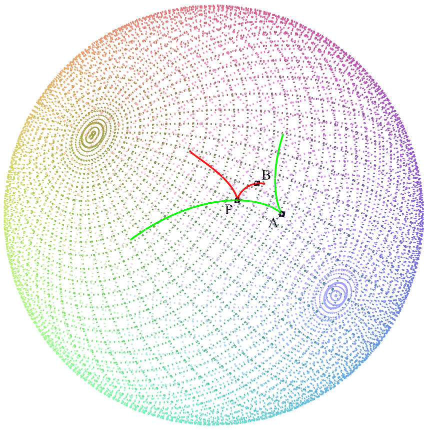

Figure 1: The green curve is the front γ, the red curve is its pedal curve. The point P is the ordinary cusp of the pedal curve. The point A is the ordinary cusp of the front γ. B is the point of the pedal curve corresponding to the point A.

If $P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$, then $P \notin \gamma(t)$. Thus

$$\mathcal{P}e_{\gamma,P} = \left(\frac{X_1}{K_1}, \frac{Y_1}{K_1}, \frac{Z_1}{K_1}\right),$$

where

$$X_1 = \frac{\sqrt{2}}{2} + \left(-\frac{3\sqrt{2}}{8}\sin(t) - \frac{\sqrt{2}}{8}\sin(3t) + \frac{3\sqrt{2}}{8}\cos(t) + \frac{\sqrt{2}}{8}\cos(3t)\right)\left(\frac{3}{4}\sin(t) + \frac{1}{4}\sin(3t)\right),$$
$$Y_1 = \frac{\sqrt{2}}{2} + \left(-\frac{3\sqrt{2}}{8}\sin(t) - \frac{\sqrt{2}}{8}\sin(3t) + \frac{3\sqrt{2}}{8}\cos(t) + \frac{\sqrt{2}}{8}\cos(3t)\right)\left(-\frac{3}{4}\cos(t) - \frac{1}{4}\cos(3t)\right),$$
$$Z_1 = -\frac{\sqrt{3}\sin(t)}{2}\left(-\frac{3\sqrt{2}}{8}\sin(t) - \frac{\sqrt{2}}{8}\sin(3t) + \frac{3\sqrt{2}}{8}\cos(t) + \frac{\sqrt{2}}{8}\cos(3t)\right),$$
$$K_1 = \sqrt{1 - \left(\frac{3\sqrt{2}}{8}\sin(t) + \frac{\sqrt{2}}{8}\sin(3t) - \frac{3\sqrt{2}}{8}\cos(t) - \frac{\sqrt{2}}{8}\cos(3t)\right)^2}.$$

According to the Theorem 3.3 and denote $t_0 = \mathcal{P}^{-1}e_{\gamma,P}(P)$, we have the map-germ $\mathcal{P}e_{\gamma,P} : (I, t_0) \to (S^2, \mathcal{P}e_{\gamma,P}(t_0))$ is C^∞ right-left equivalent to the map-germ $(\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ given by $\sigma \mapsto (\sigma, 0)$. See Figure 2.

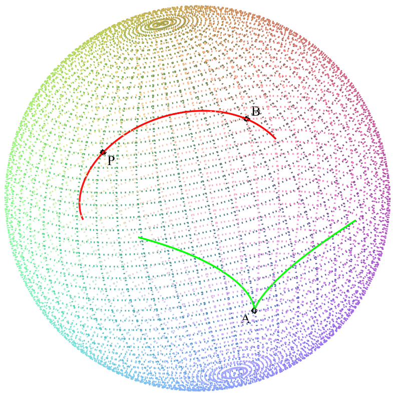

Figure 2: The green curve is the front $\boldsymbol{\gamma}$, the red curve is its pedal curve. P is the Point of the pedal curve. The point A is the ordinary cusp of the front $\boldsymbol{\gamma}$. B is the point of the pedal curve corresponding to the point A.

If $P = (\frac{\sqrt{1}}{2}, 0, \frac{\sqrt{3}}{2})$, then $P = \boldsymbol{\gamma}(0) \in \boldsymbol{\gamma}(t)$, and P is a singular point of $\boldsymbol{\gamma}$. Thus

$$\mathcal{P}e_{\boldsymbol{\gamma},P} = (X_2, Y_2, Z_2)\,,$$

where

$$X_2 = \frac{\frac{1}{2} + \left(\frac{3}{8}\sin(t) - \frac{1}{8}\sin(3t)\right)\left(\frac{3}{4}\sin(t) + \frac{1}{4}\sin(3t)\right)}{\sqrt{1 - \left(\frac{3}{8}\sin(t) + \frac{1}{8}\sin(3t)\right)^2}},$$

$$Y_2 = \frac{\left(\frac{3}{8}\sin(t) - \frac{1}{8}\sin(3t)\right)\left(-\frac{3}{4}\cos(t) - \frac{1}{4}\cos(3t)\right)}{\sqrt{1 - \left(\frac{3}{8}\sin(t) + \frac{1}{8}\sin(3t)\right)^2}},$$

$$Z_2 = \frac{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\sin(t)\left(\frac{3}{8}\sin(t) - \frac{1}{8}\sin(3t)\right)}{\sqrt{1 - \left(\frac{3}{8}\sin(t) + \frac{1}{8}\sin(3t)\right)^2}}.$$

According to the Theorem 3.3, we have the map-germ $\mathcal{P}e_{\boldsymbol{\gamma},P} : (I, 0) \to (S^2, \mathcal{P}e_{\boldsymbol{\gamma},P}(0))$ is C^∞ right-left equivalent to the map-germ $(\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ given by $\sigma \mapsto (\sigma^2, \sigma^3)$. See Figure 3.

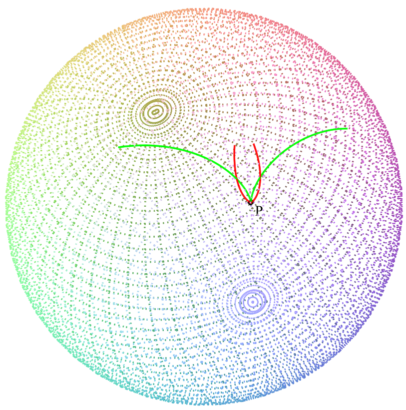

Figure 3: The green curve is the front $\boldsymbol{\gamma}$, the red curve is its pedal curve. The point P is the ordinary cusp of the front $\boldsymbol{\gamma}$ and its pedal curve.

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