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Bernoulli polynomials of the second kind and their identities arising from umbral calculus

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Abstract

In this paper, we study the Bernoulli polynomials of the second kind with umbral calculus viewpoint and derive various identities involving those polynomials by using umbral calculus.

Keywords: Bernoulli polynomial of the second kind, umbral calculus.
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1. Introduction and preliminaries

As is well known, the ordinary Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n\left(x\right)\frac{t^n}{n!},\quad \left(\text{see } [2,\ 10,\ 15,\ 17]\right). \tag{1.1}$$

When $x=0$, $B_n=B_n\left(0\right)$ are called the Bernoulli numbers. The Bernoulli polynomials of the second kind are given by the generating function to be

$$\frac{t}{\log\left(1+t\right)}\left(1+t\right)^x=\sum_{n=0}^{\infty}b_n\left(x\right)\frac{t^n}{n!},\quad \left(\text{see } [19,\ 21,\ 22]\right). \tag{1.2}$$

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When $x=0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind.
The first few Bernoulli numbers b_n of the second kind are

$$b_0 = 1,\, b_1 = \frac{1}{2},\, b_2 = -\frac{1}{12},\, b_3 = \frac{1}{24},\, b_4 = -\frac{19}{720},\, b_5 = \frac{3}{160}, \cdots .$$

By (1.2), we easily get

$$b_n(x) = \sum_{l=0}^{n} \binom{n}{l} b_l\,(x)_{n-l}\,, \tag{1.3}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 0)$, and

$$b_n(x) = B_n^{(n)}(x+1)\,, \quad \text{(see [21, 22])}\,, \tag{1.4}$$

where $B_n^{(\alpha)}(x)$ are the Bernoulli polynomials of order α.

The stirling number of the second kind is given by

$$x^n = \sum_{l=0}^{n} S_2(n,l)\,(x)_l\,, \quad (n \geq 0)\,. \tag{1.5}$$

The Stirling number of the first kind is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)\,x^l, \quad (n \geq 0)\,. \tag{1.6}$$

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{1.7}$$

Let us assume that $\mathbb{P}$ is the algebra of polynomials in the variable x over $\mathbb{C}$ and $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$. $\langle L|\, p(x)\rangle$ denotes the action of the linear functional L on a polynomial $p(x)$. For $f(t) \in \mathcal{F}$, we define the continuous linear functional $f(t)$ on $\mathbb{P}$ by

$$\langle f(t)\,|x^n\rangle = a_n, \quad (n \geq 0)\,, \quad \text{(see [21])}\,. \tag{1.8}$$

Thus, by (1.7) and (1.8), we get

$$\left\langle t^k \middle| x^n \right\rangle = n!\delta_{n,k}, \quad (n,k \geq 0)\,, \quad \text{(see [1–22])}\,, \tag{1.9}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)\,|x^n\rangle = \langle L\,|x^n\rangle\,,\,(n \geq 0)$. Thus, we see that $f_L(t) = L$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ is thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra. The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see [8, 21]). If $o(f(t)) = 1$, then $f(t)$ is called a delta series and if $o(f(t)) = 0$, then $f(t)$ is called an invertible series. For $f(t)\,, g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_n(x)$ of polynomials such that $\left\langle g(t)\, f(t)^k \middle| s_n(x) \right\rangle = n!\delta_{n,k}$, where $n,k \geq 0$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t)\,, f(t))$ which is denoted by $s_n(x) \sim (g(t)\,, f(t))$. For $p(x) \in \mathbb{P}$, we have $\left\langle e^{yt} \middle| p(x)\right\rangle = p(y)$ and $e^{yt}p(x) = p(x+y)$. Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we see that

$$f(t) = \sum_{k=0}^{\infty} \frac{\left\langle f(t)\,|x^k\right\rangle}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\left\langle t^k \middle| p(x)\right\rangle}{k!} x^k. \tag{1.10}$$

Thus, by (1.10), we get

$$p^{(k)}(0) = \left\langle t^k \middle| p(x) \right\rangle, \quad \left\langle 1 \middle| p^{(k)}(x) \right\rangle = p^{(k)}(0). \tag{1.11}$$

From (1.11), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (k \geq 0).$$

For $s_n(x) \sim (g(t), f(t))$, we have

$$\frac{ds_n(x)}{dx} = \sum_{l=0}^{n-1} \binom{n}{l} \left\langle \overline{f}(t) \middle| x^{n-l} \right\rangle s_l(x), \quad (n \geq 1), \tag{1.12}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$,

$$\frac{1}{g(\overline{f}(t))} = e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad \text{for all } x \in \mathbb{C}, \tag{1.13}$$

$$f(t) s_n(x) = n s_{n-1}(x), \quad (n \geq 1), \quad s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \tag{1.14}$$

where $p_n(x) = g(t) s_n(x)$,

$$\langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle, \tag{1.15}$$

and

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x), \quad (n \geq 0), \quad (\text{see [1, 13, 16, 21]}). \tag{1.16}$$

Assume that $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then the transfer formula is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \quad (n \geq 1). \tag{1.17}$$

For $s_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$, we have

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x), \quad (n \geq 0), \tag{1.18}$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))} \left(l(\overline{f}(t)) \right)^m \middle| x^m \right\rangle, \quad (\text{see [12, 21]}). \tag{1.19}$$

In this paper, we study the Bernoulli polynomials of the second kind with umbral calculus viewpoint and derive various identities involving those polynomials by using umbral calculus.

2. Bernoulli polynomials of the second kind

For $\alpha \in \mathbb{N}$, the Bernoulli polynomials of the second kind with order α are defined by

$$\left(\frac{t}{\log(1+t)} \right)^\alpha (1+t)^x = \sum_{n=0}^{\infty} b_n^{(\alpha)}(x) \frac{t^n}{n!}. \tag{2.1}$$

Note that $b_n(x) = b_n^{(1)}(x)$. When $x = 0$, $b_n^{(\alpha)} = b_n^{(\alpha)}(0)$ are called the Bernoulli numbers of the second kind with order α. Indeed, we note that

$$b_n^{(\alpha)}(x) = B_n^{(n-\alpha+1)}(x+1).$$

Let us consider the following two sheffer sequences :

$$q_n(x) \sim \left(1, \left(\frac{\log(1+t)}{t}\right)^{\alpha} \left(e^t - 1\right)\right)$$

and

$$(x)_n \sim \left(1, e^t - 1\right).$$

Thus, by (1.17), we get

$$\begin{aligned} q_n(x) &= x \left(\frac{t}{\log(1+t)}\right)^{\alpha n} x^{-1} (x)_n \\ &= x \left(\frac{t}{\log(1+t)}\right)^{\alpha n} (x-1)_{n-1} \\ &= x b_{n-1}^{(\alpha n)} (x-1), \quad (n \geq 1). \end{aligned}$$

That is,

$$x b_{n-1}^{(\alpha n)} (x-1) \sim \left(1, \left(\frac{\log(1+t)}{t}\right)^{\alpha} \left(e^t - 1\right)\right).$$

From (1.2) and (1.13), we have

$$b_n(x) \sim \left(\frac{t}{e^t - 1}, e^t - 1\right). \tag{2.2}$$

By (2.2), we get

$$\frac{t}{e^t - 1} b_n(x) \sim \left(1, e^t - 1\right), \quad (x)_n \sim \left(1, e^t - 1\right). \tag{2.3}$$

Thus, we see that

$$\begin{aligned} b_n(x) &= \frac{e^t - 1}{t} (x)_n = \frac{e^t - 1}{t} \sum_{l=0}^{n} S_1(n,l) x^l \\ &= \left(e^t - 1\right) \sum_{l=0}^{n} \frac{S_1(n,l)}{l+1} x^{l+1} \\ &= \sum_{l=0}^{n} \frac{S_1(n,l)}{l+1} \left((x+1)^{l+1} - x^{l+1}\right). \end{aligned} \tag{2.4}$$

When $x = 0$, we have

$$b_n = \sum_{l=0}^{n} \frac{S_1(n,l)}{l+1}.$$

By (1.12), we get

$$\begin{aligned} \frac{d}{dx} b_n(x) &= \sum_{l=0}^{n-1} \binom{n}{l} \left\langle \log(1+t) \middle| x^{n-l} \right\rangle b_l(x) \\ &= \sum_{l=0}^{n-1} \binom{n}{l} \left\langle \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} t^m \middle| x^{n-l} \right\rangle b_l(x) \\ &= \sum_{l=0}^{n-1} \binom{n}{l} (n-l-1)! (-1)^{n-l-1} b_l(x) \\ &= \sum_{l=0}^{n-1} \frac{n!}{l!(n-l)} (-1)^{n-l-1} b_l(x), \quad (n \geq 1). \end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the following lemma.

Lemma 1. *For $n \geq 1$, we have*

$$\frac{d}{dx} b_n(x) = \sum_{l=0}^{n-1} \frac{n!}{l!(n-l)} (-1)^{n-l-1} b_l(x).$$

From (1.9), we have

$$\begin{aligned} b_n(y) &= \left\langle \left(\frac{t}{\log(1+t)} \right) (1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \left(\frac{t}{\log(1+t)} \right) \middle| \sum_{m=0}^{\infty} (y)_m \frac{t^m}{m!} x^n \right\rangle \\ &= \sum_{m=0}^{n} (y)_m \binom{n}{m} \left\langle \left(\frac{t}{\log(1+t)} \right) \middle| x^{n-m} \right\rangle \\ &= \sum_{m=0}^{n} (y)_m \binom{n}{m} b_{n-m}. \end{aligned} \tag{2.6}$$

Therefore, by (2.6), we obtain the following proposition.

Proposition 2. *For $n \geq 0$, we have*

$$\begin{aligned} b_n(x) &= \sum_{m=0}^{n} \binom{n}{m} b_{n-m} (x)_m \\ &= \sum_{m=0}^{n} m! \binom{n}{m} \binom{x}{m} b_{n-m}. \end{aligned}$$

By (1.2), we get

$$\begin{aligned} b_n(x) = \frac{t}{\log(1+t)} (x)_n &= \sum_{l=0}^{n} S_1(n,l) \left(\frac{t}{\log(1+t)} \right) x^l \\ &= \sum_{l=0}^{n} S_1(n,l) \sum_{m=0}^{l} \frac{b_m}{m!} t^m x^l \\ &= \sum_{l=0}^{n} S_1(n,l) \sum_{m=0}^{l} \binom{l}{m} b_m x^{l-m} \\ &= \sum_{l=0}^{n} \sum_{m=0}^{l} S_1(n,l) \binom{l}{m} b_{l-m} x^m. \end{aligned} \tag{2.7}$$

By (1.14), we get

$$b_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} b_j(x) (y)_{n-j}. \tag{2.8}$$

Let

$$\mathbb{P}_n = \{ p(x) \in \mathbb{C}[x] | \deg p(x) \leq n \}, \quad (n \geq 0).$$

Then it is an $(n+1)$-dimensional vector space over $\mathbb{C}$. Now, we consider the polynomial $p(x)$ in $\mathbb{P}_n$ which is given by

$$p(x) = \sum_{m=0}^{n} C_m b_m(x). \tag{2.9}$$

Thus, by (2.9), we get

$$\begin{aligned}\left\langle \frac{t}{e^t-1}\left(e^t-1\right)^m \middle| p(x)\right\rangle &= \sum_{l=0}^{n} C_l \left\langle \frac{t}{e^t-1}\left(e^t-1\right)^m \middle| b_l(x)\right\rangle \\ &= \sum_{l=0}^{n} C_l l! \delta_{m,l} = m! C_m.\end{aligned} \tag{2.10}$$

From (2.10), we have

$$C_m = \frac{1}{m!}\left\langle \left(\frac{t}{e^t-1}\right)\left(e^t-1\right)^m \middle| p(x)\right\rangle. \tag{2.11}$$

Therefore, by (2.11), we obtain the following theorem.

Theorem 3. *Let $p(x) \in \mathbb{P}_n$ with*

$$p(x) = \sum_{m=0}^{n} C_m b_m(x).$$

Then, we have

$$C_m = \frac{1}{m!}\left\langle \left(\frac{t}{e^t-1}\right)\left(e^t-1\right)^m \middle| p(x)\right\rangle.$$

For example, let us take $p(x) = B_n(x) \in \mathbb{P}_n$. Then, we have

$$B_n(x) = \sum_{m=0}^{n} C_m b_m(x), \tag{2.12}$$

where

$$\begin{aligned}C_m &= \frac{1}{m!}\left\langle \left(\frac{t}{e^t-1}\right)\left(e^t-1\right)^m \middle| B_n(x)\right\rangle \\ &= \sum_{l=m}^{n} S_2(l,m)\binom{n}{l}\left\langle \left(\frac{t}{e^t-1}\right) \middle| B_{n-l}(x)\right\rangle \\ &= \sum_{l=m}^{n} S_2(l,m)\binom{n}{l}\sum_{k=0}^{n-l} B_{n-l-k}\binom{n-l}{k}\left\langle \left(\frac{t}{e^t-1}\right) \middle| x^k\right\rangle \\ &= \sum_{l=m}^{n}\sum_{k=0}^{n-l} S_2(l,m)\binom{n}{l}\binom{n-l}{k} B_{n-l-k}B_k.\end{aligned} \tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$B_n(x) = \sum_{m=0}^{n}\left\{\sum_{l=m}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k} S_2(l,m) B_{n-l-k}B_k\right\} b_m(x).$$

Remark. From (2.13), for $m \geq 1$, we have

$$\begin{aligned}C_m &= \frac{1}{m!}\left\langle \left(\frac{t}{e^t-1}\right)\left(e^t-1\right)^m \middle| B_n(x)\right\rangle \\ &= \frac{1}{m!}\left\langle \left(e^t-1\right)^{m-1} \middle| tB_n(x)\right\rangle \\ &= \frac{n}{m!}\left\langle \left(e^t-1\right)^{m-1} \middle| B_{n-1}(x)\right\rangle\end{aligned} \tag{2.14}$$

$$= \frac{n}{m!}(m-1)! \sum_{l=m-1}^{n-1} S_2(l, m-1) \frac{1}{l!} \left\langle t^l \middle| B_{n-1}(x) \right\rangle$$
$$= \frac{n}{m} \sum_{l=m-1}^{n-1} S_2(l, m-1) \binom{n-1}{l} B_{n-1-l}.$$

Therefore, by (2.12) and (2.14), we get

$$B_n(x) = \sum_{m=1}^{n} \left\{ \frac{n}{m} \sum_{l=m-1}^{n-1} S_2(l, m-1) \binom{n-1}{l} B_{n-1-l} \right\} b_m(x) + \sum_{k=0}^{n} \binom{n}{k} B_{n-k} B_k.$$

The classical polylogarithm function is given by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad (k \in \mathbb{Z},\, x > 0). \tag{2.15}$$

The poly-Bernoulli polynomials are defined by the generating function to be

$$\frac{\mathrm{Li}_k\left(1-e^t\right)}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}. \tag{2.16}$$

Thus, by (2.16), we see that

$$B_n^{(k)}(x) \sim \left(\frac{e^t - 1}{\mathrm{Li}_k\left(1 - e^{-t}\right)}, t \right). \tag{2.17}$$

Let us take $p(x) = B_n^{(k)}(x) \in \mathbb{P}_n$. Then we have

$$B_n^{(k)}(x) = \sum_{m=0}^{n} C_m b_m(x), \tag{2.18}$$

where

$$\begin{aligned} C_m &= \frac{1}{m!} \left\langle \frac{t}{e^t - 1} \left(e^t - 1\right)^m \middle| B_n^{(k)}(x) \right\rangle \\ &= \sum_{l=m}^{n} S_2(l, m) \binom{n}{l} \left\langle \frac{t}{e^t - 1} \middle| B_{n-l}^{(k)}(x) \right\rangle \\ &= \sum_{l=m}^{n} S_2(l, m) \binom{n}{l} \sum_{j=0}^{n-l} \binom{n-l}{j} B_{n-l-j}^{(k)} \left\langle \frac{t}{e^t - 1} \middle| x^j \right\rangle \\ &= \sum_{l=m}^{n} \sum_{j=0}^{n-l} \binom{n}{l} \binom{n-l}{j} S_2(l, m) B_{n-l-j}^{(k)} B_j, \end{aligned} \tag{2.19}$$

where $B_n^{(k)} = B_n^{(k)}(0)$ are the poly-Bernoulli numbers. Therefore, by (2.18) and (2.19), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$B_n^{(k)}(x) = \sum_{m=0}^{n} \left\{ \sum_{l=m}^{n} \sum_{j=0}^{n-l} \binom{n}{l} \binom{n-l}{j} S_2(l, m) B_{n-j-l}^{(k)} B_j \right\} b_m(x).$$

Let us consider $p(x) = x^n \in \mathbb{P}_n$. Then, we have

$$x^n = \sum_{m=0}^{n} C_m b_m(x), \tag{2.20}$$

where

$$\begin{aligned} C_m &= \frac{1}{m!}\left\langle \left(\frac{t}{e^t-1}\right)\left(e^t-1\right)^m \middle| x^n \right\rangle \\ &= \sum_{l=m}^{n} S_2(l,m)\binom{n}{l}\left\langle \frac{t}{e^t-1} \middle| x^{n-l} \right\rangle \\ &= \sum_{l=m}^{n} S_2(l,m)\binom{n}{l} B_{n-l}. \end{aligned} \tag{2.21}$$

Thus, by (2.20) and (2.21), we get

$$x^n = \sum_{m=0}^{n}\left\{\sum_{l=m}^{n} S_2(l,m)\binom{n}{l} B_{n-l}\right\} b_m(x). \tag{2.22}$$

Let us consider the following two Sheffer sequences :

$$b_n(x) \sim \left(\frac{t}{e^t-1}, e^t-1\right), \tag{2.23}$$

and

$$B_n^{(k)}(x) \sim \left(\frac{e^t-1}{\mathrm{Li}_k(1-e^{-t})}, t\right).$$

Then, by (1.17) and (1.18), we get

$$B_n^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} b_m(x), \tag{2.24}$$

where

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\left(\frac{t}{e^t-1}\right)\left(e^t-1\right)^m \middle| x^n \right\rangle \\ &= \sum_{l=m}^{n} S_2(l,m)\binom{n}{l}\left\langle \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} \middle| \frac{t}{e^t-1} x^{n-l} \right\rangle \\ &= \sum_{l=m}^{n} S_2(l,m)\binom{n}{l}\sum_{j=0}^{n-l}\binom{n-l}{j} B_{n-l-j}\left\langle \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} \middle| x^j \right\rangle \\ &= \sum_{l=m}^{n}\sum_{j=0}^{n-l}\binom{n}{l}\binom{n-l}{j} S_2(l,m) B_{n-l-j} B_j^{(k)}. \end{aligned} \tag{2.25}$$

Therefore, by (2.24) and (2.25), we obtain the following theorem.

Theorem 6. *For $n \geq 0$, we have*

$$B_n^{(k)}(x) = \sum_{m=0}^{n}\left\{\sum_{l=m}^{n}\sum_{j=0}^{n-l}\binom{n}{l}\binom{n-l}{j} S_2(l,m) B_{n-l-j} B_j^{(k)}\right\} b_m(x).$$

Let us consider the following Sheffer sequences:

$$b_n(x) \sim \left(\frac{t}{e^t-1}, e^t-1\right), \tag{2.26}$$

$$B_n(x) \sim \left(\frac{e^t-1}{t}, t\right).$$

Then we have

$$b_n(x) = \sum_{m=0}^{n} C_{n,m} B_m(x), \tag{2.27}$$

where

$$\begin{aligned} C_{n,m} =& \frac{1}{m!}\left\langle \frac{t}{\log(1+t)}\frac{t}{\log(1+t)}(\log(1+t))^m \middle| x^n \right\rangle \\ =& \sum_{l=m}^{n}\binom{n}{l} S_1(l,m)\left\langle \left(\frac{t}{\log(1+t)}\right)^2 \middle| x^{n-l}\right\rangle \\ =& \sum_{l=m}^{n}\binom{n}{l} S_1(l,m)\, b_{n-l}^{(2)}, \end{aligned} \tag{2.28}$$

where $b_n^{(2)}$ are di-Bernoulli numbers of the second kind.

Therefore, by (2.27) and (2.28), we get

$$b_n(x) = \sum_{m=0}^{n}\left(\sum_{l=m}^{n}\binom{n}{l} S_1(l,m)\, b_{n-l}^{(2)}\right) B_m(x). \tag{2.29}$$

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