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Hermite–Hadamard type integral inequalities via (s,m)–P-convexity on co-ordinates

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Abstract

In this paper, the notion of (s,m)–P-convex functions on the co-ordinates is introduced and several integral inequalities of the Hermite-Hadamard type for co-ordinated (s,m)–P-convex functions are established.

Keywords: Co-ordinates, (s,m)–P-convex function, Hermite–Hadamard's integral inequality, integral identity.
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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval of I of real numbers and $a, b \in I$ with $a < b$. The double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hermite-Hadamard's integral inequality.

Definition 1.1 ([9])**.** We say that a map $f : I \subseteq \mathbb{R} \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \le f(x) + f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

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Theorem 1.2 ([5, Theorem 3.1])**.** *Let $f \in P(I)$, $a, b \in I$ with $a < b$, and $f \in L([a, b])$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le 2(f(a)+f(b)). \tag{1.1}$$

Both inequalities are the best possible.

Definition 1.3 ([10])**.** For $f : [0, b] \to \mathbb{R}$ and $m \in (0, 1]$, if

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m-convex function on $[0, b]$.

Theorem 1.4 ([2])**.** *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an m-convex with $m \in (0, 1]$ and $0 \le a < b$. If $f \in L_1([a, b])$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \frac{f(x)+mf(\frac{x}{m})}{2}\,\mathrm{d}\,x \le \frac{m+1}{4}\left[\frac{f(a)+f(b)}{2} + m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}\right].$$

Definition 1.5 ([7])**.** For $s \in (0, 1]$, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be s-convex (in the second sense) if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Theorem 1.6 ([3])**.** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an s-convex function in the second sense for $s \in (0, 1)$. If $f \in L_1([a, b])$ for $a, b \in \mathbb{R}$ with $a < b$, then*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{s+1}.$$

Definition 1.7 ([8])**.** For $(s, m) \in (0, 1]^2$, a function $f : [0, b] \to \mathbb{R}$ is said to be (s, m)-convex if

$$f(\lambda x + m(1-\lambda)y) \le \lambda^s f(x) + m(1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.8 ([5])**.** A map $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class $Q(I)$ if it is nonnegative and satisfies

$$f(\lambda x + (1-\lambda)y) \le \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}$$

for all $x, y \in I$ and $\lambda \in (0, 1)$.

Theorem 1.9 ([5, Theorem 2.1])**.** *Let $f \in Q(I)$, $a, b \in I$ with $a < b$, and $f \in L_1([a, b])$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{4}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \quad \textit{and} \quad \frac{1}{b-a}\int_a^b p(x)f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2},$$

where $p(x) = \frac{(b-x)(x-a)}{(b-a)^2}$ for $x \in I$.

Definition 1.10 ([12])**.** For some $s \in [-1, 1]$, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

is valid for all $x, y \in I$ and $\lambda \in (0, 1)$.

Definition 1.11 ([1, 4]). A function $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ with $a < b$ and $c < d$ if the partial functions

$$f_y : [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y) \quad \text{and} \quad f_x : [c,d] \to \mathbb{R}, \quad f_x(v) = f(x,v)$$

are convex for all $x \in [a,b]$ and $y \in [c,d]$.

Definition 1.12 ([6, 11]). A function $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ with $a < b$ and $c < d$ if for the partial functions

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) + (1-t)(1-\lambda)f(z,w)$$

holds for all $t, \lambda \in [0,1], (x,y), (z,w) \in \Delta$.

Theorem 1.13 ([1, 4, Theorem 2.2]). *Let $f : \Delta = [a,b] \times [c,d]$ be convex on the co-ordinates on $\Delta = [a,b] \times [c,d]$ with $a < b$ and $c < d$. Then*

$$\begin{aligned}
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y\right] \\
&\le \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y) \,\mathrm{d}\, y \,\mathrm{d}\, x \\
&\le \frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x,c) \,\mathrm{d}\, x + \int_a^b f(x,d) \,\mathrm{d}\, x\right) + \frac{1}{d-c}\left(\int_c^d f(a,y) \,\mathrm{d}\, y + \int_c^d f(b,y) \,\mathrm{d}\, y\right)\right] \\
&\le \frac{f(a,c) + f(b,c) + f(a,d) + f(b,d)}{4}.
\end{aligned}$$

In this paper, we will introduce the notion of (s,m)-P-convex functions on co-ordinates and establish several integral inequalities of the Hermite-Hadamard type for co-ordinated (s,m)-P-convex functions.

2. A notion and an example

Motivated by Definitions 1.1, 1.3, 1.5, 1.7, 1.8, 1.10, 1.11, we now introduce the notion of (s,m)-P-convex functions on co-ordinates as follows.

Definition 2.1. For some $m \in (0,1]$ and $s \in [-1,1]$, a function $f : \Delta = [0,b] \times [c,d] \subseteq \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}_0$ is said to be co-ordinated (s,m)-P-convex on $[0,b] \times [c,d]$ with $0 < b$ and $c < d$, if

$$f(tx + m(1-t)z, \lambda y + (1-\lambda)w) \le t^s[f(x,y) + f(x,w)] + m(1-t)^s[f(z,y) + f(z,w)]$$

holds for all $t \in (0,1)$, $\lambda \in [0,1]$, and $(x,y), (z,w) \in [0,b] \times [c,d]$.

Remark 2.2. Let $f : \Delta = [0,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}_0$ with $0 < b$ and $c < d$ and let $s \in [-1,1]$ and $m \in (0,1]$.

1. If $f : \Delta \to \mathbb{R}_0$ be convex on the co-ordinates on Δ, then f is a co-ordinated $(s,1)$-P-convex function on Δ.
2. In Definition 2.1, if $f(x,y) = f(y)$ for all $(x,y) \in \Delta$ and $s = m = 1$, then f is a P-convex function on $[c,d]$, or say, $f \in P([c,d])$.

Example 2.3. Let $f(x,y) = x^s \ln x$ for $(x,y) \in (0,\infty) \times [1,\infty)$, $s \in [-1,0)$, and some $m \in (0,1]$. Then f is a co-ordinated (s,m)-P-convex function on $\mathbb{R}^2_+$.

In Definition 1.12, letting $x_0 = y_0 = 1$, $z_0 = 1.4$, $w_0 = 1.5$, and $t_0 = \lambda_0 = \frac{1}{2}$ yields

$$4^{-1}[f(x_0,y_0) + f(x_0,w_0) + f(z_0,y_0) + f(z_0,w_0)] - f\big(2^{-1}(x_0+z_0), 2^{-1}(y_0+w_0)\big) < 0.$$

This implies that $f(x) = x^s \ln x$ is not convex on the co-ordinates on $(0,\infty) \times [1,\infty)$.

3. Integral inequalities of the Hermite-Hadamard type

In this section, we establish integral inequalities of the Hermite-Hadamard type for (s,m)-P-convex functions on co-ordinates on the plane $\mathbb{R}_0 \times \mathbb{R}$.

Theorem 3.1. *Let $f : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}_0$ be a co-ordinated (s,m)-P-convex function on $\left[0, \frac{b}{m^2}\right] \times [c,d]$ with $0 \le a < b$ and $c < d$ for some $m \in (0,1]$ and $s \in (-1,1]$. If $f \in L_1\big(\left[0, \frac{b}{m^2}\right] \times [c,d]\big)$, then*

$$
\begin{aligned}
2^{s-2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \frac{f(x,y) + m f(\frac{x}{m}, y)}{2} \,\mathrm{d}\, x \,\mathrm{d}\, y \\
&\le \frac{f(a,c) + f(a,d) + m\big[f(\frac{a}{m}, c) + f(\frac{a}{m}, d)\big]}{2(s+1)} \\
&\quad + m \frac{f(\frac{b}{m}, c) + f(\frac{b}{m}, d) + m\big[f(\frac{b}{m^2}, c) + f(\frac{b}{m^2}, d)\big]}{2(s+1)}.
\end{aligned}
$$

Proof. Using the co-ordinated (s,m)-P-convexity of f, we obtain

$$
\begin{aligned}
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &= f\left(\frac{ta + (1-t)b + (1-t)a + tb}{2}, \frac{\lambda c + (1-\lambda)d + (1-\lambda)c + \lambda d}{2}\right) \\
&\le \frac{1}{2^s}\bigg\{ f(ta + (1-t)b, \lambda c + (1-\lambda)d) + f(ta + (1-t)b, (1-\lambda)c + \lambda d) \\
&\quad + m\left[f\left(\frac{(1-t)a + tb}{m}, \lambda c + (1-\lambda)d\right) + f\left(\frac{(1-t)a + tb}{m}, (1-\lambda)c + \lambda d\right)\right]\bigg\}
\end{aligned}
$$

for all $(t,\lambda) \in [0,1]^2$. Further integrating with respect to t and λ over $[0,1] \times [0,1]$, we have

$$
\begin{aligned}
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2^s} \int_0^1 \int_0^1 \bigg\{ f(ta + (1-t)b, \lambda c + (1-\lambda)d) + f(ta + (1-t)b, (1-\lambda)c + \lambda d) \\
&\quad + m\left[f\left(\frac{(1-t)a + tb}{m}, \lambda c + (1-\lambda)d\right) + f\left(\frac{(1-t)a + tb}{m}, (1-\lambda)c + \lambda d\right)\right]\bigg\} \,\mathrm{d}\, t \,\mathrm{d}\, \lambda \\
&= \frac{1}{2^{s-1}(b-a)(d-c)} \int_c^d \int_a^b \left[f(x,y) + m f\left(\frac{x}{m}, y\right)\right] \mathrm{d}\, x \,\mathrm{d}\, y.
\end{aligned}
\tag{3.1}
$$

By similar argument, we obtain

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \left[f(x,y) + m f\left(\frac{x}{m}, y\right)\right] \mathrm{d}\, x \,\mathrm{d}\, y \\
&\quad = \int_0^1 \int_0^1 \left[f(ta + (1-t)b, \lambda c + (1-\lambda)d) + m f\left(t\frac{a}{m} + (1-t)\frac{b}{m}, \lambda c + (1-\lambda)d\right)\right] \mathrm{d}\, t \,\mathrm{d}\, \lambda \\
&\quad \le \int_0^1 \int_0^1 \bigg\{ t^s[f(a,c) + f(a,d)] + m(1-t)^s\left[f\left(\frac{b}{m}, c\right) + f\left(\frac{b}{m}, d\right)\right] \\
&\qquad + m t^s\left[f\left(\frac{a}{m}, c\right) + f\left(\frac{a}{m}, d\right)\right] + m^2(1-t)^s\left[f\left(\frac{b}{m^2}, c\right) + f\left(\frac{b}{m^2}, d\right)\right]\bigg\} \,\mathrm{d}\, t \,\mathrm{d}\, \lambda \qquad (3.2) \\
&\quad = \frac{1}{s+1}\bigg\{ f(a,c) + f(a,d) + m\left[f\left(\frac{a}{m}, c\right) + f\left(\frac{a}{m}, d\right)\right] \\
&\qquad + m\left[f\left(\frac{b}{m}, c\right) + f\left(\frac{b}{m}, d\right)\right] + m^2\left[f\left(\frac{b}{m^2}, c\right) + f\left(\frac{b}{m^2}, d\right)\right]\bigg\}.
\end{aligned}
$$

From the inequalities (3.1) and (3.2), Theorem 3.1 is proved. □

Corollary 3.2. *Under the assumptions of* Theorem 3.1, if $m = 1$, then

$$\begin{aligned} 2^{s-2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \\ &\le \frac{1}{s+1}\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big]. \end{aligned} \tag{3.3}$$

Remark 3.3. Under the assumptions of Corollary 3.2, putting $f(x,y) = f(y)$ for all $(x,y) \in \Delta$ and $s = 1$ in the inequality (3.3) yields the inequality (1.1).

Theorem 3.4. *Suppose that $f : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}_0$ is a co-ordinated (s,m)-P-convex function on $\left[0, \frac{b}{m^2}\right] \times [c,d]$ with $0 \le a < b$ and $c < d$ for some $m \in (0,1]$ and $s \in [-1,1]$. If $f \in L_1\big(\left[0, \frac{b}{m^2}\right] \times [c,d]\big)$, then*

$$\begin{aligned} &2^{2s-4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\quad\le 2^{s-3}\left[\frac{1}{b-a}\int_a^b \frac{f(x,\frac{c+d}{2}) + m f(\frac{x}{m},\frac{c+d}{2})}{2}\,\mathrm{d}\,x + \frac{1}{d-c}\int_c^d \frac{f(\frac{a+b}{2},y) + m f(\frac{a+b}{2m},y)}{2}\,\mathrm{d}\,y\right] \\ &\quad\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b \frac{f(x,y) + 2m f(\frac{x}{m},y) + m^2 f(\frac{x}{m^2},y)}{4}\,\mathrm{d}\,x\,\mathrm{d}\,y. \end{aligned} \tag{3.4}$$

Proof. Since f is co-ordinated (s,m)-P-convex on $\left[0, \frac{b}{m^2}\right] \times [c,d]$, for all $t \in [0,1]$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &= f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}, \frac{(c+d)/2}{2} + \frac{(c+d)/2}{2}\right) \\ &\le \frac{1}{2^{s-1}}\left[f\left(ta+(1-t)b, \frac{c+d}{2}\right) + m f\left(\frac{(1-t)a+tb}{m}, \frac{c+d}{2}\right)\right]. \end{aligned}$$

Integrating this inequality on $[0,1]$ with respect to t arrives at

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2^{s-1}}\int_0^1 \left[f\left(ta+(1-t)b, \frac{c+d}{2}\right) + m f\left(\frac{(1-t)a+tb}{m}, \frac{c+d}{2}\right)\right]\mathrm{d}\,t \\ &= \frac{1}{2^{s-1}(b-a)}\int_a^b \left[f\left(x, \frac{c+d}{2}\right) + m f\left(\frac{x}{m}, \frac{c+d}{2}\right)\right]\mathrm{d}\,x. \end{aligned} \tag{3.5}$$

For all $u \in \left[0, \frac{b}{m}\right]$ and $\lambda \in [0,1]$, by the co-ordinated (s,m)-P-convexity of f, we obtain

$$\begin{aligned} f\left(u, \frac{c+d}{2}\right) \le \frac{1}{2^s}\bigg\{ & f(u, \lambda c + (1-\lambda)d) + f(u, (1-\lambda)c + \lambda d) \\ & + m\left[f\left(\frac{u}{m}, \lambda c + (1-\lambda)d\right) + f\left(\frac{u}{m}, (1-\lambda)c + \lambda d\right)\right]\bigg\}. \end{aligned} \tag{3.6}$$

Putting $u = x$ and $u = \frac{x}{m}$ in (3.6) and substituting into (3.5) acquire

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2^{s-1}}\int_0^1 \left[f\left(ta+(1-t)b, \frac{c+d}{2}\right) + m f\left(\frac{(1-t)a+tb}{m}, \frac{c+d}{2}\right)\right]\mathrm{d}\,t \\ &= \frac{1}{2^{s-1}(b-a)}\int_a^b \left[f\left(x, \frac{c+d}{2}\right) + m f\left(\frac{x}{m}, \frac{c+d}{2}\right)\right]\mathrm{d}\,x \\ &= \frac{1}{2^{s-1}(b-a)}\int_0^1\int_a^b \left[f\left(x, \frac{c+d}{2}\right) + m f\left(\frac{x}{m}, \frac{c+d}{2}\right)\right]\mathrm{d}\,x\,\mathrm{d}\,\lambda \\ &\le \frac{1}{2^{2s-1}(b-a)}\int_0^1\int_a^b \Big\{ f(x, \lambda c + (1-\lambda)d) + f(x, (1-\lambda)c + \lambda d) \end{aligned}$$

$$
\begin{aligned}
&+ m\left[f\left(\frac{x}{m}, \lambda c+(1-\lambda)d\right)+f\left(\frac{x}{m},(1-\lambda)c+\lambda d\right)\right]\\
&+ m\left[f\left(\frac{x}{m}, \lambda c+(1-\lambda)d\right)+f\left(\frac{x}{m},(1-\lambda)c+\lambda d\right)\right.\\
&\left.\left.+ m\left(f\left(\frac{x}{m^2}, \lambda c+(1-\lambda)d\right)+f\left(\frac{x}{m^2},(1-\lambda)c+\lambda d\right)\right)\right]\right\}\,\mathrm{d}\,x\,\mathrm{d}\,\lambda\\
&= \frac{1}{2^{2s-2}(b-a)(d-c)}\int_c^d\int_a^b\left[f(x,y)+2mf\left(\frac{x}{m},y\right)+m^2f\left(\frac{x}{m^2},y\right)\right]\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}
$$

A similar argument applied for the co-ordinated (s,m)-P-convexity of f leads to

$$
\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2^s}\int_0^1\left\{f\left(\frac{a+b}{2}, \lambda c+(1-\lambda)d\right)+f\left(\frac{a+b}{2},(1-\lambda)c+\lambda d\right)\right.\\
&\left.+ m\left[f\left(\frac{a+b}{2m}, \lambda c+(1-\lambda)d\right)+f\left(\frac{a+b}{2m},(1-\lambda)c+\lambda d\right)\right]\right\}\mathrm{d}\,\lambda\\
&= \frac{1}{2^{s-1}(d-c)}\int_c^d\left[f\left(\frac{a+b}{2},y\right)+mf\left(\frac{a+b}{2m},y\right)\right]\mathrm{d}\,y\\
&= \frac{1}{2^{s-1}(d-c)}\int_c^d\int_0^1\left[f\left(\frac{a+b}{2},y\right)+mf\left(\frac{a+b}{2m},y\right)\right]\mathrm{d}\,t\,\mathrm{d}\,y\\
&\le \frac{1}{2^{2s-2}(d-c)}\int_c^d\int_0^1\left[f(ta+(1-t)b,y)+mf\left(\frac{(1-t)a+tb}{m},y\right)\right.\\
&\left.+ mf\left(\frac{ta+(1-t)b}{m},y\right)+m^2f\left(\frac{(1-t)a+tb}{m^2},y\right)\right]\mathrm{d}\,t\,\mathrm{d}\,y\\
&= \frac{1}{2^{2s-2}(b-a)(d-c)}\int_c^d\int_a^b\left[f(x,y)+2mf\left(\frac{x}{m},y\right)+m^2f\left(\frac{x}{m^2},y\right)\right]\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}
$$

Adding the above inequalities results in (3.4). Theorem 3.4 is proved. □

Corollary 3.5. *Under the assumptions of* Theorem 3.4,

1. if $m=1$, then

$$
\begin{aligned}
2^{2s-4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le 2^{s-3}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y;
\end{aligned}
$$

2. if $m=s=1$, then

$$
\begin{aligned}
\frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y;
\end{aligned}
$$

3. if $m=1, s=0$, then

$$
\begin{aligned}
\frac{1}{16}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{8}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y;
\end{aligned}
$$

4. if $m = 1, s = -1$, then

$$\frac{1}{64} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{16}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y\right] \le \frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x,y) \,\mathrm{d}\, x \,\mathrm{d}\, y.$$

Theorem 3.6. *Suppose that $f : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}_0$ is a co-ordinated (s,m)-P-convex function on $\left[0, \frac{b}{m^2}\right] \times [c,d]$ with $0 \le a < b$ and $c < d$ for some $m \in (0,1]$ and $s \in (-1,1]$. If $f \in L_1\left(\left[0, \frac{b}{m^2}\right] \times [c,d]\right)$, then*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x,y) \,\mathrm{d}\, x \,\mathrm{d}\, y \\
&\le \frac{1}{b-a}\int_a^b \frac{f(x,c) + f(x,d) + mf(\frac{x}{m}, c) + mf(\frac{x}{m}, d)}{2^{s+1}} \,\mathrm{d}\, x + \frac{1}{d-c}\int_c^d \frac{f(a,y) + mf(\frac{b}{m}, y)}{s+1} \,\mathrm{d}\, y \\
&\le 2^{2-s}\left[\frac{f(a,c) + f(a,d) + m\left[f(\frac{a}{m}, c) + f(\frac{a}{m}, d)\right]}{2(s+1)} + m\frac{f(\frac{b}{m}, c) + f(\frac{b}{m}, d) + m\left[f(\frac{b}{m^2}, c) + f(\frac{b}{m^2}, d)\right]}{2(s+1)}\right].
\end{aligned}$$

Proof. By the co-ordinated (s,m)-P-convexity of f, we obtain

$$f(x, \lambda c + (1-\lambda)d) \le \frac{1}{2^s}\left[f(x,c) + f(x,d) + mf\left(\frac{x}{m}, c\right) + mf\left(\frac{x}{m}, d\right)\right] \tag{3.7}$$

for all $\lambda \in [0,1]$ and $x \in [a,b]$. Setting $y = \lambda c + (1-\lambda)d$ for $0 \le \lambda \le 1$ and using (3.7) gives

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x,y) \,\mathrm{d}\, x \,\mathrm{d}\, y \\
&= \frac{1}{b-a}\int_0^1 \int_a^b f(x, \lambda c + (1-\lambda)d) \,\mathrm{d}\, x \,\mathrm{d}\, \lambda \\
&\le \frac{1}{2^s(b-a)}\int_0^1 \int_a^b \left[f(x,c) + f(x,d) + mf\left(\frac{x}{m}, c\right) + mf\left(\frac{x}{m}, d\right)\right] \mathrm{d}\, x \,\mathrm{d}\, \lambda \\
&= \frac{1}{2^s(b-a)}\int_a^b \left[f(x,c) + f(x,d) + mf\left(\frac{x}{m}, c\right) + mf\left(\frac{x}{m}, d\right)\right] \mathrm{d}\, x.
\end{aligned} \tag{3.8}$$

If $(u,v) \in [a,b] \times [c,d]$, choose $u = ta + (1-t)b$ for all $0 < t < 1$, and

$$f(u,v) \le 2\left[t^s f(a,v) + m(1-t)^s f\left(\frac{b}{m}, v\right)\right]. \tag{3.9}$$

Letting $(u,v) = (x,c)$, (x,d) and $(\frac{u}{m}, v) = (\frac{x}{m}, c)$, $(\frac{x}{m}, d)$ respectively in (3.9) and taking them into the inequality (3.8) reveal

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x,y) \,\mathrm{d}\, x \,\mathrm{d}\, y \\
&\le \frac{1}{2^s(b-a)}\int_a^b \left[f(x,c) + f(x,d) + mf\left(\frac{x}{m}, c\right) + mf\left(\frac{x}{m}, d\right)\right] \mathrm{d}\, x \\
&\le \frac{1}{2^{s-1}}\int_0^1 \left\{t^s f(a,c) + m(1-t)^s f\left(\frac{b}{m}, c\right) + t^s f(a,d) + m(1-t)^s f\left(\frac{b}{m}, d\right)\right. \\
&\quad \left. + m\left[t^s f\left(\frac{a}{m}, c\right) + m(1-t)^s f\left(\frac{b}{m^2}, c\right) + t^s f\left(\frac{a}{m}, d\right) + m(1-t)^s f\left(\frac{b}{m^2}, d\right)\right]\right\} \mathrm{d}\, t \\
&= \frac{1}{2^{s-1}(s+1)}\left\{f(a,c) + f(a,d) + mf\left(\frac{a}{m}, c\right) + mf\left(\frac{a}{m}, d\right)\right. \\
&\quad \left. + mf\left(\frac{b}{m}, c\right) + mf\left(\frac{b}{m}, d\right) + m^2 f\left(\frac{b}{m^2}, c\right) + m^2 f\left(\frac{b}{m^2}, d\right)\right]\right\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y &\le \frac{2}{d-c}\int_c^d\int_0^1\left[t^s f(a,y)+m(1-t)^s f\left(\frac{b}{m},y\right)\right]\mathrm{d}\,t\,\mathrm{d}\,y\\ &= \frac{2}{(s+1)(d-c)}\int_c^d\left[f(a,y)+mf\left(\frac{b}{m},y\right)\right]\mathrm{d}\,y\\ &\le \frac{1}{2^{s-1}(s+1)}\left\{f(a,c)+f(a,d)+mf\left(\frac{a}{m},c\right)+mf\left(\frac{a}{m},d\right)\right.\\ &\quad\left.+mf\left(\frac{b}{m},c\right)+mf\left(\frac{b}{m},d\right)+m^2f\left(\frac{b}{m^2},c\right)+m^2f\left(\frac{b}{m^2},d\right)\right]\right\}.\end{aligned}$$

The proof of Theorem 3.6 is completed. □

Corollary 3.7. *Under the conditions of* Theorem 3.6 and Corollary 3.2,

1. if $m=1$, then

$$\begin{aligned}2^{2s-4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le 2^{s-3}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\ &\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\ &\le \frac{1}{2^s(b-a)}\int_a^b\big[f(x,c)+f(x,d)\big]\mathrm{d}\,x+\frac{1}{(s+1)(d-c)}\int_c^d\big[f(a,y)+f(b,y)\big]\,\mathrm{d}\,y\\ &\le \frac{1}{2^{s-2}(s+1)}\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big];\end{aligned}$$

2. if $m=s=1$, then

$$\begin{aligned}\frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\ &\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\ &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b\big[f(x,c)+f(x,d)\big]\mathrm{d}\,x+\frac{1}{d-c}\int_c^d\big[f(a,y)+f(b,y)\big]\,\mathrm{d}\,y\right]\\ &\le f(a,c)+f(a,d)+f(b,c)+f(b,d);\end{aligned}$$

3. if $m=1$ and $s=0$, then

$$\begin{aligned}\frac{1}{16}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{8}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\ &\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\ &\le \frac{1}{b-a}\int_a^b\big[f(x,c)+f(x,d)\big]\mathrm{d}\,x+\frac{1}{d-c}\int_c^d\big[f(a,y)+f(b,y)\big]\,\mathrm{d}\,y\\ &\le 4\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big].\end{aligned}$$

Theorem 3.8. *Let $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}_0$ be a co-ordinated (s,m)-P-convex function on $\big[0,\frac{b}{m}\big]\times[c,d]$ for $m\in(0,1]$ and $s=-1$ with $0\le a<b$ and $c<d$. If $f\in L_1\big(\big[0,\frac{b}{m}\big]\times[c,d]\big)$, then*

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b p(x)f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\le\frac{1}{2}\left[f(a,c)+f(a,d)+mf\left(\frac{b}{m},c\right)+mf\left(\frac{b}{m},d\right)\right],$$

where $p(x)=\frac{(b-x)(x-a)}{(b-a)^2}$ for $x\in[a,b]$.

Proof. Putting $x = ta+(1-t)b$ and $y = \lambda c+(1-\lambda)d$ for $0 < t < 1$ and $0 \le \lambda \le 1$ and using the co-ordinated (s, m)-P-convexity of f reveal

$$\begin{aligned}\frac{1}{(b-a)(d-c)} &\int_c^d \int_a^b p(x)f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \\ &= \int_0^1 \int_0^1 t(1-t)f(ta+(1-t)b, \lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda \\ &\le \int_0^1 \int_0^1 \left\{(1-t)[f(a,c)+f(a,d)] + tm\left[f\left(\frac{b}{m},c\right)+f\left(\frac{b}{m},d\right)\right]\right\}\mathrm{d}\,t\,\mathrm{d}\,\lambda \\ &= \frac{1}{2}\left[f(a,c)+f(a,d)+mf\left(\frac{b}{m},c\right)+mf\left(\frac{b}{m},d\right)\right].\end{aligned}$$

Theorem 3.8 is proved. □

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