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A fixed point theorem on soft G-metric spaces

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Abstract

We introduce soft G-metric spaces via soft element. Then, we obtain soft convergence and soft continuity by using soft G-metric. Also, we prove a fixed point theorem for mappings satisfying sufficient conditions in soft G-metric spaces.

Keywords: Soft set, soft metric space, generalized metric space, soft G-metric space, fixed point.
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1. Introduction and Preliminaries

There are uncertainties in many complicated problems in the fields of engineering, physics, computer science, medical science, social science and economics. These problems can not be solved by classical methods such as probability theory, fuzzy set theory, intuitionistic fuzzy sets, rough set etc. In [10], Molodtsov introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties. Then, Maji et al. [8] studied soft set theory in detail. Also, Ali et al. [1] established new algebraic operations on soft sets. Shabir and Naz [15] introduced soft topological spaces and investigated their fundamental properties. Zorlutuna [18] also studied those spaces. Das and Samanta [5] introduced the notions of soft real set and soft real number and gave their properties. Recently, soft set theory have a high potential for applications in several areas, see for example [3, 4, 9].

Fixed point theory plays an important role and has many applications in mathematics. A basic result in fixed point theory is the Banach contraction principle. There has been a lot of activity in this area since the appearance of this principle.

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Many researchers obtained fixed point theorems for various mapping in different metric spaces. Among them are G-metric spaces [13] which are a generalization of metric spaces. Several new fixed point theorems were inferred in this new structure [2, 11, 12, 14, 16].

Recently, Wardowski [17] introduced a notion of soft mapping and its fixed points. Also, Das and Samanta [6] gave the concept of soft metric spaces and Banach fixed point theorem in soft metric space settings.

In the first section, we define the concept of soft G-metric, according to soft element, and describe some of its properties. Later, we give a relation between the soft metric and soft G-metric. Further, we introduce soft G-ball and get its basic features. In the second section, we investigate soft convergence and soft continuity with the help of soft G-metric. Then, we prove existence and uniqueness of fixed point in soft G-metric spaces.

Definition 1.1 ([10])**.** Let E be a set of parameters, a pair (F, E), where F is a mapping from E to $\mathcal{P}(X)$, is called a soft set over X.

Definition 1.2 ([8])**.** Let (F_1, E) and (F_2, E) be two soft sets over a common universe X. Then, (F_1, E) is said to be a soft subset of (F_2, E) if $F_1(\lambda) \subseteq F_2(\lambda)$, for all $\lambda \in E$. This is denoted by $(F_1, E)\widetilde{\subseteq}(F_2, E)$. (F_1, E) is said to be soft equal to (F_2, E) if $F_1(\lambda) = F_2(\lambda)$, for all $\lambda \in E$. This is denoted by $(F_1, E) = (F_2, E)$.

Definition 1.3 ([1])**.** The complement of a soft set (F, E) is defined as $(F, E)^c = (F^c, E)$, where $F^c : E \to \mathcal{P}(X)$ is a mapping given by $F^c(\lambda) = X \backslash F(\lambda)$, for all $\lambda \in E$.

Definition 1.4 ([8])**.** Let (F, E) be a soft set over X. Then

(a) (F, E) is said to be a null soft set if $F(\lambda) = \emptyset$, for all $\lambda \in E$. This is denoted by $\widetilde{\emptyset}$.
(b) (F, E) is said to be an absolute soft set if $F(\lambda) = X$, for all $\lambda \in E$. This is denoted by $\widetilde{X}$.

Clearly, we have $(\widetilde{X})^c = \widetilde{\emptyset}$ and $(\widetilde{\emptyset})^c = \widetilde{X}$.

Definition 1.5 ([15])**.** The difference (H, E) of two soft sets (F_1, E) and (F_2, E) over X, denoted by $(F_1, E)\backslash(F_2, E)$, is defined by $H(\lambda) = F_1(\lambda)\backslash F_2(\lambda)$, for all $\lambda \in E$.

Definition 1.6 ([8])**.** The union (H, E) of two soft sets (F, E) and (G, E) over a common universe X, denoted by $(F, E)\widetilde{\cup}(G, E)$ is defined by $H(\lambda) = F(\lambda) \cup G(\lambda)$, for all $\lambda \in E$.

The following definition of intersection of two soft sets is given in [7], where it was named bi-intersection.

Definition 1.7 ([7])**.** The intersection (H, E) of two soft sets (F, E) and (G, E) over a common universe X, denoted by $(F, E)\widetilde{\cap}(G, E)$, is defined by $H(\lambda) = F(\lambda) \cap G(\lambda)$, for all $\lambda \in E$.

Definition 1.8 ([18])**.** Let I be an index set and $\{(F_i, E)\}_{i\in I}$ a nonempty family of soft sets over a common universe X. Then

(a) The union of these soft sets is the soft set (H, E), where $H(\lambda) = \bigcup_{i\in I} F_i(\lambda)$ for all $\lambda \in E$. This is denoted by $\widetilde{\bigcup}_{i\in I}(F_i, E) = (H, E)$.
(b) The intersection of these soft sets is the soft set (H, E), where $H(\lambda) = \bigcap_{i\in I} F_i(\lambda)$ for all $\lambda \in E$. This is denoted by $\widetilde{\bigcap}_{i\in I}(F_i, E) = (H, E)$.

Definition 1.9 ([5])**.** A function $\varepsilon : E \to X$ is said to be a soft element of X. A soft element ε of X is said to belong to a soft set (F, E) of X, denoted by $\varepsilon\widehat{\in}(F, E)$, if $\varepsilon(e) \in F(e)$ for each $e \in E$. In that case, ε is also said to be a soft element of the soft set (F, E). Thus, every singleton soft set (a soft set (F, E) for which $F(e)$ is a singleton set for each $e \in E$) can be identified with a soft element by simply identifying the singleton set with the element that contains each $e \in E$.

Definition 1.10 ([5])**.** Let $\mathbf{R}$ be the set of real numbers, $\mathcal{B}(\mathbf{R})$ the collection of all nonempty bounded subsets of $\mathbf{R}$, and E a set of parameters. Then a mapping $F : E \to \mathcal{B}(\mathbf{R})$ is called a soft real set. It is denoted by (F, E) and $\mathbf{R}(E)$ denotes the set of all soft real sets. Also, $\mathbf{R}(E)^*$ denotes the set of all nonnegative soft real sets ((F, E) is said to be a nonnegative soft real set if $F(\lambda)$ is a subset of the set of nonnegative real numbers for each $\lambda \in E$).

If, in particular, (F, E) is a singleton soft set, then identifying (F, E) with the corresponding soft element, we will call it a soft real number. $\mathbb{R}(E)$ denotes the set of all soft real numbers.

Definition 1.11 ([5])**.** Let $(F, E), (G, E) \in \mathbf{R}(E)$. We define

(a) $(F, E) = (G, E)$ if $F(\lambda) = G(\lambda)$ for each $\lambda \in E$.
(b) $(F + G)(\lambda) = \{a + b : a \in F(\lambda), b \in G(\lambda)\}$ for each $\lambda \in E$.
(c) $(F - G)(\lambda) = \{a - b : a \in F(\lambda), b \in G(\lambda)\}$ for each $\lambda \in E$.
(d) $(F.G)(\lambda) = \{a.b : a \in F(\lambda), b \in G(\lambda)\}$ for each $\lambda \in E$.
(e) $(F/G)(\lambda) = \{a/b : a \in F(\lambda), b \in G(\lambda)\backslash\{0\}\}$ provided $0 \notin G(\lambda)$ for each $\lambda \in E$.

In this paper, following [6], $S(\widetilde{X})$ denotes the set of soft sets (F, E) over X for which $F(\lambda) \neq \emptyset$ for all $\lambda \in E$ and $SE(F, E)$ denotes the collection of all soft elements of (F, E) for any soft set $(F, E) \in S(\widetilde{X})$. Also, $\widetilde{x}, \widetilde{y}, \widetilde{z}$ denote soft elements of a soft set and $\widetilde{r}, \widetilde{s}, \widetilde{t}$ denote soft real numbers, while $\overline{r}, \overline{s}, \overline{t}$ will denote a particular type of soft real numbers such that $\overline{r}(\lambda) = r$ for all $\lambda \in E$.

Definition 1.12 ([6])**.** For two soft real numbers $\widetilde{r}, \widetilde{s}$, we define

(a) $\widetilde{r} \widetilde{\leq} \widetilde{s}$ if $\widetilde{r}(\lambda) \leq \widetilde{s}(\lambda)$ for all $\lambda \in E$.
(b) $\widetilde{r} \widetilde{\geq} \widetilde{s}$ if $\widetilde{r}(\lambda) \geq \widetilde{s}(\lambda)$ for all $\lambda \in E$.
(c) $\widetilde{r} \widetilde{<} \widetilde{s}$ if $\widetilde{r}(\lambda) < \widetilde{s}(\lambda)$ for all $\lambda \in E$.
(d) $\widetilde{r} \widetilde{>} \widetilde{s}$ if $\widetilde{r}(\lambda) > \widetilde{s}(\lambda)$ for all $\lambda \in E$

Proposition 1.13 ([6])**.** *Any collection of soft elements of a soft set can generate a soft subset of that soft set.*

Remark 1.14 ([6]). Let (F, E) be a soft set. Y generates a soft subset of (F, E) by considering for each $\lambda \in E$. Considering the set $\{\widetilde{x}(\lambda) : \widetilde{x} \in Y\}$ and associating a soft set with its λ-levels to be this set.

The soft set constructed from a collection $\mathcal{B}$ of soft elements is denoted by $SS(\mathcal{B})$.

Proposition 1.15 ([6])**.** *For any soft subsets $(F, E), (G, E) \in S(\widetilde{X})$, we have $(F, E) \widetilde{\subseteq} (G, E)$ if and only if every soft element of (F, E) is also a soft element of (G, E).*

Definition 1.16 ([6])**.** A mapping $d : SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$, is said to be a soft metric on $\widetilde{X}$ if d satisfies

(M1) $d(\widetilde{x}, \widetilde{y}) \widetilde{\geq} \overline{0}$ for all $\widetilde{x}, \widetilde{y} \widetilde{\in} \widetilde{X}$.
(M2) $d(\widetilde{x}, \widetilde{y}) = \overline{0}$ if and only if $\widetilde{x} = \widetilde{y}$.
(M3) $d(\widetilde{x}, \widetilde{y}) = d(\widetilde{y}, \widetilde{x})$ for all $\widetilde{x}, \widetilde{y} \widetilde{\in} \widetilde{X}$.
(M4) $d(\widetilde{x}, \widetilde{z}) \widetilde{\leq} d(\widetilde{x}, \widetilde{y}) + d(\widetilde{y}, \widetilde{z})$ for all $\widetilde{x}, \widetilde{y}, \widetilde{z} \widetilde{\in} \widetilde{X}$.

The soft set $\widetilde{X}$ with a soft metric d on $\widetilde{X}$ is said to be a soft metric space and is denoted by $(\widetilde{X}, d, E)$.

Definition 1.17 ([13])**.** Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality),

then the function G is called a generalized metric or a G-metric on X, and the pair (X, G) is a G-metric space.

2. Soft G-Metric Spaces

Definition 2.1. Let X be a nonempty set and E be the nonempty set of parameters. A mapping $\widetilde{G} : SE(\widetilde{X}) \times SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ is said to be a soft generalized metric or soft G-metric on $\widetilde{X}$ if $\widetilde{G}$ satisfies the following conditions:

($\widetilde{G}_1$) $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \overline{0}$ if $\widetilde{x} = \widetilde{y} = \widetilde{z}$.
($\widetilde{G}_2$) $\overline{0} \widetilde{<} \widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y})$ for all $\widetilde{x}, \widetilde{y} \in SE(\widetilde{X})$ with $\widetilde{x} \neq \widetilde{y}$.
($\widetilde{G}_3$) $\widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y}) \widetilde{\leq} \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z})$ for all $\widetilde{x}, \widetilde{y}, \widetilde{z} \in SE(\widetilde{X})$ with $\widetilde{y} \neq \widetilde{z}$.
($\widetilde{G}_4$) $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \widetilde{G}(\widetilde{x}, \widetilde{z}, \widetilde{y}) = \widetilde{G}(\widetilde{y}, \widetilde{z}, \widetilde{x}) = \cdots$.
($\widetilde{G}_5$) $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{x}, \widetilde{a}, \widetilde{a}) + \widetilde{G}(\widetilde{a}, \widetilde{y}, \widetilde{z})$ for all $\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{a} \in SE(\widetilde{X})$.

The soft set $\widetilde{X}$ with a soft G-metric $\widetilde{G}$ on $\widetilde{X}$ is said to be a soft G-metric space and is denoted by $(\widetilde{X}, \widetilde{G}, E)$.

Example 2.2. Let X be a nonempty set and E the nonempty set of parameters. We define $\widetilde{G} : SE(\widetilde{X}) \times SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ by

$$\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \begin{cases} \overline{0}, & \text{if all of the variables are equal;} \\ \overline{1}, & \text{if two of the variables are equal, and the remaining one is distinct;} \\ \overline{2}, & \text{if all of the variables are distinct.} \end{cases}$$

for all $\widetilde{x}, \widetilde{y}, \widetilde{z} \in SE(\widetilde{X})$. Then $\widetilde{G}$ satisfies all the soft G-metric axioms.

Definition 2.3. A soft G-metric space $(\widetilde{X}, \widetilde{G}, E)$ is symmetric if

($\widetilde{G}_6$) $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y}) = \widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y})$ for all $\widetilde{x}, \widetilde{y} \in SE(\widetilde{X})$.

Remark 2.4. A soft G-metric space $(\widetilde{X}, \widetilde{G}, E)$ given in Example 2.2 is also symmetric.

Remark 2.5.

(a) Parameterized family of crisp G-metrics $\{G_\lambda : \lambda \in E\}$ on a crisp set X may not be soft G-metric on the soft set $\widetilde{X}$.
(b) Any soft G-metric is not just a parameterized family of crisp G-metrics on a crisp set X.

The following example relates to Remark 2.5.

Example 2.6.

(a) Let $X = \{a, b\}$, $G(a, a, a) = G(b, b, b) = 0$, $G(a, a, b) = 1$, and $G(a, b, b) = 2$. Then, we know that $G : X \times X \times X \to \mathbb{R}^+$ is a G-metric, from Example 1 of [13]. Let $E = \{e_1, e_2\}$ and $SE(\widetilde{X}) = \{\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{t}\}$ be defined by $\widetilde{x}(e_1) = a$, $\widetilde{x}(e_2) = a$, $\widetilde{y}(e_1) = b$, $\widetilde{y}(e_2) = b$, $\widetilde{z}(e_1) = b$, $\widetilde{z}(e_2) = a$, $\widetilde{t}(e_1) = a$, $\widetilde{t}(e_2) = b$. Then, $\widetilde{G} : SE(\widetilde{X}) \times SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ where $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z})(\lambda) = G(\widetilde{x}(\lambda), \widetilde{y}(\lambda), \widetilde{z}(\lambda))$ for each $\lambda \in E$, is not a soft G-metric.
(b) Let $X = \{a, b\}$, $E = \{e_1, e_2\}$ and $SE(\widetilde{X}) = \{\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{t}\}$ with $\widetilde{x}(e_1) = a$, $\widetilde{x}(e_2) = a$, $\widetilde{y}(e_1) = b$, $\widetilde{y}(e_2) = b$, $\widetilde{z}(e_1) = a$, $\widetilde{z}(e_2) = b$, $\widetilde{t}(e_1) = b$, $\widetilde{t}(e_2) = a$. Consider the soft G-metric of Example 2.2. We decompose $\widetilde{G}$ parameter wise as $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z})(\lambda) = G_\lambda(\widetilde{x}(\lambda), \widetilde{y}(\lambda), \widetilde{z}(\lambda))$ for each $\lambda \in E$ and $\widetilde{x}, \widetilde{y}, \widetilde{z} \in SE(\widetilde{X})$. Then, $G_\lambda : X \times X \times X \to \mathbb{R}^+$ is not a well defined mapping. So, it is not G-metric on X.

Theorem 2.7. *If $\{G_\lambda : \lambda \in E\}$ are G-metrics on X and if for $\lambda \in E$, $\widetilde{G} : SE(\widetilde{X}) \times SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ is defined by $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z})(\lambda) = G_\lambda(\widetilde{x}(\lambda), \widetilde{y}(\lambda), \widetilde{z}(\lambda))$ where $\widetilde{x}(\lambda)$, $\widetilde{y}(\lambda)$, $\widetilde{z}(\lambda)$ are constant, then $\widetilde{G}$ is a soft G-metric on $\widetilde{X}$.*

Proof. The proof is obvious since $\{G_\lambda : \lambda \in E\}$ are G-metrics on X. □

Theorem 2.8. *Let $\widetilde{G}$ be a soft G-metric on $\widetilde{X}$. If for $(r,s,m) \in X \times X \times X$ and $\lambda \in E$, $\{\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z})(\lambda) : \widetilde{x}(\lambda) = r, \widetilde{y}(\lambda) = s, \widetilde{z}(\lambda) = m\}$ is a singleton set and if for $\lambda \in E$, $G_\lambda : X \times X \times X \to \mathbb{R}^+$ is defined by $G_\lambda(\widetilde{x}(\lambda),\widetilde{y}(\lambda),\widetilde{z}(\lambda)) = \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z})(\lambda)$ for $\widetilde{x},\widetilde{y},\widetilde{z} \in SE(\widetilde{X})$, then G_λ is a G-metric on X.*

Proof. The proof is clear from the hypothesis and soft G-metric axioms. □

Proposition 2.9. *Let $(\widetilde{X},\widetilde{G},E)$ be a soft G-metric space; then the following hold for all $\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{a} \in SE(\widetilde{X})$.*

(a) *If $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) = \overline{0}$, then $\widetilde{x} = \widetilde{y} = \widetilde{z}$.*
(b) $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{y}) + \widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{z})$.
(c) $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{y}) \widetilde{\leq} \overline{2}\widetilde{G}(\widetilde{y},\widetilde{x},\widetilde{x})$.
(d) $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{z}) + \widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{z})$.
(e) $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \overline{\frac{2}{3}}(\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{a}) + \widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{z}) + \widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{z}))$.
(f) $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{a}) + \widetilde{G}(\widetilde{y},\widetilde{a},\widetilde{a}) + \widetilde{G}(\widetilde{z},\widetilde{a},\widetilde{a})$.
(g) $|\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) - \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{a})| \widetilde{\leq} \max\{\widetilde{G}(\widetilde{a},\widetilde{z},\widetilde{z}),\widetilde{G}(\widetilde{z},\widetilde{a},\widetilde{a})\}$.
(h) $|\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) - \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{a})| \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{z})$.
(i) $|\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) - \widetilde{G}(\widetilde{y},\widetilde{z},\widetilde{z})| \widetilde{\leq} \max\{\widetilde{G}(\widetilde{x},\widetilde{z},\widetilde{z}),\widetilde{G}(\widetilde{z},\widetilde{x},\widetilde{x})\}$.
(j) $|\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{y}) - \widetilde{G}(\widetilde{y},\widetilde{x},\widetilde{x})| \widetilde{\leq} \max\{\widetilde{G}(\widetilde{y},\widetilde{x},\widetilde{x}),\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{y})\}$.

Proof.

(a) Case 1: Let all of the variables be distinct. Then, we have $\overline{0} \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{y}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z})$ from $(\widetilde{G}_2)$ and $(\widetilde{G}_3)$, respectively.

Case 2: Let two of the variables be equal and the remaining one be distinct. Thus, we have $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{>} \overline{0}$ by $(\widetilde{G}_4)$ and $(\widetilde{G}_2)$. From the two cases, we obtain $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \neq \overline{0}$.

(b) We have $\widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{y}) + \widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{z}) = \widetilde{G}(\widetilde{y},\widetilde{x},\widetilde{x}) + \widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{z}) \widetilde{\geq} \widetilde{G}(\widetilde{y},\widetilde{x},\widetilde{z}) = \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z})$ from $(\widetilde{G}_4)$ and $(\widetilde{G}_5)$.

(c) By using (b), we have $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{y}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{y}) + \widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{y}) = \overline{2}\widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{y})$. Thus, $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{y}) \widetilde{\leq} \overline{2}\widetilde{G}(\widetilde{y},\widetilde{x},\widetilde{x})$ from $(\widetilde{G}_4)$.

(d) Case 1: Let $\widetilde{x} \neq \widetilde{z}$. Then, we have $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{a},\widetilde{a},\widetilde{x}) + \widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{a},\widetilde{x},\widetilde{z}) + \widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{z})$ from $(\widetilde{G}_5)$, $(\widetilde{G}_4)$ and $(\widetilde{G}_3)$, respectively. So, we obtain $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{z}) + \widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{z})$ by $(\widetilde{G}_4)$.

Case 2: Let $\widetilde{x} = \widetilde{z}$ and $\widetilde{y} \neq \widetilde{a}$. Thus, we get $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) = \widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{y}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{a}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{a}) + \widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{x}) = \widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{x}) + \widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{x})$ from $(\widetilde{G}_4)$ and $(\widetilde{G}_3)$.

Case 3: Let $\widetilde{x} = \widetilde{z}$ and $\widetilde{y} = \widetilde{a}$. It is obvious.

(e) By (d) and $(\widetilde{G}_4)$, we have $\overline{3}\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \overline{2}\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{a}) + \overline{2}\widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{z}) + \overline{2}\widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{z})$. Thus,

$$\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \overline{\frac{2}{3}}(\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{a}) + \widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{z}) + \widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{z})).$$

(f) By $(\widetilde{G}_5)$ and (b), we obtain $\overline{3}\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \overline{3}\widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{a}) + \overline{3}\widetilde{G}(\widetilde{y},\widetilde{a},\widetilde{a}) + \overline{3}\widetilde{G}(\widetilde{z},\widetilde{a},\widetilde{a})$. Hence, $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{a}) + \widetilde{G}(\widetilde{y},\widetilde{a},\widetilde{a}) + \widetilde{G}(\widetilde{z},\widetilde{a},\widetilde{a})$.

(g) We have $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{z},\widetilde{a},\widetilde{a}) + \widetilde{G}(\widetilde{a},\widetilde{x},\widetilde{y})$ from $(\widetilde{G}_5)$. Then, $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) - \widetilde{G}(\widetilde{a},\widetilde{x},\widetilde{y}) \widetilde{\leq} \widetilde{G}(\widetilde{z},\widetilde{a},\widetilde{a}) \widetilde{\leq} \max\{\widetilde{G}(\widetilde{z},\widetilde{a},\widetilde{a}),\widetilde{G}(\widetilde{a},\widetilde{z},\widetilde{z})\}$. In a similar way, we get $-\max\{\widetilde{G}(\widetilde{z},\widetilde{a},\widetilde{a}),\widetilde{G}(\widetilde{a},\widetilde{z},\widetilde{z})\} \widetilde{\leq} -\widetilde{G}(\widetilde{a},\widetilde{z},\widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) - \widetilde{G}(\widetilde{a},\widetilde{x},\widetilde{y})$. Thus, we obtain

$$|\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) - \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{a})| \widetilde{\leq} \max\{\widetilde{G}(\widetilde{a},\widetilde{z},\widetilde{z}),\widetilde{G}(\widetilde{z},\widetilde{a},\widetilde{a})\}.$$

(h) The proof is clear, by (d) and $(\widetilde{G}_4)$.

(i) It is obvious, by $(\widetilde{G}_5)$ and $(\widetilde{G}_4)$.

(j) It follows from (c) and $(\widetilde{G}_4)$. □

Proposition 2.10. *Let $(\widetilde{X},\widetilde{G},E)$ be a soft G-metric space; then the following are equivalent.*

(a) $(\widetilde{X},\widetilde{G},E)$ *is symmetric.*
(b) $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{y}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{a})$ *for all* $\widetilde{x},\widetilde{y},\widetilde{a} \in SE(\widetilde{X})$.
(c) $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{a}) + \widetilde{G}(\widetilde{z},\widetilde{y},\widetilde{b})$ *for all* $\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{a},\widetilde{b} \in SE(\widetilde{X})$.

Proof.

(a) $\Rightarrow$ (b) Since $(\widetilde{X}, \widetilde{G}, E)$ is symmetric, we have $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y}) = \widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y})$.
Case 1: Let $\widetilde{x} \neq \widetilde{a}$. We have $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y}) \widetilde{\leq} \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{a})$ by $(\widetilde{G}_3)$.
Case 2: Let $\widetilde{x} = \widetilde{a}$. It is clear by $(\widetilde{G}_4)$.

(b) $\Rightarrow$ (c) We obtain $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{y}, \widetilde{y}, \widetilde{x}) + \widetilde{G}(\widetilde{y}, \widetilde{y}, \widetilde{z}) \widetilde{\leq} \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{a}) + \widetilde{G}(\widetilde{z}, \widetilde{y}, \widetilde{b})$ by using Proposition 2.9(b) and the hypothesis.

(c) $\Rightarrow$ (a) By hypothesis and $(\widetilde{G}_4)$, we have $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y}) \widetilde{\leq} \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{x}) + \widetilde{G}(\widetilde{y}, \widetilde{y}, \widetilde{y}) = \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{x}) = \widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y})$. In a similar way, we get $\widetilde{G}(\widetilde{y}, \widetilde{x}, \widetilde{x}) \widetilde{\leq} \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y})$. □

Proposition 2.11. *For any soft metric d on $\widetilde{X}$, we can construct a soft G-metric by the following mappings $\widetilde{G}_s$ and $\widetilde{G}_m$.*

(a) $\widetilde{G}_s(d)(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \overline{\tfrac{1}{3}}(d(\widetilde{x}, \widetilde{y}) + d(\widetilde{y}, \widetilde{z}) + d(\widetilde{x}, \widetilde{z}))$,

(b) $\widetilde{G}_m(d)(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \max\{d(\widetilde{x}, \widetilde{y}), d(\widetilde{y}, \widetilde{z}), d(\widetilde{x}, \widetilde{z})\}$.

Proof.

(a) Since d is a soft metric, the proofs of $(\widetilde{G}_1)$, $(\widetilde{G}_2)$, and $(\widetilde{G}_4)$ are obvious. The proof of $(\widetilde{G}_5)$ follows from $(M4)$.
$(\widetilde{G}_3)$ Case 1: Let $\widetilde{y} \neq \widetilde{z}$ and $\widetilde{x} = \widetilde{y}$. Since $\widetilde{G}_s(d)(\widetilde{x}, \widetilde{x}, \widetilde{y}) = \overline{0}$, the assertion is clear.
Case 2: Let $\widetilde{x} = \widetilde{z}$, $\widetilde{z} \neq \widetilde{y}$, and $\widetilde{x} \neq \widetilde{y}$. Then $\widetilde{G}_s(d)(\widetilde{x}, \widetilde{x}, \widetilde{y}) = \widetilde{G}_s(d)(\widetilde{x}, \widetilde{y}, \widetilde{z})$.
Case 3: Let $\widetilde{x} \neq \widetilde{z}$, $\widetilde{z} \neq \widetilde{y}$, and $\widetilde{x} \neq \widetilde{y}$. From (M4), we have $\overline{2}d(\widetilde{x}, \widetilde{y}) \widetilde{\leq} d(\widetilde{x}, \widetilde{y}) + d(\widetilde{x}, \widetilde{z}) + d(\widetilde{z}, \widetilde{y})$. Then, $\widetilde{G}_s(d)(\widetilde{x}, \widetilde{x}, \widetilde{y}) = \overline{\tfrac{2}{3}}d(\widetilde{x}, \widetilde{y}) \widetilde{\leq} \overline{\tfrac{1}{3}}[d(\widetilde{x}, \widetilde{y}) + d(\widetilde{x}, \widetilde{z}) + d(\widetilde{z}, \widetilde{y})] = \widetilde{G}_s(d)(\widetilde{x}, \widetilde{y}, \widetilde{z})$.

(b) The proofs of $(\widetilde{G}_1)$, $(\widetilde{G}_2)$ and $(\widetilde{G}_4)$ are obvious.
$(\widetilde{G}_3)$ Let $\widetilde{z} \neq \widetilde{y}$. $\widetilde{G}_m(d)(\widetilde{x}, \widetilde{x}, \widetilde{y}) = d(\widetilde{x}, \widetilde{y}) \widetilde{\leq} \max\{d(\widetilde{x}, \widetilde{y}), d(\widetilde{x}, \widetilde{z}), d(\widetilde{y}, \widetilde{z})\} = \widetilde{G}_m(d)(\widetilde{x}, \widetilde{y}, \widetilde{z})$.
$(\widetilde{G}_5)$ Case 1: $\widetilde{G}_m(d)(\widetilde{x}, \widetilde{y}, \widetilde{z}) = d(\widetilde{x}, \widetilde{y})$. From (M4), we have $d(\widetilde{x}, \widetilde{y}) \widetilde{\leq} d(\widetilde{x}, \widetilde{a}) + d(\widetilde{a}, \widetilde{y}) = \widetilde{G}_m(d)(\widetilde{x}, \widetilde{a}, \widetilde{a}) + d(\widetilde{a}, \widetilde{y}) \widetilde{\leq} \widetilde{G}_m(d)(\widetilde{x}, \widetilde{a}, \widetilde{a}) + \widetilde{G}_m(d)(\widetilde{a}, \widetilde{y}, \widetilde{z})$.
The other cases can be proved in a similar way. □

Proposition 2.12. *For any soft G-metric $\widetilde{G}$ on $\widetilde{X}$, we can construct a soft metric $d_0\widetilde{G}$ on $\widetilde{X}$ defined by*

$$d_{\widetilde{G}}(\widetilde{x}, \widetilde{y}) = \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y}) + \widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y})$$

Proof. The proofs of (M1), (M3), and (M4) obviously follow from $(\widetilde{G}_1)$ and $(\widetilde{G}_2)$, $(\widetilde{G}_4)$ and $(\widetilde{G}_5)$, respectively.
(M2) Let $d_{\widetilde{G}}(\widetilde{x}, \widetilde{y}) = \overline{0}$. Assume $\widetilde{x} \neq \widetilde{y}$. Since $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y}) + \widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y}) = \overline{0}$, we would have $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y}) \widetilde{\leq} \overline{0}$ by Proposition 2.9 (c). This contradicts to $(\widetilde{G}_2)$. Thus, $\widetilde{x} = \widetilde{y}$. The converse is clear. □

Proposition 2.13. *Let d be a soft metric on $\widetilde{X}$; then the following hold.*

(a) $d_{\widetilde{G}_s(d)}(\widetilde{x}, \widetilde{y}) = \overline{\tfrac{4}{3}}d(\widetilde{x}, \widetilde{y})$.

(b) $d_{\widetilde{G}_m(d)}(\widetilde{x}, \widetilde{y}) = \overline{2}d(\widetilde{x}, \widetilde{y})$.

Proof. The proofs follow from the definitions of $d_{\widetilde{G}}$, $\widetilde{G}_m$, and $\widetilde{G}_s$. □

Proposition 2.14. *Let $\widetilde{G}$ be a soft G-metric on $\widetilde{X}$; then the following hold.*

(a) $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) \widetilde{\leq} \widetilde{G}_s(d_{\widetilde{G}})(\widetilde{x}, \widetilde{y}, \widetilde{z}) \widetilde{\leq} \overline{2}\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z})$.

(b) $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) \widetilde{\leq} \widetilde{G}_m(d_{\widetilde{G}})(\widetilde{x}, \widetilde{y}, \widetilde{z}) \widetilde{\leq} \overline{3}\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z})$.

Proof. (a) By Proposition 2.9 (b), we have $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) \widetilde{\leq} \overline{\tfrac{1}{3}}[d_{\widetilde{G}}(\widetilde{x}, \widetilde{y}) + d_{\widetilde{G}}(\widetilde{y}, \widetilde{z}) + d_{\widetilde{G}}(\widetilde{x}, \widetilde{z})]$. On the other hand, by Proposition 2.9 (c) and $(\widetilde{G}_3)$, we obtain $\overline{\tfrac{1}{3}}[d_{\widetilde{G}}(\widetilde{x}, \widetilde{y}) + d_{\widetilde{G}}(\widetilde{y}, \widetilde{z}) + d_{\widetilde{G}}(\widetilde{x}, \widetilde{z})] \widetilde{\leq} \overline{2}\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z})$.

(b) By Proposition 2.9 (b), we obtain $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z})\widetilde{\leq}\widetilde{G}_m(d_{\widetilde{G}})(\widetilde{x},\widetilde{y},\widetilde{z})$. Also, from Proposition 2.9 (c) and $(\widetilde{G}_3)$, we get $\widetilde{G}_m(d_{\widetilde{G}})(\widetilde{x},\widetilde{y},\widetilde{z})\widetilde{\leq}\overline{3}\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z})$. □

Definition 2.15. Let $(\widetilde{X},\widetilde{G},E)$ be a soft G-metric space. For $\widetilde{a}\in SE(\widetilde{X})$ and $\widetilde{r}\widetilde{>}\overline{0}$, the $\widetilde{G}$-ball with a center $\widetilde{a}$ and radius $\widetilde{r}$ is

$$B_{\widetilde{G}}(\widetilde{a},\widetilde{r})=\{\widetilde{x}\in SE(\widetilde{X})\mid \widetilde{G}(\widetilde{a},\widetilde{x},\widetilde{x})\widetilde{<}\widetilde{r}\}\subseteq SE(\widetilde{X}).$$

$SS(B_{\widetilde{G}}(\widetilde{a},\widetilde{r}))$ will be called a soft $\widetilde{G}$-ball with center $\widetilde{a}$ and radius $\widetilde{r}$.

Example 2.16. Consider the soft G-metric space $(\widetilde{X},\widetilde{G},E)$ given in Example 2.2. We have

$$B_{\widetilde{G}}(\widetilde{a},\widetilde{r})=\begin{cases} SE(\widetilde{X}), & \widetilde{r}\widetilde{>}\overline{1} \\ \{\widetilde{a}\}, & \widetilde{r}\widetilde{\leq}\overline{1}\end{cases}\quad \text{for any } \widetilde{a}\in SE(\widetilde{X}).$$

Proposition 2.17. *Let $(\widetilde{X},\widetilde{G},E)$ be a soft G-metric space. For $\widetilde{a}\in SE(\widetilde{X})$ and $\widetilde{r}\widetilde{>}\overline{0}$, the following hold.*

(a) If $\widetilde{G}(\widetilde{a},\widetilde{x},\widetilde{y})\widetilde{<}\widetilde{r}$, then $\widetilde{x},\widetilde{y}\in B_{\widetilde{G}}(\widetilde{a},\widetilde{r})$.

(b) If $\widetilde{x}\in B_{\widetilde{G}}(\widetilde{a},\widetilde{r})$, then there exists a $\widetilde{\delta}\widetilde{>}\overline{0}$ such that $B_{\widetilde{G}}(\widetilde{x},\widetilde{\delta})\subseteq B_{\widetilde{G}}(\widetilde{a},\widetilde{r})$.

Proof.

(a) It obviously follows from $(\widetilde{G}_3)$.

(b) Let $\widetilde{x}\in B_{\widetilde{G}}(\widetilde{a},\widetilde{r})$. Assume that $\widetilde{y}\in B_{\widetilde{G}}(\widetilde{x},\widetilde{\delta})$. Then, we have $\widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{y})\widetilde{\leq}\widetilde{G}(\widetilde{a},\widetilde{x},\widetilde{x})+\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{y})\widetilde{<}\widetilde{G}(\widetilde{a},\widetilde{x},\widetilde{x})+\widetilde{\delta}$ by $(\widetilde{G}_5)$. Say $\widetilde{\delta}=\widetilde{r}-\widetilde{G}(\widetilde{a},\widetilde{x},\widetilde{x})$. Thus, we get $\widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{y})\widetilde{<}\widetilde{r}$. Hence, $\widetilde{y}\in B_{\widetilde{G}}(\widetilde{a},\widetilde{r})$. □

Proposition 2.18. *Let $(\widetilde{X},\widetilde{G},E)$ be a soft G-metric space. For $\widetilde{x_0}\in SE(\widetilde{X})$ and $\widetilde{r}\widetilde{>}\overline{0}$, we have*

$$B_{\widetilde{G}}(\widetilde{x_0},\overline{\frac{1}{3}}\widetilde{r})\subseteq B_{d_{\widetilde{G}}}(\widetilde{x_0},\widetilde{r})\subseteq B_{\widetilde{G}}(\widetilde{x_0},\widetilde{r})$$

Proof. It obviously follows from Proposition 2.9 (c). □

3. Soft G-convergence

Definition 3.1. Let $(\widetilde{X},\widetilde{G},E)$ be a soft G-metric space and $(\widetilde{x_n})$ a sequence of soft elements in $\widetilde{X}$. The sequence $(\widetilde{x_n})$ is said to be soft G-convergent at $\widetilde{x}$ in $\widetilde{X}$ if for every $\widetilde{\epsilon}\widetilde{\geq}\overline{0}$, there exists a natural number N=N($\widetilde{\epsilon}$) such that $\overline{0}\widetilde{\leq}\ \widetilde{G}(\widetilde{x_n},\widetilde{x_n},\widetilde{x})\ \widetilde{<}\widetilde{\epsilon}$ whenever $n\geq N$ i.e., $n\geq N\Rightarrow(\widetilde{x_n})\in B_{\widetilde{G}}(\widetilde{x},\widetilde{\epsilon})$. We denote this by $\widetilde{x_n}\to\widetilde{x}$ as $n\to\infty$ or by $\lim_{n\to\infty}(\widetilde{x_n})=\widetilde{x}$.

Proposition 3.2. *Let $(\widetilde{X},\widetilde{G},E)$ be a soft G-metric space. For a sequence $(\widetilde{x_n})$ in $\widetilde{X}$ and a soft element $\widetilde{x}$ the following are equivalent:*

(a) $(\widetilde{x_n})$ is soft G-convergent to $\widetilde{x}$.

(b) $d_{\widetilde{G}}(\widetilde{x_n},\widetilde{x})\to\overline{0}$ as $n\to\infty$.

(c) $\widetilde{G}(\widetilde{x_n},\widetilde{x_n},\widetilde{x})\to\overline{0}$ as $n\to\infty$.

(d) $\widetilde{G}(\widetilde{x_n},\widetilde{x},\widetilde{x})\to\overline{0}$ as $n\to\infty$.

(e) $\widetilde{G}(\widetilde{x_n},\widetilde{x_m},\widetilde{x})\to\overline{0}$ as $n,m\to\infty$.

Proof. That (a) implies (b) follows from Proposition 2.18. The other implications can be proved by using Propositions 2.9 and 2.12. □

Definition 3.3. Let $(\widetilde{X}, \widetilde{G}, E)$, $(\widetilde{X'}, \widetilde{G'}, E')$ be two soft G-metric spaces. Then a function f: $\widetilde{X} \to \widetilde{X'}$ is a soft G-continuous at a soft element $\widetilde{a} \in SE(\widetilde{X})$ if and only if for every $\widetilde{\epsilon}\widetilde{>}\overline{0}$, there exists $\widetilde{\delta}\widetilde{\geq}\overline{0}$ such that $\widetilde{x}, \widetilde{y}\widetilde{\in}\widetilde{X}$ and $\widetilde{G}(\widetilde{a}, \widetilde{x}, \widetilde{y})\widetilde{<}\widetilde{\delta}$ implies that $\widetilde{G'}(\widetilde{f(a)}, \widetilde{f(x)}, \widetilde{f(y)})\widetilde{<}\widetilde{\epsilon}$.

A function f is soft G-continuous if and only if it is soft G-continuous at all $\widetilde{a} \in SE(\widetilde{X})$.

Proposition 3.4. *Let $(\widetilde{X}, \widetilde{G}, E)$, $(\widetilde{X'}, \widetilde{G'}, E')$ be two soft G-metric spaces. Then a function f: $\widetilde{X} \to \widetilde{X'}$ is a soft G-continuous at a soft element $\widetilde{a} \in SE(\widetilde{X})$ if and only if it is soft G-sequentially continuous at a soft element $\widetilde{a} \in SE(\widetilde{X})$, i.e., whenever $(\widetilde{x_n})$ is soft G-convergent to $\widetilde{a}$, $(\widetilde{f(x_n)})$ is soft G'-convergent to $\widetilde{f(a)}$.*

Proof. Necessity. Assume that f is soft G-continuous. Given $\widetilde{x_n} \to \widetilde{a}$, we wish to show that $\widetilde{f(x_n)} \to \widetilde{f(a)}$. Let $\widetilde{\epsilon}\widetilde{>}\overline{0}$. By hypothesis, there exists a $\widetilde{\delta}\widetilde{\geq}\overline{0}$ such that $\widetilde{x}, \widetilde{y}\widetilde{\in}\widetilde{X}$ and

$$\widetilde{G}(\widetilde{a}, \widetilde{x}, \widetilde{y})\widetilde{<}\widetilde{\delta} \Rightarrow \widetilde{G'}(\widetilde{f(a)}, \widetilde{f(x)}, \widetilde{f(y)})\widetilde{<}\widetilde{\epsilon}.$$

Since $\widetilde{x_n} \to \widetilde{a}$ corresponds to $\widetilde{\delta}\widetilde{>}\overline{0}$, where $\widetilde{\delta}(\lambda) = \delta_\lambda$, there exists a natural number N such that

$$n \geq N \Rightarrow \widetilde{G}(\widetilde{a}, \widetilde{x_n}, \widetilde{x_n})\widetilde{<}\widetilde{\delta} \Rightarrow \widetilde{G}(\widetilde{a}, \widetilde{x_n}, \widetilde{x_n})(\lambda) < \delta_\lambda.$$

Hence we have

$$n \geq N \Rightarrow \widetilde{G'}(\widetilde{f(a)}, \widetilde{f(x_n)}, \widetilde{f(x_n)})\widetilde{<}\widetilde{\epsilon} \Rightarrow \widetilde{G}(\widetilde{f(a)}, \widetilde{f(x_n)}, \widetilde{f(x_n)})(\lambda) < \epsilon_\lambda.$$

This implies that $\widetilde{f(x_n)} \to \widetilde{f(a)}$.

Sufficiency. This can be clearly proved assuming that f is not soft G-continuous at a soft element $\widetilde{a}$. □

Definition 3.5. Let $(\widetilde{X}, \widetilde{G}, E)$ be a soft G-metric space having at least two soft elements. Then $(\widetilde{X}, \widetilde{G}, E)$ is said to poses soft G-Hausdorff property if for any two soft elements $\widetilde{x}$, $\widetilde{y}$ such that $x(\lambda) \neq y(\lambda)$ for all λ $\in$ E, there are two soft open balls $B_{\widetilde{G}}(\widetilde{x}, \widetilde{r})$ and $B_{\widetilde{G}}(\widetilde{y}, \widetilde{r})$ with centers $\widetilde{x}$ and $\widetilde{y}$ respectively, such that

$$B_{\widetilde{G}}(\widetilde{x}, \widetilde{r}) \cap B_{\widetilde{G}}(\widetilde{y}, \widetilde{r}) = \widetilde{\emptyset}.$$

Theorem 3.6. *Every soft G-metric space poses soft G-Hausdorff property.*

Proof. Let $(\widetilde{X}, \widetilde{G}, E)$ be a soft G-metric space having at least two soft elements. Let $\widetilde{x}$, $\widetilde{y}$ be two soft elements in $\widetilde{X}$ such that $x(\lambda) \neq y(\lambda)$ for all $\lambda \in$ E. Then $\widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y})\widetilde{>}\overline{0}$ i.e., $\widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y})(\lambda) > 0$ for all $\lambda \in$ E. Let us choose a real number $\widetilde{\epsilon}(\lambda) = \frac{1}{2}\widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y})(\lambda)$ for each $\lambda \in E$. Suppose that there exists $\widetilde{z}\widetilde{\in}\widetilde{X}$ such that $\widetilde{z} \in B_{\widetilde{G}}(\widetilde{x}, \widetilde{\epsilon})$ and $\widetilde{z} \in B_{\widetilde{G}}(\widetilde{y}, \widetilde{\epsilon})$. Then $\widetilde{G}(\widetilde{z}, \widetilde{z}, \widetilde{x})\widetilde{<}\widetilde{\epsilon}$ and $\widetilde{G}(\widetilde{z}, \widetilde{z}, \widetilde{y})\widetilde{<}\widetilde{\epsilon}$. By (b) of Proposition 2.9, we have

$$\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z})\widetilde{\leq}\widetilde{G}(\widetilde{z}, \widetilde{z}, \widetilde{x}) + \widetilde{G}(\widetilde{z}, \widetilde{z}, \widetilde{y})\widetilde{<}\widetilde{\epsilon} + \widetilde{\epsilon} = 2\widetilde{\epsilon}.$$

It is a contradiction. So, X poses the soft G-Hausdorff property. □

Theorem 3.7. *If a sequence in a soft G-metric space has a limit, then it is unique.*

Proof. the assertion is similar to that of Theorem 3.6. □

Definition 3.8. Let $(\widetilde{X}, \widetilde{G}, E)$ be a soft G-metric space. Let $T : (\widetilde{X}, \widetilde{G}, E) \to (\widetilde{X}, \widetilde{G}, E)$ be a mapping. If there exists a soft element $\widetilde{x_0} \in SE(\widetilde{X})$ such that $\mathrm{T}(\widetilde{x_0}) = \widetilde{x_0}$, then $\widetilde{x_0}$ is called a fixed point of T.

Definition 3.9. Let $(\widetilde{X}, \widetilde{G}, E)$ be a soft G-metric space and $T : (\widetilde{X}, \widetilde{G}, E) \to (\widetilde{X}, \widetilde{G}, E)$ a mapping. For every $x_0 \in SE(\widetilde{X})$, we can construct the sequence $\widetilde{x_n}$ of soft elements by choosing $\widetilde{x_0}$ and continuing by: $\widetilde{x_1} = T(\widetilde{x_0})$, $\widetilde{x_2} = T(\widetilde{x_1}) = T^2(\widetilde{x_0})$,..., $\widetilde{x_n} = T(\widetilde{x_{n-1}}) = T^n(\widetilde{x_0})$. We say that the sequence is constructed by iteration method.

Theorem 3.10. *Let $(\widetilde{X}, \widetilde{G}, E)$ be a soft G-metric space and $T : (\widetilde{X}, \widetilde{G}, E) \to (\widetilde{X}, \widetilde{G}, E)$ a mapping satisfying*

(i) *$\widetilde{G}(T\widetilde{x}, T\widetilde{y}, T\widetilde{z}) \widetilde{\leq} \overline{a}\widetilde{G}(\widetilde{x}, T\widetilde{x}, T\widetilde{x}) + \overline{b}\widetilde{G}(\widetilde{y}, T\widetilde{y}, T\widetilde{y}) + \overline{c}\widetilde{G}(\widetilde{z}, T\widetilde{z}, T\widetilde{z})$ for all $\widetilde{x}, \widetilde{y}, \widetilde{z} \in SE(\widetilde{X})$ where $\overline{0} \widetilde{<} \overline{a} + \overline{b} + \overline{c} \widetilde{<} \overline{1}$*

(ii) *T is soft G-continuous at a soft element $\widetilde{u} \in SE(\widetilde{X})$.*

(iii) *There is an $\widetilde{x} \in SE(\widetilde{X})$ such that $T^n(\widetilde{x})$ has a subsequence $T^{ni}(\widetilde{x})$ soft G-converging to $\widetilde{u}$.*

Then $\widetilde{u}$ is a unique fixed point. (i.e., $T\widetilde{u} = \widetilde{u}$).

Proof. $T^{ni+1}(\widetilde{x})$ is soft G-convergent to $T(\widetilde{u})$ by (ii) and Proposition 3.4. Assume contrary: that $T\widetilde{u} \neq \widetilde{u}$ i.e., $T\widetilde{u}(\lambda_0) \neq \widetilde{u}(\lambda_0)$ for a $\lambda_0 \in E$. Consider the two soft $\widetilde{G}$-balls $B_1 = SS(B_{\widetilde{G}}(\widetilde{u}, \widetilde{\epsilon}))$ and $B_2 = SS(B_{\widetilde{G}}(T\widetilde{u}, \widetilde{\epsilon}))$ where $\widetilde{\epsilon} < \frac{1}{6}\min\{\widetilde{G}(\widetilde{u}, T\widetilde{u}, T\widetilde{u}), \widetilde{G}(\widetilde{u}, \widetilde{u}, T\widetilde{u})\}$. We would have $T^{ni}(\widetilde{x}) \to \widetilde{u}$ and $T^{ni+1}(\widetilde{x}) \to T\widetilde{u}$, so that there would exist an N such that $T^{ni}(\widetilde{x}) \in B_1$ and $T^{ni+1}(\widetilde{x}) \in B_2$. Hence we would have

$$\widetilde{G}(T^{ni}(\widetilde{x}), T^{ni+1}(\widetilde{x}), T^{ni+1}(\widetilde{x})) > \widetilde{\epsilon}, \forall i > N. \tag{3.1}$$

We would have from (i),

$$\begin{aligned} \widetilde{G}(T^{ni+1}(\widetilde{x}), T^{ni+2}(\widetilde{x}), T^{ni+3}(\widetilde{x})) \quad \widetilde{\leq} \quad & \overline{a}\widetilde{G}(T^{ni}(\widetilde{x}), T^{ni+1}(\widetilde{x}), T^{ni+1}(\widetilde{x})) + \\ & \overline{b}\widetilde{G}(T^{ni+1}(\widetilde{x}), T^{ni+2}(\widetilde{x}), T^{ni+2}(\widetilde{x})) + \\ & \overline{c}\widetilde{G}(T^{ni+2}(\widetilde{x}), T^{ni+3}(\widetilde{x}), T^{ni+3}(\widetilde{x})) \end{aligned} \tag{3.2}$$

but, by axiom $(\widetilde{G}_3)$ of the definition of the soft G-metric, we would obtain

$$\widetilde{G}(T^{ni+1}(\widetilde{x}), T^{ni+2}(\widetilde{x}), T^{ni+2}(\widetilde{x})) \widetilde{\leq} \widetilde{G}(T^{ni+1}(\widetilde{x}), T^{ni+2}(\widetilde{x}), T^{ni+3}(\widetilde{x})) \tag{3.3}$$

$$\widetilde{G}(T^{ni+2}(\widetilde{x}), T^{ni+3}(\widetilde{x}), T^{ni+3}(\widetilde{x})) \widetilde{\leq} \widetilde{G}(T^{ni+1}(\widetilde{x}), T^{ni+2}(\widetilde{x}), T^{ni+3}(\widetilde{x})). \tag{3.4}$$

Then

$$\widetilde{G}(T^{ni+1}(\widetilde{x}), T^{ni+2}(\widetilde{x}), T^{ni+3}(\widetilde{x})) \widetilde{\leq} \overline{q}\widetilde{G}(T^{ni}(\widetilde{x}), T^{ni+1}(\widetilde{x}), T^{ni+1}(\widetilde{x})) \tag{3.5}$$

where $\overline{q} = \overline{a}/(\overline{1} - (\overline{b} + \overline{c}))$ and $\overline{q} \widetilde{<} \overline{1}$.

Hence by inequality(3.3) and (3.5), we would get

$$\widetilde{G}(T^{ni+1}(\widetilde{x}), T^{ni+2}(\widetilde{x}), T^{ni+2}(\widetilde{x})) \widetilde{\leq} \overline{q}\widetilde{G}(T^{ni}(\widetilde{x}), T^{ni+1}(\widetilde{x}), T^{ni+1}(\widetilde{x})) \tag{3.6}$$

For $l > j > N$ and by (3.5), we would have

$$\begin{aligned} \widetilde{G}(T^{n_l}(\widetilde{x}), T^{n_l+1}(\widetilde{x}), T^{n_l+1}(\widetilde{x})) \quad & \widetilde{\leq} \quad \overline{q}\widetilde{G}(T^{n_l-1}(\widetilde{x}), T^{n_l}(\widetilde{x}), T^{n_l}(\widetilde{x})) \\ & \widetilde{\leq} \quad \overline{q^2}\widetilde{G}(T^{n_l-2}(\widetilde{x}), T^{n_l-1}(\widetilde{x}), T^{n_l-1}(\widetilde{x})) \\ & \widetilde{\leq} \quad \ldots \widetilde{\leq} \overline{q^{n_l-n_j}}\widetilde{G}(T^{n_j}(\widetilde{x}), T^{n_j+1}(\widetilde{x}), T^{n_j+1}(\widetilde{x})). \end{aligned} \tag{3.7}$$

So, as $l \to \infty$ we would have $\lim \widetilde{G}(T^{n_l}(\widetilde{x}), T^{n_l+1}(\widetilde{x}), T^{n_l+1}(\widetilde{x})) \widetilde{\leq} \overline{0}$ which contradicts (3.1), hence $T\widetilde{u} = \widetilde{u}$.

To prove uniqueness, suppose that for a $\widetilde{v} \neq \widetilde{u}$ we have $T\widetilde{v} = \widetilde{v}$. Then

$$\widetilde{G}(\widetilde{u}, \widetilde{v}, \widetilde{v}) = \widetilde{G}(T(\widetilde{u}), T(\widetilde{v}), T(\widetilde{v})) \widetilde{\leq} \overline{a}\widetilde{G}(\widetilde{u}, T\widetilde{u}, T\widetilde{u}) + (\overline{b} + \overline{c})\widetilde{G}(\widetilde{v}, T\widetilde{v}, T\widetilde{v}) = \overline{0},$$

which implies that $\widetilde{u}$=$\widetilde{v}$. □

4. Conclusion

We introduced soft G-metric spaces as generalizations of G-metric spaces and soft G-metric spaces, and discussed the result about existence and uniqueness of fixed point in the setting of these spaces. We believe that this study will help researchers to upgrade and support the further studies on soft topology, soft metric and fixed point theory to carry out a general framework for their applications in real life.

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