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# On the multilevel nonlinear problem and its convergence algorithms

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**Abstract**

In this paper, applying the geometrical knowledge of Hilbert spaces, we investigate and analyze a system of multilevel split fixed point problems (**MSFP**). New split solution algorithms are introduced and strong convergence theorems for (**MSFP**) are established. At the end of this paper, as an application of our results, we investigate and analyze a system of multilevel split variational inclusion problems (**MSVIP**) and some strong convergence solution for (**MSVIP**) are obtained. These results obtained by this paper improve and develop some known ones in the literature. 

*Keywords:* Multilevel nonlinear problem, nonexpansive mapping, variational inclusion problem, split solution algorithm, strong convergence theorem.
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## 1. Introduction

In this paper, let $H$ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, respectively. If without special note, in this paper, all the spaces denote real Hilbert spaces. A point $x$ is called a fixed point of a mapping $T$ if $Tx = x$ ( when $T$ is a single-valued mapping) or $x \in Tx$ (when $T$ is a set-valued mapping).

Let $T : H_1 \supset Q \to H_1$ and $S : H_2 \supset K \to H_2$ be two nonlinear mappings. $A : H_1 \to H_2$ is a linear and bounded operator. A split common fixed point problem(**(SCFP)**, for short) for $T$ and $S$ is to find,

$$p \in Q \text{ such that } Tp = p \text{ and } SAp = Ap (\text{when } T, S \text{ are single-valued mappings })$$

or

$$p \in Tp \text{ and } Ap \in SAp (\text{when } T, S \text{ are set-valued mappings}).$$

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Generally, a nonlinear problem is solved by an iterative algorithm. Traditionally, many algorithms can only solve a nonlinear problem or a common solution of some nonlinear problems. However, many nonlinear problems don't have common solutions generally. In this case, can we get approximation solutions to different nonlinear problems by an algorithm? In recent years, around this important science problem, many scholars began to study this class of problem. They obtained some interesting works. Especially, applying the geometrical knowledge of Hilbert spaces, professor C. Byrne and professor Y. Censor converted some nonlinear problems into fixed point problems of mappings and put forward the concept of split solution problems and gave some ground-breaking works [2, 6]. Along this idea, professor A. Moudafi et al. boosted vastly the development in this field [7, 8, 20, 21, 23]. All these works provided by them stimulated many correlation researches. Scholars gave some iterative algorithms and obtained some approximation solutions to **(SCFP)** [5, 13, 18, 31, 30, 32]. But, among these works, they only studied two nonlinear problems. So, naturally, an important science problem is whether their researched results can be generalized to more nonlinear problems or not. Based on this consideration and inspired by their works, in this paper, we study and investigate the following multilevel problem:

$$\textbf{(MLSCFP)}\ \text{Find } u \in H_1, v \in H_2, w \in H_3 \text{ such that } T_1u = u, T_2v = v, T_3w = w \\ \text{and } t := Au = Bv = Cw, St = t,$$

where $A : H_1 \to H$, $B : H_2 \to H$ and $C : H_3 \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. $T_i : H_i \to H_i$ and $S : H \to H(i = 1, 2, 3)$ are single-valued mappings. We regard the problem as a multilevel split common fixed point problem(**(MLSCFP)**, for short).

The following examples are some special cases for **(MLSCFP)**.

**Example 1.1.** If $i = 1$ in **(MLSCFP)**, then **(MLSCFP)** reduces to **(SCFP)**.

**Example 1.2.** If $H_1 = H_2$ (or $H_2 = H_3$), then **(MLSCFP)** reduces to find

$$u, v \in H_1, w \in H_2(\text{ or } u \in H_1, v, w \in H_2) \text{ such that } T_1u = u, T_2v = v, T_3w = w \\ \text{and } t := Au = Bv = Cw, St = t,$$

where $A, B : H_1 \to H$, $C : H_2 \to H$(or $A : H_1 \to H$, $B, C : H_2 \to H$) are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. $T_1, T_2 : H_1 \to H_1$, $T_3 : H_2 \to H_2$ (or $T_1 : H_1 \to H_1$, $T_2, T_3 : H_2 \to H_2$) and $S : H \to H$ are single-valued mappings.

**Example 1.3.** If $H_1 = H_2 = H_3$, then **(MLSCFP)** reduces to find

$$u, v, w \in H_1 \text{ such that } T_1u = u,\ T_2v = v,\ T_3w = w \text{ and } t := Au = Bv = Cw,\ St = t,$$

where $A : H_1 \to H$, $B : H_1 \to H$ and $C : H_1 \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. $T_1, T_2, T_3 : H_1 \to H_1$, and $S : H \to H$ are single-valued mappings.

In essence, Example 1.3 is still **(SCFP)**, but it isn't equal to **(SCFP)** completely.

**Example 1.4.** If $H_1 = H_2 = H_3 = H$, then **(MLSCFP)** reduces to find

$$u, v, w \in H_1 \text{ such that } T_1u = u, T_2v = v, T_3w = w \text{ and } t := Au = Bv = Cw, St = t,$$

where $A : H \to H$, $B : H \to H$ and $C : H \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. $T_1, T_2, T_3, S : H \to H$ are single-valued mappings.

In essence, the case in Example 1.4 is a multilevel split common fixed point problem under the same space. But until now, we don't find some researched results of this problem.

**Example 1.5.** If $H_1 = H_2 = H_3 = H$ and $A = B = C$ is an identity operator, then **(MLSCFP)** reduces to find

$$p \in H \text{ such that } T_1p = T_2p = T_3p = Sp = p,$$

where $T_1, T_2, T_3, S : H \to H$ are single-valued mappings.

In essence, the case in Example 1.5 belongs to a class of problems to find a common fixed point for some single-valued nonlinear mappings. It has been investigated by [1, 10, 15, 26, 28].

*Remark* 1.6. Although Example 1.2 and Example 1.4 are the special cases of **(MLSCFP)**, they are different from **(SCFP)** obviously. So, these special cases are also new problems.

For convenience, in this paper, we regard an approximation solution as a weak convergence solution if it is obtained by a weak convergence sequence. Conversely, an approximation solution is called a strong convergence solution if it is obtained by a strong convergence sequence.

In this paper, we will establish strong convergence algorithms for **(MLSCFP)** which implies that some strong convergence solutions of **(MLSCFP)** are obtained. Our results improve and generalize many ones in the literature. At the end of this paper, we apply our results to a multilevel split variational inclusion problem(**(MSVIP)**, for short). Some strong convergence theorems for **(MSVIP)** are established, which implies that many results for variational inclusion problems are generalized.

## 2. Preliminaries

In this section, we recall some known concepts and conclusions.

Let $Q$ is a closed convex subset of a real Hilbert space $H$. A mapping $T : Q \to H$ is called a *nonexpansive mapping* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in Q$. A mapping $T : Q \to H$ is called a *firmly nonexpansive mapping* if $\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle$ for all $x, y \in Q$. A projection operator is a classical firmly nonexpansive mapping. That is, if $P_Q$ denotes the *projection operator* (or *metric projection*) from $H$ onto $Q$, then $P_Q$ satisfies $\|P_Q(x) - P_Q(y)\|^2 \le \langle P_Q(x) - P_Q(y), x - y\rangle, \quad \forall\ x, y \in H$. Besides, the projection operator has an important property that is

$$\|y - P_Q(x)\|^2 + \|x - P_Q(x)\|^2 \le \|x - y\|^2, \text{ for } x \in H \text{ and } y \in Q. \tag{2.1}$$

*Remark* 2.1. Obviously, a firmly nonexpansive mapping must be nonexpansive.

A set-valued mapping $T : H \to 2^H$ is said to be *monotone*, if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply that $\langle f - g, x - y\rangle \ge 0$. Let $D(T)$ and $G(T)$ denote the domain and the graph for $T$, respectively. A monotone mapping $T : H \to H$ is said to be *maximal*, if the graph $G(T)$ of $T$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $T$ is maximal, if and only if for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \ge 0$ for every $(y, g) \in G(T)$ implies that $f \in Tx$.

**Lemma 2.2** ([29]). *For a given $z \in H$, $x \in Q$ satisfies the inequality $\langle x - z, y - x\rangle \ge 0$, $\forall\, y \in Q$ if and only if $x = P_Q(z)$, where $P_Q$ is a projection operator from $H$ onto $Q$.*

**Lemma 2.3.** *The following results are well known. They can be found in* [24] *or* [29].

*(a)* $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$, *$x, y \in H$ and $\lambda \in [0, 1]$;*
*(b)* $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2$, *$x, y \in H$;*
*(c)* $\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2$, *$x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1], \alpha + \beta + \gamma = 1$.*

The following result is crucial in this paper.

**Lemma 2.4** ([11]). *Let $H$ be a real Hilbert space. Let $T : H \to 2^H$ be a set-valued maximal monotone mapping, $\beta > 0$, and let $J^T_\beta$ be a resolvent mapping of $T$ (that is $J^T_\beta = (I + \beta T)^{-1}$).*

*(i)* *For each $\beta > 0$, $J^T_\beta$ is a single-valued and firmly nonexpansive mapping;*
*(ii)* *$D(J^T_\beta) = H$ and $Fix(J^T_\beta) = \{x \in D(T) : 0 \in Tx\}$;*
*(iii)* *$\|x - J^T_\beta x\| \le \|x - J^T_\gamma x\|$ for all $0 < \beta \le \gamma$ and for all $x \in H$;*

(iv) $(I - J_\beta^T)$ *is a firmly nonexpansive mapping for each* $\beta > 0$;

(v) *Suppose that* $T^{-1}(0) \neq \emptyset$, *then* $\|x - J_\beta^T x\| + \|J_\beta^T x - \bar{x}\|^2 \leq \|x - \bar{x}\|^2$ *for each* $x \in H$, *each* $\bar{x} \in T^{-1}(0)$, *and each* $\beta > 0$;

(vi) *Suppose that* $T^{-1}(0) \neq \emptyset$, *then* $\langle x - J_\beta^T x, J_\beta^T x - w\rangle \geq 0$ *for each* $x \in H$ *and each* $w \in T^{-1}(0)$, *each* $\beta > 0$.

*Remark* 2.5. By Lemma 2.4 and Remark 2.1, if $T$ is a set-valued maximal monotone mapping and $J_\beta^T$ denotes a resolvent mapping of $T$, then $J_\beta^T$ is a nonexpansive mapping.

In this paper, the symbols $\to$ and $\rightharpoonup$ are used to denote strong and weak convergence, respectively. $F(T)$ is used to denote a fixed point set of a mapping $T$.

## 3. Strong convergence solutions for (MLSCFP)

In this section, we construct an iteration scheme for **(MLSCFP)** provided that mappings are single-valued and nonexpansive.

**Theorem 3.1.** *Let* $H, H_1, H_2, H_3$ *be real Hilbert spaces. Let* $T_i : H_i \to H_i (i = 1, 2, 3)$ *and* $T : H \to H$ *be nonexpansive mappings.* $A : H_1 \to H$, $B : H_2 \to H$ *and* $C : H_3 \to H$ *are three linear and bounded operators with their adjoint operators* $A^*$, $B^*$ *and* $C^*$, *respectively. Let* $C_1 = H_1$, $Q_1 = H_2$, $K_1 = H_3$, $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by the following algorithm:*

$$\begin{cases} x_1 \in H_1,\ y_1 \in H_2,\ z_1 \in H_3 \quad \textit{chosen arbitrarily}, \\ w_n = T(\frac{Ax_n+By_n+Cz_n}{3}),\ t_n = T_1(x_n - \tau A^*(Ax_n - w_n)), \\ u_n = T_2(y_n - \tau B^*(By_n - w_n)),\ v_n = T_3(z_n - \tau C^*(Cz_n - w_n)), \\ C_{n+1} \times Q_{n+1} \times K_{n+1} = \{(x, y, z) \in C_n \times Q_n \times K_n : \|t_n - x\|^2 + \|u_n - y\|^2 + \|v_n - z\|^2 \\ \qquad\qquad \leq \|x_n - x\|^2 + \|y_n - y\|^2 + \|z_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1),\ y_{n+1} = P_{C_{n+1}}(y_1),\ z_{n+1} = P_{C_{n+1}}(z_1),\ n \in \mathbb{N}, \end{cases} \tag{3.1}$$

*where* $\xi > 0$, $\tau \in (0, \min\{\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}, \frac{1}{\|C\|^2}\})$ *are constants. If*

$$\Omega = \{t = (p, q, r) \in F(T_1) \times F(T_2) \times F(T_3) : Ap = Bq = Cr \in F(T)\} \neq \emptyset,$$

*then the following statements hold.*

(a) $\{(x_n, y_n, z_n)\}$ *converges strongly to* $(x^*, y^*, z^*)$;

(b) $\{w_n\}$ *converges strongly to* $w^*$, *where* $w^* := Ax^* = By^* = Cz^*$, $(x^*, y^*, z^*) \in \Omega$.

*Proof.* Let $t = (p, q, r) \in \Omega$ and $w := Ap = Bq = Cr$. Then

$$\begin{aligned} \|T_1(x_n - \tau A^*(Ax_n - w_n)) - p\|^2 &\leq \|x_n - \tau A^*(Ax_n - w_n) - p\|^2 \\ &= \|x_n - p\|^2 + \|\tau A^*(Ax_n - w_n)\|^2 - 2\tau\langle x_n - p, A^*(Ax_n - w_n)\rangle \\ &= \|x_n - p\|^2 + \|\tau A^*(Ax_n - w_n)\|^2 - 2\tau\langle Ax_n - Ap, Ax_n - w_n\rangle \\ &= \|x_n - p\|^2 + \|\tau A^*(Ax_n - w_n)\|^2 - \tau\|Ax_n - Ap\|^2 - \tau\|Ax_n - w_n\|^2 + \tau\|w_n - Ap\|^2 \\ &= \|x_n - p\|^2 - \tau(1 - \tau\|A^*\|^2)\|Ax_n - w_n\|^2 - \tau\|Ax_n - Ap\|^2 + \tau\|w_n - Ap\|^2 \\ &= \|x_n - p\|^2 - \tau(1 - \tau\|A^*\|^2)\|Ax_n - w_n\|^2 - \tau\|Ax_n - w\|^2 + \tau\|w_n - w\|^2, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \|T_2(y_n - \tau B^*(By_n - w_n)) - q\|^2 &\leq \|y_n - \tau B^*(By_n - w_n) - q\|^2 \\ &= \|y_n - q\|^2 + \|\tau B^*(By_n - w_n)\|^2 - 2\tau\langle y_n - q, B^*(By_n - w_n)\rangle \\ &= \|y_n - q\|^2 + \|\tau B^*(By_n - w_n)\|^2 - 2\tau\langle By_n - Bp, By_n - w_n\rangle \\ &= \|y_n - q\|^2 + \|\tau B^*(By_n - w_n)\|^2 - \tau\|By_n - Bp\|^2 - \tau\|By_n - w_n\|^2 + \tau\|w_n - Bq\|^2 \\ &= \|y_n - q\|^2 - \tau(1 - \tau\|B^*\|^2)\|By_n - w_n\|^2 - \tau\|By_n - Bq\|^2 + \tau\|w_n - Bq\|^2 \\ &= \|y_n - q\|^2 - \tau(1 - \tau\|B^*\|^2)\|By_n - w_n\|^2 - \tau\|By_n - w\|^2 + \tau\|w_n - w\|^2, \end{aligned} \tag{3.3}$$

$$\begin{aligned}\|T_3(z_n-\tau C^*(Cz_n-w_n))-r\|^2 &\leq \|z_n-\tau C^*(Cz_n-w_n)-r\|^2\\ &=\|z_n-r\|^2+\|\tau C^*(Cz_n-w_n)\|^2-2\tau\langle z_n-r, C^*(Cz_n-w_n)\rangle\\ &=\|z_n-r\|^2+\|\tau C^*(Cz_n-w_n)\|^2-2\tau\langle Cz_n-Cr, Cz_n-w_n\rangle\\ &=\|z_n-r\|^2+\|\tau C^*(Cz_n-w_n)\|^2-\tau\|Cz_n-Cr\|^2-\tau\|Cz_n-w_n\|^2+\tau\|w_n-Cr\|^2\\ &=\|z_n-r\|^2-\tau(1-\tau\|C^*\|^2)\|Cz_n-w_n\|^2-\tau\|Cz_n-Cr\|^2+\tau\|w_n-Cr\|^2\\ &=\|z_n-r\|^2-\tau(1-\tau\|C^*\|^2)\|Cz_n-w_n\|^2-\tau\|Cz_n-w\|^2+\tau\|w_n-w\|^2,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\|w_n-w\|^2 &= \|T(\tfrac{Ax_n+By_n+Cz_n}{3})-w\|^2 \leq \|\tfrac{Ax_n+By_n+Cz_n}{3}-w\|^2\\ &\leq \tfrac{1}{3}\|Ax_n-w\|^2+\tfrac{1}{3}\|By_n-w\|^2+\tfrac{1}{3}\|Cz_n-w\|^2.\end{aligned} \tag{3.5}$$

By (3.2)-(3.5), we have the following results:

$$\begin{aligned}&\|T_1(x_n-\tau A^*(Ax_n-w_n))-p\|^2\\ &\quad\leq \|x_n-p\|^2-\tau(1-\tau\|A^*\|^2)\|Ax_n-w_n\|^2-\tau\|Ax_n-w\|^2+\tau\|w_n-w\|^2\\ &\quad\leq \|x_n-p\|^2-\tau(1-\tau\|A^*\|^2)\|Ax_n-w_n\|^2-\tau\|Ax_n-w\|^2\\ &\qquad+\tau\tfrac{1}{3}\|Ax_n-w\|^2+\tau\tfrac{1}{3}\|By_n-w\|^2+\tau\tfrac{1}{3}\|Cz_n-w\|^2,\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\|T_2(y_n-\tau B^*(By_n-w_n))-q\|^2\\ &\quad\leq \|y_n-q\|^2-\tau(1-\tau\|B^*\|^2)\|By_n-w_n\|^2-\tau\|By_n-w\|^2+\tau\|w_n-w\|^2\\ &\quad\leq \|y_n-q\|^2-\tau(1-\tau\|B^*\|^2)\|By_n-w_n\|^2-\tau\|By_n-w\|^2\\ &\qquad+\tau\tfrac{1}{3}\|Ax_n-w\|^2+\tau\tfrac{1}{3}\|By_n-w\|^2+\tau\tfrac{1}{3}\|Cz_n-w\|^2,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\|T_3(z_n-\tau C^*(Cz_n-w_n))-r\|^2\\ &\leq \|z_n-r\|^2-\tau(1-\tau\|C^*\|^2)\|Cz_n-w_n\|^2-\tau\|Cz_n-w\|^2+\tau\|w_n-w\|^2\\ &\leq \|z_n-r\|^2-\tau(1-\tau\|C^*\|^2)\|Cz_n-w_n\|^2-\tau\|Cz_n-w\|^2\\ &\quad+\tau\tfrac{1}{3}\|Ax_n-w\|^2+\tau\tfrac{1}{3}\|By_n-w\|^2+\tau\tfrac{1}{3}\|Cz_n-w\|^2.\end{aligned} \tag{3.8}$$

By Eqs. (3.6)-(3.8), one can easily obtain

$$\begin{aligned}&\|T_1(x_n-\tau A^*(Ax_n-w_n))-p\|^2+\|T_2(y_n-\tau B^*(By_n-w_n))-q\|^2+\|T_3(z_n-\tau C^*(Cz_n-w_n))-r\|^2\\ &\quad\leq \|x_n-p\|^2-\tau(1-\tau\|A^*\|^2)\|Ax_n-w_n\|^2+\|y_n-q\|^2-\tau(1-\tau\|B^*\|^2)\|By_n-w_n\|^2\\ &\qquad+\|z_n-r\|^2-\tau(1-\tau\|C^*\|^2)\|Cz_n-w_n\|^2\\ &\quad=\|x_n-p\|^2+\|y_n-q\|^2+\|z_n-r\|^2-\tau(1-\tau\|A^*\|^2)\|Ax_n-w_n\|^2\\ &\qquad-\tau(1-\tau\|B^*\|^2)\|By_n-w_n\|^2-\tau(1-\tau\|C^*\|^2)\|Cz_n-w_n\|^2,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\|t_n-p\|^2+\|u_n-q\|^2+\|v_n-r\|^2\\ &\quad=\|T_1(x_n-\tau A^*(Ax_n-w_n))-p\|^2+\|T_2(y_n-\tau B^*(By_n-w_n))-q\|^2\\ &\qquad+\|T_3(z_n-\tau C^*(Cz_n-w_n))-r\|^2\\ &\quad\leq \|x_n-p\|^2+\|y_n-q\|^2+\|z_n-r\|^2-\tau(1-\tau\|A^*\|^2)\|Ax_n-w_n\|^2\\ &\qquad-\tau(1-\tau\|B^*\|^2)\|By_n-w_n\|^2-\tau(1-\tau\|C^*\|^2)\|Cz_n-w_n\|^2.\end{aligned} \tag{3.10}$$

That is

$$\|t_n-p\|^2+\|u_n-q\|^2+\|v_n-r\|^2 \leq \|x_n-p\|^2+\|y_n-q\|^2+\|z_n-r\|^2, \tag{3.11}$$

which implies that $t=(p,q,r)\in C_n\times Q_n\times K_n$, $\Omega\subset C_n\times Q_n\times K_n$ and $C_n\times Q_n\times K_n\neq\emptyset$ for all $n\in\mathbb{N}$. Further, the following relationships are obvious:

$$\begin{aligned}&C_{n+1}\subset C_n,\quad Q_{n+1}\subset Q_n,\quad K_{n+1}\subset K_n,\\ &x_{n+1}=P_{C_{n+1}}(x_1)\in C_n, y_{n+1}=P_{Q_{n+1}}(y_1)\in Q_n, z_{n+1}=P_{K_{n+1}}(z_1)\in K_n.\end{aligned} \tag{3.12}$$

Hence, again from Eq. (3.1) we have

$$\|x_{n+1}-x_1\|\leq\|x_1-p\|, \|y_{n+1}-y_1\|\leq\|y_1-q\|,\ \|z_{n+1}-z_1\|\leq\|z_1-r\|, \tag{3.13}$$

which yields that $\{x_n\}, \{y_n\}, \{z_n\}$ are all bounded. On the other hand, by Eq. (2.1), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\|^2 + \|x_1 - x_n\|^2 = \|x_{n+1} - P_{C_n}(x_1)\|^2 + \|x_1 - P_{C_n}(x_1)\|^2 \le \|x_{n+1} - x_1\|^2,\\
&\|y_{n+1} - y_n\|^2 + \|y_1 - y_n\|^2 = \|y_{n+1} - P_{Q_n}(y_1)\|^2 + \|y_1 - P_{Q_n}(y_1)\|^2 \le \|y_{n+1} - y_1\|^2,\\
&\|z_{n+1} - z_n\|^2 + \|z_1 - z_n\|^2 = \|z_{n+1} - P_{K_n}(z_1)\|^2 + \|z_1 - P_{K_n}(z_1)\|^2 \le \|z_{n+1} - z_1\|^2.
\end{aligned} \tag{3.14}$$

So,

$$\|x_1 - x_n\| \le \|x_{n+1} - x_1\|, \|y_1 - y_n\| \le \|y_{n+1} - y_1\|, \|z_1 - z_n\| \le \|z_{n+1} - z_1\|,$$

which shows the limits of $\{\|x_n - x_1\|\}$, $\{\|y_n - y_1\|\}$ and $\{\|z_n - z_1\|\}$ exist. We can also obtain easily $\lim_{n\to\infty} \|x_n - x_m\| = \lim_{n\to\infty} \|y_n - y_m\| = \lim_{n\to\infty} \|z_n - z_m\| = 0$ for $m > n$, since it just needs to replace $n+1$ with $m$ for some $m > n$ in Eqs. (3.12) and (3.14). So, all $\{x_n\}, \{y_n\}, \{z_n\}$ are Cauchy sequences.

Setting $x_n \to x^*$, $y_n \to y^*$, $z_n \to z^*$, we prove $(x^*, y^*, z^*) \in \Omega$. Firstly, we say $\|x_n - T_1 x_n\| \to 0$, $\|y_n - T_2 y_n\| \to 0$, $\|z_n - T_3 z_n\| \to 0$. Thanks to Eqs. (3.1) and (3.12),

$$\|t_n - x_{n+1}\|^2 + \|u_n - y_{n+1}\|^2 + \|v_n - z_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \|y_n - y_{n+1}\|^2 + \|z_n - z_{n+1}\|^2. \tag{3.15}$$

Hence, $\lim_{n\to\infty} \|t_n - x_{n+1}\| = \lim_{n\to\infty} \|u_n - y_{n+1}\| = \lim_{n\to\infty} \|v_n - z_{n+1}\| = 0$. Further, we have

$$\lim_{n\to\infty} \|t_n - x_n\| = \lim_{n\to\infty} \|u_n - y_n\| = \lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{3.16}$$

By Eq. (3.10), we have

$$\begin{aligned}
L\|Ax_n - w_n\|^2 &+ L\|By_n - w_n\|^2 + L\|Cz_n - w_n\|^2\\
&\le \|x_n - p\|^2 + \|y_n - q\|^2 + \|z_n - r\|^2 - \|t_n - p\|^2 - \|u_n - q\|^2 - \|v_n - r\|^2\\
&\le M\|x_n - t_n\| + M\|y_n - u_n\| + M\|z_n - v_n\|,
\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}
&L = \min\{\tau(1 - \tau\|A^*\|^2), \tau(1 - \tau\|C^*\|^2), \tau(1 - \tau\|B^*\|^2)\},\\
&M = \sup_{n\in\mathbb{N}}\{\|x_n - p\| + \|t_n - p\| + \|y_n - q\| + \|u_n - q\| + \|z_n - r\| + \|v_n - r\|\}.
\end{aligned} \tag{3.18}$$

By Eqs. (3.16) and (3.17), we obtain

$$\lim_{n\to\infty} \|Ax_n - w_n\| = 0, \ \lim_{n\to\infty} \|By_n - w_n\| = 0, \ \lim_{n\to\infty} \|Cz_n - w_n\| = 0. \tag{3.19}$$

Further, because $T_1$, $T_2$ and $T_3$ are all nonexpansive mappings, we have

$$\begin{aligned}
\|x_n - T_1 x_n\| &= \|x_n - t_n + t_n - T_1 x_n\| \le \|x_n - t_n\| + \|t_n - T_1 x_n\|\\
&= \|x_n - t_n\| + \|T_1(x_n - \tau A^*(Ax_n - w_n)) - T_1 x_n\|\\
&\le \|x_n - t_n\| + \|\tau A^*(Ax_n - w_n))\|,\\[1em]
\|y_n - T_2 y_n\| &= \|y_n - u_n + u_n - T_2 y_n\| \le \|y_n - u_n\| + \|u_n - T_2 y_n\|\\
&= \|y_n - u_n\| + \|T_2(y_n - \tau B^*(By_n - w_n)) - T_2 y_n\|\\
&\le \|y_n - u_n\| + \|\tau B^*(By_n - w_n))\|,\\[1em]
\|z_n - T_3 z_n\| &= \|z_n - v_n + v_n - T_3 z_n\| \le \|z_n - v_n\| + \|v_n - T_3 z_n\|\\
&= \|z_n - v_n\| + \|T_3(z_n - \tau C^*(Cz_n - w_n)) - T_3 z_n\|\\
&\le \|z_n - v_n\| + \|\tau C^*(Cz_n - w_n)\|.
\end{aligned} \tag{3.20}$$

So, from Eqs. (3.16), (3.19) and (3.20) we have

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0, \ \lim_{n\to\infty} \|y_n - T_2 y_n\| = 0, \ \lim_{n\to\infty} \|z_n - T_3 z_n\| = 0. \tag{3.21}$$

Noting $x_n \to x^*$, $y_n \to y^*$, $z_n \to z^*$, it follows from (3.21) that

$$x^* \in F(T_1), y^* \in F(T_2), z^* \in F(T_3). \tag{3.22}$$

Further, by virtue of Eq. (3.19) and $Ax_n \to Ax^*, By_n \to By^*, Cz_n \to Cz^*$, we have

$$w_n \to w^* : Ax^* = By^* = Cz^*. \tag{3.23}$$

Finally, we prove $w^* \in F(T)$. By Eqs. (3.1) and (3.19), the following inequality holds.

$$\begin{aligned} \|w_n - Tw_n\| &= \|T(\tfrac{Ax_n+By_n+Cz_n}{3}) - Tw_n\| \le \|\tfrac{Ax_n+By_n+Cz_n}{3} - w_n\| \\ &\le \tfrac{1}{3}\|Ax_n - w_n\| + \tfrac{1}{3}\|By_n - w_n\| + \tfrac{1}{3}\|Cz_n - w_n\|. \end{aligned} \tag{3.24}$$

So,

$$\lim_{n\to\infty} \|w_n - Tw_n\| = 0. \tag{3.25}$$

Thus $w^* \in F(T)$. This completes the proof of Theorem 3.1. □

The following convergence theorems can be established by applying Theorem 3.1.

If $H_1 = H_2$ in Theorem 3.1, then Theorem 3.2 holds.

**Theorem 3.2.** *Let $H, H_1, H_3$ be real Hilbert spaces. Let $T_i : H_1 \to H_1 (i = 1, 2)$, $T_3 : H_3 \to H_3$ and $T : H \to H$ be nonexpansive mappings. $A, B : H_1 \to H$ and $C : H_3 \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. Let $C_1 = Q_1 = H_1$, $K_1 = H_3$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the algorithm* (3.1)*. If*

$$\Omega = \{t = (p, q, r) \in F(T_1) \times F(T_2) \times F(T_3) : Ap = Bq = Cr \in F(T)\} \neq \emptyset,$$

*then the following statements hold.*

*(a) $\{(x_n, y_n, z_n)\}$ converges strongly to $(x^*, y^*, z^*)$;*
*(b) $\{w_n\}$ converges strongly to $w^*$, where $w^* := Ax^* = By^* = Cz^*$, $(x^*, y^*, z^*) \in \Omega$.*

If $H_1 = H_2 = H_3$ in Theorem 3.1, we have Theorem 3.3.

**Theorem 3.3.** *Let $H, H_1$ be real Hilbert spaces. Let $T_i : H_1 \to H_1 (i = 1, 2, 3)$ and $T : H \to H$ be nonexpansive mappings. $A, B, C : H_1 \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. Let $C_1 = Q_1 = K_1 = H_1$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the algorithm* (3.1)*. If*

$$\Omega = \{t = (p, q, r) \in F(T_1) \times F(T_2) \times F(T_3) : Ap = Bq = Cr \in F(T)\} \neq \emptyset,$$

*then the following statements hold.*

*(a) $\{(x_n, y_n, z_n)\}$ converges strongly to $(x^*, y^*, z^*)$;*
*(b) $\{w_n\}$ converges strongly to $w^*$, where $w^* := Ax^* = By^* = Cz^*$, $(x^*, y^*, z^*) \in \Omega$.*

If $H = H_1 = H_2$ and $A = I$ in Theorem 3.1, then Theorem 3.4 holds.

**Theorem 3.4.** *Let $H, H_3$ be real Hilbert spaces. Let $T, T_i : H \to H (i = 1, 2)$, $T_3 : H_3 \to H_3$ be nonexpansive mappings. $B : H \to H$, $C : H_3 \to H$ are two linear and bounded operators with their adjoint operators $B^*$ and $C^*$, respectively. Let $C_1 = Q_1 = H$, $K_1 = H_3$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the following algorithm:*

$$\begin{cases} x_1, y_1 \in H, z_1 \in H_3 \quad \textit{chosen arbitrarily}, \\ w_n = T(\frac{x_n+By_n+Cz_n}{3}), t_n = T_1(x_n - \tau(x_n - w_n)), \\ u_n = T_2(y_n - \tau B^*(By_n - w_n)),\ v_n = T_3(z_n - \tau C^*(Cz_n - w_n)), \\ C_{n+1} \times Q_{n+1} \times K_{n+1} = \{(x, y, z) \in C_n \times Q_n \times K_n : \|t_n - x\|^2 + \|u_n - y\|^2 + \|v_n - z\|^2 \\ \qquad\qquad \le \|x_n - x\|^2 + \|y_n - y\|^2 + \|z_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1),\ y_{n+1} = P_{C_{n+1}}(y_1),\ z_{n+1} = P_{C_{n+1}}(z_1),\ n \in \mathbb{N}, \end{cases}$$

*where* $\xi > 0, \tau \in (0, \min\{1, \frac{1}{\|B\|^2}, \frac{1}{\|C\|^2}\})$ *are constants. If*

$$\Omega = \{t = (p, q, r) \in F(T_1) \times F(T_2) \times F(T_3) : p = Bq = Cr \in F(T)\} \neq \emptyset,$$

*then the following statements hold.*

(a) $\{(x_n, y_n, z_n)\}$ *converges strongly to* $(x^*, y^*, z^*)$;

(b) $\{w_n\}$ *converges strongly to* $x^*$, *where* $x^* = By^* = Cz^*$, $(x^*, y^*, z^*) \in \Omega$.

If $H = H_1 = H_2$ and $A = B = I$ in Theorem 3.1, then Theorem 3.5 holds.

**Theorem 3.5.** *Let* $H, H_3$ *be real Hilbert spaces. Let* $T, T_i : H \to H\,(i = 1, 2)$, $T_3 : H_3 \to H_3$ *be nonexpansive mappings.* $C : H_3 \to H$ *is linear and bounded operators with its adjoint operators* $C^*$. *Let* $C_1 = Q_1 = H$, $K_1 = H_3$, $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by the following algorithm:*

$$\begin{cases} x_1, y_1 \in H, z_1 \in H_3 \quad \text{chosen arbitrarily,} \\ w_n = T(\frac{x_n+y_n+Cz_n}{3}), t_n = T_1(x_n - \tau(x_n - w_n)), \\ u_n = T_2(y_n - \tau(y_n - w_n)),\ v_n = T_3(z_n - \tau C^*(Cz_n - w_n), \\ C_{n+1} \times Q_{n+1} \times K_{n+1} = \{(x, y, z) \in C_n \times Q_n \times K_n : \|t_n - x\|^2 + \|u_n - y\|^2 + \|v_n - z\|^2 \\ \qquad\qquad\qquad\qquad \leq \|x_n - x\|^2 + \|y_n - y\|^2 + \|z_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1),\ y_{n+1} = P_{C_{n+1}}(y_1),\ z_{n+1} = P_{C_{n+1}}(z_1),\ n \in \mathbb{N}, \end{cases}$$

*where* $\xi > 0, \tau \in (0, \min\{1, \frac{1}{\|C\|^2}\})$ *are constants. If* $\Omega = \{r \in F(T_3) : p = Cr \in F(T) \bigcap F(T_1) \bigcap F(T_2)\} \neq \emptyset$, *then the following statements hold.*

(a) $\{(x_n, y_n, z_n)\}$ *converges strongly to* $(x^*, x^*, z^*)$;

(b) $\{w_n\}$ *converges strongly to* $x^*$, *where* $x^* = Cz^*$, $z^* \in \Omega$.

*Proof.* $\Omega$ can be rewritten as

$$\Omega = \{t = (p, p, r) \in F(T_1) \times F(T_2) \times F(T_3) : p = Cr \in F(T)\} \neq \emptyset,$$

hence Theorem 3.5 is correct by Theorem 3.1. □

If $H = H_1 = H_2 = H_3$ in Theorem 3.1, we have Theorem 3.6.

**Theorem 3.6.** *Let* $H, H_1$ *be real Hilbert spaces. Let* $T_1, T_2, T_3, T : H \to H$ *be nonexpansive mappings.* $A, B, C : H \to H$ *is linear and bounded operators with their adjoint operators* $A^*, B^*, C^*$, *respectively. Let* $C_1 = Q_1 = K_1 = H$, $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by the algorithm* (3.1). *If*

$$\Omega = \{t = (p, q, r) \in F(T_1) \times F(T_2) \times F(T_3) : Ap = Bq = Cr \in F(T)\} \neq \emptyset,$$

*then the following statements hold.*

(a) $\{(x_n, y_n, z_n)\}$ *converges strongly to* $(x^*, y^*, z^*)$;

(b) $\{w_n\}$ *converges strongly to* $w^*$, *where* $w^* := Ax^* = By^* = Cz^*$, $(x^*, y^*, z^*) \in \Omega$.

If $H = H_1 = H_2 = H_3$ and $A = B = C = I$ in Theorem 3.1, we have

**Theorem 3.7.** *Let* $H$ *be real Hilbert spaces. Let* $T_1, T_2, T_3, T : H \to H$ *be nonexpansive mappings. Let* $C_1 = Q_1 = K_1 = H$, $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by the following algorithm:*

$$\begin{cases} x_1, y_1, z_1 \in H \quad \text{chosen arbitrarily,} \\ w_n = T(\frac{x_n+y_n+z_n}{3}), t_n = T_1(x_n - \tau(x_n - w_n)), \\ u_n = T_2(y_n - \tau(y_n - w_n)),\ v_n = T_3(z_n - \tau(z_n - w_n), \\ C_{n+1} \times Q_{n+1} \times K_{n+1} = \{(x, y, z) \in C_n \times Q_n \times K_n : \|t_n - x\|^2 + \|u_n - y\|^2 + \|v_n - z\|^2 \\ \qquad\qquad\qquad\qquad \leq \|x_n - x\|^2 + \|y_n - y\|^2 + \|z_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1),\ y_{n+1} = P_{C_{n+1}}(y_1),\ z_{n+1} = P_{C_{n+1}}(z_1),\ n \in \mathbb{N}, \end{cases}$$

*where* $\xi > 0, \tau \in (0, 1)$ *are constants. If* $\Omega = F(T_1) \bigcap F(T_2) \bigcap F(T_3) \bigcap F(T) \neq \emptyset$, *then* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *and* $\{w_n\}$ *all converge strongly to* $p$, *where* $p \in \Omega$.

*Proof.* $\Omega$ can be rewritten as

$$\Omega = \{t = (p, q, r) \in F(T_1) \times F(T_2) \times F(T_3) : p = q = r \in F(T)\} \neq \emptyset.$$

Hence, Theorem 3.7 can be deduced by Theorem 3.1. □

*Remark* 3.8. The problem studied by Theorem 3.6 is a multilevel split common fixed point problem under the same space. And the problem studied by Theorem 3.7 is one to find a common fixed point to some nonlinear mappings.

## 4. Applications to multilevel split variational inclusion problems

In this section, we apply the algorithm (3.1) to multilevel split variational inclusion problems.

Let $T : H \to 2^H$ be a set-valued mapping. The *classical variational inclusion problem*(**(CVIP)**, for short) is to find $x^* \in H$ such that $0 \in Tx^*$ or $x^* \in T^{-1}(0)$. The point $x^*$ is also called a zero point of $T$. When $T$ is a set-valued maximal monotone mapping, a well known method to solve the **(CVIP)** is the proximal point algorithm established by the resolvent mapping $J_r^T = (I + rT)^{-1}, r > 0$. For more detail, see the References [9, 11, 19, 25]. Besides of the proximal point algorithm, some other iterative algorithms are also introduced in [4, 16, 17], which are used to find the approximation solution of the **(CVIP)**.

In 2011, A. Moudafi [22] generalized the **(CVIP)** to the *split variational inclusion problems*(**SFVIP** , for short ). The so-called **SFVIP** is the following problem:

$$\text{Find } p \in H_1 \text{ such that } 0 \in T_1(p) \text{ and } 0 \in T_2(Ap)( \text{ or } p \in T_1^{-1}(0), Ap \in T_2^{-1}(0)),$$

where $A : H_1 \to H_2$ is a linear and bounded operator with its adjoint operator $A^*$. $T_1 : H_1 \to 2^{H_1}$ and $T_2 : H_2 \to 2^{H_2}$ are two set-valued maximal monotone mappings.

In [22], prof. Moudafi obtained a weak convergence solution of the **(SFVIP)** by the iterative sequence $\{x_n\}$ defined

$$x_{n+1} = J_\lambda^{T_1}(x_n + \gamma A^*(J_\lambda^{T_2} - I)Ax_n),$$

where $\lambda$ and $\gamma$ are fixed numbers. To obtain a strong convergence solution of the **(SFVIP)**, prof. Chuang [21] introduced the following Halpern-Mann type iterative process with perturbation:

$$x_{n+1} = a_n u + b_n x_n + c_n J_{\beta_n}^{T_1}(x_n - \rho_n A^*(I - J_{\beta_n}^{T_2})Ax_n) + d_n v_n.$$

Then he proved the above sequence $\{x_n\}$ converges strongly to a solution of the **(SFVIP)** under some appropriate conditions.

Very recently, the **(SFVIP)** has been generalized to the *general split variational inclusion problem*(**(GSVIP)**, for short) by prof. Shih-sen Chang et al. The so-called **(GSVIP)** is the following problem:

$$\text{Find } p \in H_1 \text{ such that } 0 \in \bigcap_{i=1}^{\infty} T_i(p),\ 0 \in \bigcap_{i=1}^{\infty} S_i(Ap)(p \in \bigcap_{i=1}^{\infty} T_i^{-1}(0),\ Ap \in \bigcap_{i=1}^{\infty} S_i^{-1}(0)),$$

where $A : H_1 \to H_2$ is a linear and bounded operator with its adjoint operator $A^*$. $T_i : H_1 \to 2^{H_1}$ and $S_i : H_2 \to 2^{H_2}$ $(i \in \mathbb{N})$ are two families of set-valued maximal monotone mappings. Let $\{x_n\}$ be defined by

$$x_{n+1} = \alpha_n x_n + \xi_n f(x_n) + \sum_{i=1}^{\infty} c_{n,i} J_{\beta_{n,i}}^{T_i}(x_n - \gamma_{n,i} A^*(I - J_{\beta_{n,i}}^{S_i})Ax_n).$$

Then Chang and Wang [9] proved that $\{x_n\}$ converges strongly to a solution of the **(GSVIP)** under some appropriate conditions.

We note that both the **(SFVIP)** and the **(GSVIP)** are confined to two real Hilbert spaces. Naturally, an important problem is whether both of them can be generalized to more set-valued maximal monotone mappings under more different Hilbert spaces or not. Based on this question, in this paper, we study and investigate the following new problem:

$$\textbf{(MSVIP)}\ \text{Find } u \in H_1, v \in H_2, w \in H_3 \text{ such that } 0 \in S_1(u), 0 \in S_2(v), 0 \in S_3(w) \\ \text{and } t := Au = Bv = Cw, 0 \in S(t),$$

which can also be rewritten as follows.

$$\textbf{(MSVIP)}\ \text{Find } u \in H_1, v \in H_2, w \in H_3 \text{ such that } u \in S_1^{-1}(0), v \in S_2^{-1}(0), w \in S_3^{-1}(0) \\ \text{and } t := Au = Bv = Cw, t \in S^{-1}(0),$$

where $A : H_1 \to H$, $B : H_2 \to H$ and $C : H_3 \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. $S_i : H_i \to 2^{H_i}$ and $S : H \to 2^H (i = 1, 2, 3)$ are set-valued maximal monotone mappings. We regard the problem as a multilevel split variational inclusion problem(**(MSVIP)**, for short). The following examples are some special cases of **(MSVIP)**.

**Example 4.1.** If $i = 1$ in **(MSVIP)**, then **(MSVIP)** reduces to **(SFVIP)**.

**Example 4.2.** If $H_1 = H_2$(or $H_2 = H_3$), then **(MSVIP)** reduces to find

$$u, v \in H_1, w \in H_2(\text{ or } u \in H_1, v, w \in H_2) \text{ such that } 0 \in S_1(u), 0 \in S_2(v), 0 \in S_3(w) \\ \text{and } t := Au = Bv = Cw, 0 \in S(t),$$

where $A, B : H_1 \to H$, $C : H_2 \to H$(or $A : H_1 \to H$, $B, C : H_2 \to H$) are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. $S_1, S_2 : H_1 \to 2^{H_1}$, $S_3 : H_2 \to 2^{H_2}$ (or $S_1 : H_1 \to 2^{H_1}$, $S_2, S_3 : H_2 \to 2^{H_2}$) and $S : H \to 2^H$ are set-valued maximal monotone mappings.

**Example 4.3.** If $H_1 = H_2 = H_3$, then **(MSVIP)** reduces to find

$$u, v, w \in H_1 \text{ such that } 0 \in S_1(u) \bigcap S_2(v) \bigcap S_3(w) \text{ and } t := Au = Bv = Cw, 0 \in S(t),$$

where $A : H_1 \to H$, $B : H_1 \to H$ and $C : H_1 \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. $S_1, S_2, S_3 : H_1 \to 2^{H_1}$, and $S : H \to 2^H$ are set-valued maximal monotone mappings.

**Example 4.4.** If $H_1 = H_2 = H_3 = H$, then **(MSVIP)** reduces to find

$$u, v, w \in H_1 \text{ such that } 0 \in S_1(u) \bigcap S_2(v) \bigcap S_3(w) \text{ and } t := Au = Bv = Cw, 0 \in S(t),$$

where $A : H \to H$, $B : H \to H$ and $C : H \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. $S_1, S_2, S_3, S : H \to 2^H$ are set-valued maximal monotone mappings.

In essence, the case in Example 4.4 is a multilevel split solution problem under the same space. But until now, we haven't found any researched results related to this problem.

**Example 4.5.** If $H_1 = H_2 = H_3 = H$ and $A = B = C$ is an identity operator, then **(MSVIP)** reduces to find $p \in H$ such that $0 \in S_1(p) \bigcap S_2(p) \bigcap S_3(p) \bigcap S(p)$, where $S_1, S_2, S_3, S : H \to 2^H$ are set-valued maximal monotone mappings.

In essence, the case in Example 4.5 belongs to a class of problems to find a common solution to some variational inclusion problems. It has been investigated by [16, 17].

*Remark* 4.6. Although Examples 4.2 - 4.4 are all the special cases of **(MSVIP)**, they are still different from **(SFVIP)** obviously. So, these special cases are also new problems.

Let $S_1, S_2, S_3, S$ be set-valued maximal monotone mappings, their resolvent mappings are $J_\xi^{S_1}, J_\xi^{S_2}, J_\xi^{S_3}, J_\xi^{S} (\xi > 0)$, respectively. Next, we will give some strong convergence algorithms for **(MSVIP)**.

**Theorem 4.7.** *Let $H, H_1, H_2, H_3$ be real Hilbert spaces. Let $S_i : H_i \to 2^{H_i}(i = 1, 2, 3)$ and $S : H \to 2^H$ be set-valued maximal monotone mappings. $A : H_1 \to H$, $B : H_2 \to H$ and $C : H_3 \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. Let $C_1 = H_1$, $Q_1 = H_2$, $K_1 = H_3$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the following algorithm:*

$$\begin{cases} x_1 \in H_1, y_1 \in H_2, z_1 \in H_3 \quad \text{chosen arbitrarily}, \\ w_n = J_\xi^S(\frac{Ax_n+By_n+Cz_n}{3}), t_n = J_\xi^{S_1}(x_n - \tau A^*(Ax_n - w_n)), \\ u_n = J_\xi^{S_2}(y_n - \tau B^*(By_n - w_n)),\ v_n = J_\xi^{S_3}(z_n - \tau C^*(Cz_n - w_n)), \\ C_{n+1}\times Q_{n+1}\times K_{n+1} = \{(x,y,z)\in C_n\times Q_n\times K_n : \|t_n - x\|^2 + \|u_n - y\|^2 + \|v_n - z\|^2 \\ \qquad\qquad \leq \|x_n - x\|^2 + \|y_n - y\|^2 + \|z_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1),\ y_{n+1} = P_{C_{n+1}}(y_1),\ z_{n+1} = P_{C_{n+1}}(z_1),\ n \in \mathbb{N}, \end{cases} \tag{4.1}$$

*where $\xi > 0, \tau \in (0, \min\{\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}, \frac{1}{\|C\|^2}\})$ are constants. If*

$$\Omega = \{t = (p, q, r) \in S_1^{-1}(0) \times S_2^{-1}(0) \times S_3^{-1}(0) : Ap = Bq = Cr \in S^{-1}(0)\} \neq \emptyset,$$

*then the following statements hold.*

*(a) $\{(x_n, y_n, z_n)\}$ converges strongly to $(p, q, r)$;*
*(b) $\{w_n\}$ converges strongly to $w$, where $w := Ap = Bq = Cr$, $(p, q, r) \in \Omega$.*

*Proof.* By Remark 2.5, all $J_\xi^{S_1}, J_\xi^{S_2}, J_\xi^{S_3}, J_\xi^S$ are nonexpansive. So, by Theorem 3.1, Theorem 4.7 is correct. This completes the proof of Theorem 4.7. □

The following convergence theorems can be established by applying Theorem 4.7.
If $H_1 = H_2$ in Theorem 4.7, then Theorem 4.8 holds.

**Theorem 4.8.** *Let $H, H_1, H_3$ be real Hilbert spaces. Let $S_i : H_1 \to 2^{H_1}(i = 1, 2)$, $S_3 : H_3 \to 2^{H_3}$ and $S : H \to 2^H$ be set-valued maximal monotone mappings. $A, B : H_1 \to H$ and $C : H_3 \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. Let $C_1 = Q_1 = H_1$, $K_1 = H_3$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the algorithm* (4.1) *. If*

$$\Omega = \{t = (p, q, r) \in S_1^{-1}(0) \times S_2^{-1}(0) \times S_3^{-1}(0) : Ap = Bq = Cr \in S^{-1}(0)\} \neq \emptyset,$$

*then the following statements hold.*

*(a) $\{(x_n, y_n, z_n)\}$ converges strongly to $(p, q, r)$;*
*(b) $\{w_n\}$ converges strongly to $w$, where $w := Ap = Bq = Cr$, $(p, q, r) \in \Omega$.*

If $H_1 = H_2 = H_3$ in Theorem 4.7, we have Theorem 4.9.

**Theorem 4.9.** *Let $H, H_1$ be real Hilbert spaces. Let $S_i : H_1 \to 2^{H_1}(i = 1, 2, 3)$ and $S : H \to 2^H$ be set-valued maximal monotone mappings. $A, B, C : H_1 \to H$ are three linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. Let $C_1 = Q_1 = K_1 = H_1$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the algorithm* (4.1)*. If*

$$\Omega = \{t = (p, q, r) \in S_1^{-1}(0) \times S_2^{-1}(0) \times S_3^{-1}(0) : Ap = Bq = Cr \in S^{-1}(0)\} \neq \emptyset,$$

*then the following statements hold.*

*(a) $\{(x_n, y_n, z_n)\}$ converges strongly to $(p, q, r)$;*
*(b) $\{w_n\}$ converges strongly to $w$, where $w := Ap = Bq = Cr$, $(p, q, r) \in \Omega$.*

If $H = H_1 = H_2$ and $A = I$ in Theorem 4.7, then Theorem 4.10 holds.

**Theorem 4.10.** *Let $H, H_3$ be real Hilbert spaces. Let $S, S_i : H \to 2^H\,(i = 1, 2)$, $S_3 : H_3 \to 2^{H_3}$ be set-valued maximal monotone mappings. $B : H \to H$, $C : H_3 \to H$ are two linear and bounded operators with their adjoint operators $B^*$ and $C^*$, respectively. Let $C_1 = Q_1 = H$, $K_1 = H_3$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the following algorithm:*

$$\begin{cases} x_1, y_1 \in H, z_1 \in H_3 \quad \text{chosen arbitrarily}, \\ w_n = J_\xi^S\left(\frac{x_n + By_n + Cz_n}{3}\right), t_n = J_\xi^{S_1}(x_n - \tau(x_n - w_n)), \\ u_n = J_\xi^{S_2}(y_n - \tau B^*(By_n - w_n)),\ v_n = J_\xi^{S_3}(z_n - \tau C^*(Cz_n - w_n)), \\ C_{n+1} \times Q_{n+1} \times K_{n+1} = \{(x, y, z) \in C_n \times Q_n \times K_n : \|t_n - x\|^2 + \|u_n - y\|^2 + \|v_n - z\|^2 \\ \qquad\qquad \le \|x_n - x\|^2 + \|y_n - y\|^2 + \|z_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1),\ y_{n+1} = P_{C_{n+1}}(y_1),\ z_{n+1} = P_{C_{n+1}}(z_1),\ n \in \mathbb{N}, \end{cases}$$

*where $\xi > 0, \tau \in (0, \min\{1, \frac{1}{\|B\|^2}, \frac{1}{\|C\|^2}\})$ are constants. If*

$$\Omega = \{t = (p, q, r) \in S_1^{-1}(0) \times S_2^{-1}(0) \times S_3^{-1}(0) : p = Bq = Cr \in S^{-1}(0)\} \neq \emptyset,$$

*then the following statements hold.*

*(a) $\{(x_n, y_n, z_n)\}$ converges strongly to $(p, q, r)$;*
*(b) $\{w_n\}$ converges strongly to $p$, where $p = Bq = Cr$, $(p, q, r) \in \Omega$.*

If $H = H_1 = H_2$ and $A = B = I$ in Theorem 4.7, then Theorem 4.11 holds.

**Theorem 4.11.** *Let $H, H_3$ be real Hilbert spaces. Let $S, T_i : H \to 2^H\,(i = 1, 2)$, $T_3 : H_3 \to 2^{H_3}$ be set-valued maximal monotone mappings. $C : H_3 \to H$ is linear and bounded operators with its adjoint operators $C^*$. Let $C_1 = Q_1 = H$, $K_1 = H_3$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the following algorithm:*

$$\begin{cases} x_1, y_1 \in H, z_1 \in H_3 \quad \text{chosen arbitrarily}, \\ w_n = J_\xi^S\left(\frac{x_n + y_n + Cz_n}{3}\right), t_n = J_\xi^{T_1}(x_n - \tau(x_n - w_n)), \\ u_n = J_\xi^{T_2}(y_n - \tau(y_n - w_n)),\ v_n = J_\xi^{T_3}(z_n - \tau C^*(Cz_n - w_n)), \\ C_{n+1} \times Q_{n+1} \times K_{n+1} = \{(x, y, z) \in C_n \times Q_n \times K_n : \|t_n - x\|^2 + \|u_n - y\|^2 + \|v_n - z\|^2 \\ \qquad\qquad \le \|x_n - x\|^2 + \|y_n - y\|^2 + \|z_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1),\ y_{n+1} = P_{C_{n+1}}(y_1),\ z_{n+1} = P_{C_{n+1}}(z_1),\ n \in \mathbb{N}, \end{cases}$$

*where $\xi > 0, \tau \in (0, \min\{1, \frac{1}{\|C\|^2}\})$ are constants. If*

$$\Omega = \{t = (p, p, r) \in T_1^{-1}(0) \times T_2^{-1}(0) \times T_3^{-1}(0) : p = Cr \in S^{-1}(0)\} \neq \emptyset,$$

*then the following statements hold.*

*(a) $\{(x_n, y_n, z_n)\}$ converges strongly to $(p, p, r)$;*
*(b) $\{w_n\}$ converges strongly to $p$, where $p = Cr$, $(p, p, r) \in \Omega$.*

If $H = H_1 = H_2 = H_3$ in Theorem 4.7, we have Theorem 4.12.

**Theorem 4.12.** *Let $H, H_1$ be real Hilbert spaces. Let $T_1, T_2, T_3, S : H \to 2^H$ be set-valued maximal monotone mappings. $A, B, C : H \to H$ is linear and bounded operators with their adjoint operators $A^*$, $B^*$ and $C^*$, respectively. Let $C_1 = Q_1 = K_1 = H$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the following algorithm:*

$$\begin{cases} x_1, y_1, z_1 \in H \quad \text{chosen arbitrarily}, \\ w_n = J_\xi^S\left(\frac{Ax_n + By_n + Cz_n}{3}\right), t_n = J_\xi^{T_1}(x_n - \tau A^*(Ax_n - w_n)), \\ u_n = J_\xi^{T_2}(y_n - \tau B^*(By_n - w_n)),\ v_n = J_\xi^{T_3}(z_n - \tau C^*(Cz_n - w_n)), \\ C_{n+1} \times Q_{n+1} \times K_{n+1} = \{(x, y, z) \in C_n \times Q_n \times K_n : \|t_n - x\|^2 + \|u_n - y\|^2 + \|v_n - z\|^2 \\ \qquad\qquad \le \|x_n - x\|^2 + \|y_n - y\|^2 + \|z_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1),\ y_{n+1} = P_{C_{n+1}}(y_1),\ z_{n+1} = P_{C_{n+1}}(z_1),\ n \in \mathbb{N}, \end{cases}$$

*where* $\xi > 0, \tau \in (0, \min\{\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}, \frac{1}{\|C\|^2}\})$ *are constants. If*

$$\Omega = \{t = (p,q,r) \in T_1^{-1}(0) \times T_2^{-1}(0) \times T_3^{-1}(0) : Ap = Bq = Cr \in S^{-1}(0)\} \neq \emptyset,$$

*then the following statements hold.*

*(a)* $\{(x_n, y_n, z_n)\}$ *converges strongly to* $(p,q,r)$*;*
*(b)* $\{w_n\}$ *converges strongly to* $w$*, where* $w := Ap = Bq = Cr$*,* $(p,q,r) \in \Omega$*.*

If $H = H_1 = H_2 = H_3$ and $A = B = C = I$ in Theorem 4.7, we have Theorem 4.13.

**Theorem 4.13.** *Let* $H$ *be real Hilbert spaces. Let* $T_1, T_2, T_3, S : H \to 2^H$ *be set-valued maximal monotone mappings. Let* $C_1 = Q_1 = K_1 = H$*,* $\{x_n\}$*,* $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by the following algorithm:*

$$\begin{cases} x_1, y_1, z_1 \in H \quad \text{chosen arbitrarily,} \\ w_n = J_\xi^S(\frac{x_n+y_n+z_n}{3}), t_n = J_\xi^{T_1}(x_n - \tau(x_n - w_n)), \\ u_n = J_\xi^{T_2}(y_n - \tau(y_n - w_n)), \; v_n = J_\xi^{T_3}(z_n - \tau(z_n - w_n)), \\ C_{n+1} \times Q_{n+1} \times K_{n+1} = \{(x,y,z) \in C_n \times Q_n \times K_n : \|t_n - x\|^2 + \|u_n - y\|^2 + \|v_n - z\|^2 \\ \qquad\qquad \leq \|x_n - x\|^2 + \|y_n - y\|^2 + \|z_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \; y_{n+1} = P_{C_{n+1}}(y_1), \; z_{n+1} = P_{C_{n+1}}(z_1), \; n \in \mathbb{N}, \end{cases}$$

*where* $\xi > 0, \tau \in (0,1)$ *are constants. If* $\Omega = T_1^{-1}(0) \bigcap T_2^{-1}(0) \bigcap T_3^{-1}(0) \bigcap S^{-1}(0) \neq \emptyset$*, then* $\{x_n\}$*,* $\{y_n\}$*,* $\{z_n\}$ *and* $\{w_n\}$ *converge all strongly to* $p$*, where* $p \in \Omega$*.*

*Proof.* $\Omega$ can be rewritten as

$$\Omega = \{t = (p,q,r) \in T_1^{-1}(0) \times T_2^{-1}(0) \times T_3^{-1}(0) : p = q = r \in S^{-1}(0)\} \neq \emptyset.$$

Hence, Theorem 4.13 can be deduced by Theorem 4.7. □

*Remark* 4.14. The problem studied by Theorem 4.12 is a multilevel split variational inclusion problem under the same space. And the problem studied by Theorem 4.13 is one to find a common solution to some variational inclusion problems. Theorem 4.7 and Theorem 4.12 generalize many known results in the literature, for example, [3, 4, 9, 11, 12, 16, 17, 19, 22, 24, 25] and references therein.

## 5. Conclusion

It is well known that many nonlinear problems can be solved by numerical methods. In general, a numerical method only solves a problem. However, if some nonlinear problems can be converted into the fixed point problem of mappings, then by the geometrical knowledge of Hilbert spaces and our experience, these problems can be solved by an iterative method of fixed point of nonlinear mappings, for example[14]. This shows that the iterative method can provide an approach to solve different problems which implies that it is important.

The iterative method is an important method to solve some nonlinear problems. However, for a problem with logic choice, for example, the differential systems with impulsive effects suffered by logic choice, see [27], can we solve such a problem by the iterative method?

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