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Common fixed point theorems for compatible and weakly compatible maps in Menger probabilistic G-metric spaces

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Abstract

In this paper, we prove some new common fixed point theorems for compatible and weakly compatible self-maps under ϕ-contractive conditions in Menger probabilistic G-metric spaces. Our results improve and generalize many comparable results in existing literature. Finally, an example is given as an application of our main results.

Keywords: Menger probabilistic G-metric space, common fixed point, compatible maps, weakly compatible maps.
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1. Introduction

The concept of a probabilistic metric space was introduced and studied by Menger [10, 13]. Since then, many fixed point results for maps satisfying different contractive conditions have been studied [4, 5, 6, 15, 17]. Mustafa and Sims [12] defined the concept of a G-metric space and many fixed point theorems for contractive maps in G-metric spaces have been studied [1, 2]. Zhou et al. [16] defined the notion of a generalized probabilistic metric space (or a PGM-space), which was a generalization of a PM-space and a G-metric space. Since then, some results in Menger probabilistic G-metric spaces have been studied [18].

Jungck [7] initiated the concept of compatible maps in metric spaces and obtained some common fixed point theorems. In [8], the concept of weakly compatible maps was given. Mishra [11] introduced the concept of compatible maps in a Menger space, then, other authors have obtained many fixed point results for compatible maps and weakly compatible maps [3, 9, 14].

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In this paper, we first introduce the notion of compatible maps and weakly compatible maps in Menger probabilistic G-metric spaces. Then, we prove some new common fixed point theorems for compatible maps and weakly compatible maps satisfying ϕ-contractive conditions in Menger probabilistic G-metric spaces with a continuous t-norm Δ of H-type. As an application, we present an example to illustrate the validity of our main results. Our results generalize the results of [3] and many other results in corresponding literatures.

2. Preliminaries

Let $\mathbb{R}$ denote the set of reals, $\mathbb{R}^+$ the nonnegative reals and $\mathbb{Z}^+$ be the set of all positive integers. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$. We will denote by $\mathcal{D}$ the set of all distribution functions, while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied:

(1) $\Delta(a,1) = a$;
(2) $\Delta(a,b) = \Delta(b,a)$;
(3) $a \geq b, c \geq d \Rightarrow \Delta(a,c) \geq \Delta(b,d)$;
(4) $\Delta(a,\Delta(b,c)) = \Delta(\Delta(a,b),c)$.

A typical example of a t-norm is $\Delta_{\min}$, where $\Delta_{\min}(a,b) = \min\{a,b\}$, for each $a,b \in [0,1]$.

Definition 2.1 ([5]). A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$\Delta^1(t) = \Delta(t,t), \qquad \Delta^m(t) = \Delta(t, \Delta^{m-1}(t)), \qquad$ for $m = 2,3,..., t \in [0,1]$.

The t-norm $\Delta_{\min}$ is a trivial example of H-type, but there are other t-norms Δ of H-type with $\Delta \neq \Delta_{\min}$ (see, e.g., [5]).

Definition 2.2 ([12]). Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions:

(G-1) $G(x,y,z) = 0$ if $x = y = z$ for all $x,y,z \in X$;
(G-2) $G(x,x,y) > 0$ for all $x,y \in X$ with $x \neq y$;
(G-3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;
(G-4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ for all $x,y,z \in X$;
(G-5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then G is called a generalized metric or a G-metric on X and the pair (X,G) is a G-metric space.

Definition 2.3 ([16]). A Menger probabilistic G-metric space (shortly, a PGM-space) is a triple (X,G^*,Δ), where X is a nonempty set, Δ is a continuous t-norm and G^* is a mapping from $X \times X \times X$ into $\mathcal{D}$ ($G^*_{x,y,z}$ denotes the value of G^* at the point (x,y,z)) satisfying the following conditions:

(PGM-1) $G^*_{x,y,z}(t) = 1$ for all $x,y,z \in X$ and $t > 0$ if and only if $x = y = z$;
(PGM-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for all $x,y,z \in X$ with $z \neq y$ and $t > 0$;
(PGM-3) $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = ...$(symmetry in all three variables);
(PGM-4) $G^*_{x,y,z}(t+s) \geq \Delta(G^*_{x,a,a}(s), G^*_{a,y,z}(t))$ for all $x,y,z,a \in X$ and $s,t \geq 0$.

Example 2.4. Let (X,G) be a G-metric space. Define a mapping $G^* : X \times X \times X \to \mathcal{D}$ by

$$G^*(x,y,z)(t) = G^*_{x,y,z}(t) = H(t - G(x,y,z)) \tag{2.1}$$

for $x,y,z \in X$ and $t > 0$. Then (X,G^*,Δ) is a Menger PGM-space called the induced Menger PGM-space by (X,G).

Definition 2.5 ([16]). Let (X, G^*, Δ) be a Menger PGM-space and x_0 be any point in X. For any $\epsilon > 0$ and δ with $0 < \delta < 1$, and (ϵ, δ)-neighborhood of x_0 is the set of all points y in X for which $G^*_{x_0,y,y}(\epsilon) > 1-\delta$ and $G^*_{y,x_0,x_0}(\epsilon) > 1-\delta$. We write

$$N_{x_0}(\epsilon, \delta) = \{y \in X : G^*_{x_0,y,y}(\epsilon) > 1-\delta, G^*_{y,x_0,x_0}(\epsilon) > 1-\delta\},$$

which means that $N_{x_0}(\epsilon, \delta)$ is the set of all points y in X for which the probability of the distance from x_0 to y being less than ϵ is greater than $1-\delta$.

Definition 2.6 ([16]). Let (X, G^*, Δ) be a PGM-space, $\{x_n\}$ is a sequence in X.

(1) $\{x_n\}$ is said to be convergent to a point $x \in X$ (write $x_n \to x$), if for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $x_n \in N_{x_0}(\epsilon, \delta)$ whenever $n > M_{\epsilon,\delta}$;

(2) $\{x_n\}$ is called a *Cauchy* sequence, if for any $\epsilon > 0$ and $0 < \delta < 1$,there exists a positive integer $M_{\epsilon,\delta}$ such that $G^*_{x_n,x_m,x_l}(\epsilon) > 1-\delta$ whenever $n, m, l > M_{\epsilon,\delta}$;

(3) (X, G^*, Δ) is said to be complete if every *Cauchy* sequence in X converges to a point in X.

Definition 2.7. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function and $\phi^n(t)$ be the nth iteration of $\phi(t)$,

(i) ϕ is non-decreasing;

(i)$'$ ϕ is strictly increasing;

(ii) ϕ is upper semi-continuous from the right;

(iii) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ and $\phi(t) < t/2$ for all $t > 0$.

We define Φ_0 the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying conditions (i), (ii), (iii) and Φ_1 the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying conditions (i)$'$, (ii), (iii).

Definition 2.8 ([4]). Let $F_1, F_2 \in \mathcal{D}$. The algebraic sum $F_1 \oplus F_2$ of F_1 and F_2 is defined by

$$(F_1 \oplus F_2)(t) = \sup_{t_1+t_2=t} \min\{F_1(t_1), F_2(t_2)\},$$

for all $t \in \mathbb{R}$.

We can analogously prove the following lemma as in Menger PM-spaces.

Lemma 2.9. *Let (X, G^*, Δ) be a Menger PGM-space with Δ a continuous t-norm, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in X and $x, y, z \in X$, if $\{x_n\} \to x$, $\{y_n\} \to x$ and $\{z_n\} \to x$ as $n \to \infty$. Then*

(1) $\liminf_{n\to\infty} G^*_{x_n,y_n,z_n}(t) \geq G^*_{x,y,z}(t)$ *for all* $t > 0$;

(2) $G^*_{x,y,z}(t + o) \geq \limsup_{n\to\infty} G^*_{x_n,y_n,z_n}(t)$ *for all* $t > 0$.

Lemma 2.10 ([5]). *Let (X, G^*, Δ) be a Menger PGM-space. For each $\lambda \in (0, 1]$, define a function G^*_λ by*

$$G^*_\lambda(x, y, z) = \inf_t\{t \geq 0 : G^*_{x,y,z}(t) > 1-\lambda\} \tag{2.2}$$

for any $x, y, z \in X$, then

(1) $G^*_\lambda(x, y, z) < t$ *if and only if* $G^*_{x,y,z}(t) > 1-\lambda$;

(2) $G^*_\lambda(x, y, z) = 0$ *for all* $\lambda \in (0, 1]$ *if and only if* $x = y = z$;

(3) $G^*_\lambda(x, y, z) = G^*_\lambda(y, x, z) = G^*_\lambda(y, z, x) = ...$;

(4) If $\Delta = \Delta_{\min}$, *then for every* $\lambda \in (0, 1]$, $G^*_\lambda(x, y, z) \leq G^*_\lambda(x, a, a) + G^*_\lambda(a, y, z)$.

Lemma 2.11 ([18]). *Let (X, G^*, Δ) be a Menger PGM-space and let $\{G^*_\lambda\}$, $\lambda \in (0, 1]$ be a family of functions on X defined by (2.2). If Δ is a t-norm of H-type, then for each $\lambda \in (0, 1]$, there exists $\mu \in [0, \lambda]$, such that for each $m \in \mathbb{Z}^+$,*

$$G^*_\lambda(x_0, x_m, x_m) \leq \sum_{i=0}^{m-1} G^*_\mu(x_i, x_{i+1}, x_{i+1}),$$

$$G^*_\lambda(x_0, x_0, x_m) \leq \sum_{i=0}^{m-1} G^*_\mu(x_i, x_i, x_{i+1})$$

for all $x_0, x_1, ..., x_m \in X$.

Lemma 2.12 ([18])**.** *Let (X, G^*, Δ) be a Menger PGM-space and Δ be a continuous t-norm. Then the following statements are equivalent:*

(i) the sequence $\{x_n\}$ is a Cauchy sequence;

(ii) for any $\epsilon > 0$ and $0 < \lambda < 1$, there exists $M \in \mathbb{Z}^+$ such that $G^*_{x_n,x_m,x_m}(\epsilon) > 1 - \lambda$, for all $n, m > M$.

3. Main results

In this section, we will establish some new common fixed point theorems for compatible maps and weakly compatible maps in Menger PGM-spaces. To this end, we first introduce the concepts of compatible maps and weakly compatible maps in Menger PGM-spaces.

Definition 3.1. Let S and T be two self-maps of a Menger PGM-space (X, G^*, Δ). S and T are said to be compatible if $G^*_{STx_n,TSx_n,TSx_n}(t) \to 1$ and $G^*_{STx_n,STx_n,TSx_n}(t) \to 1$ for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$ for some $u \in X$.

Definition 3.2. Let S and T be two self-maps of a Menger PGM-space (X, G^*, Δ). S and T are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $Tu = Su$ for some $u \in X$ implies that $TSu = STu$.

The following lemmas will be useful in proving our main results.

Lemma 3.3. *Let $\{y_n\}$ be a sequence in a Menger PGM-space (X, G^*, Δ), where Δ is a t-norm of H-type. If there exists a function $\phi \in \Phi_0$, such that*

$$G^*_{y_n,y_{n+1},y_{n+1}}(\phi(t)) \geq \min\{G^*_{y_{n-1},y_n,y_n}(t), G^*_{y_n,y_{n+1},y_{n+1}}(t)\} \tag{3.1}$$

for all $t > 0$ and $n \in \mathbb{Z}^+$. Then $\{y_n\}$ is a Cauchy sequence in X.

Proof. Let $\{G^*_\lambda\}$, $\lambda \in (0, 1]$ be the family of pseudo-metrics defined by (2.2). For each $\lambda \in (0, 1]$ and $n \in \mathbb{Z}^+$, putting $a_n = G^*_\lambda(y_{n-1}, y_n, y_n)$, we will prove that

$$a_{n+1} \leq \phi(a_n) \tag{3.2}$$

for all $n \in \mathbb{Z}^+$. In fact, since ϕ is upper semi-continuous from the right, for given $\epsilon > 0$ and each a_n, there exist $p_n > a_n$ such that $\phi(p_n) < \phi(a_n) + \epsilon$. By Lemma 2.10, it follows from $p_n > a_n = G^*_\lambda(y_{n-1}, y_n, y_n)$ that $G^*_{y_{n-1},y_n,y_n}(p_n) > 1 - \lambda$ for all $n \in \mathbb{Z}^+$. Thus, by (3.1), we get

$$G^*_{y_n,y_{n+1},y_{n+1}}(\phi(\max\{p_n, p_{n+1}\})) \geq \min\{G^*_{y_{n-1},y_n,y_n}(p_n), G^*_{y_n,y_{n+1},y_{n+1}}(p_{n+1})\} > 1 - \lambda.$$

Similarly by Lemma 2.10, we can have

$$G^*_\lambda(y_n, y_{n+1}, y_{n+1}) < \phi(\max\{p_n, p_{n+1}\}) = \max\{\phi(p_n), \phi(p_{n+1})\} \leq \phi(\max\{a_n, a_{n+1}\}) + \epsilon.$$

By the arbitrariness of ϵ, we have

$$a_{n+1} = G^*_\lambda(y_n, y_{n+1}, y_{n+1}) \leq \phi(\max\{a_n, a_{n+1}\}). \tag{3.3}$$

So, we can infer that $a_{n+1} \leq a_n$. If not, then by (3.3), we have $a_{n+1} \leq \phi(a_{n+1}) < a_{n+1}/2 < a_{n+1}$, which is a contradiction. Hence, (3.3) implies that $a_{n+1} \leq \phi(a_n)$, and (3.2) is proved.

Repeatedly using (3.2), we get

$$G^*_\lambda(y_n, y_{n+1}, y_{n+1}) \leq \phi(G^*_\lambda(y_{n-1}, y_n, y_n)) \leq \cdots \leq \phi^n(G^*_\lambda(y_0, y_1, y_1)) \tag{3.4}$$

for all $n \in \mathbb{Z}^+$. Noting that Δ is a t-norm of H-type. By Lemma 2.11, for each $\lambda \in (0,1]$, there exists $\mu \in (0,\lambda]$, such that

$$G^*_\lambda(y_n, y_m, y_m) \le \sum_{i=n}^{m-1} G^*_\mu(y_i, y_{i+1}, y_{i+1}) \tag{3.5}$$

for all $m, n \in \mathbb{Z}^+$ with $m > n$. Since $\phi \in \Phi_0$, we have $\phi^n(G^*_\mu(y_0, y_1, y_1)) < +\infty$. So for given $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that $\sum_{i=n}^{\infty} \phi^n(G^*_\mu(y_0, y_1, y_1)) < \epsilon$ for all $n \ge n_0$. Thus, it follows from (3.5) that

$$G^*_\lambda(y_n, y_m, y_m) \le \sum_{i=n}^{\infty} \phi^n(G^*_\mu(y_0, y_1, y_1)) < \epsilon$$

for all $n \ge n_0$, which implies that $G^*_{y_n, y_m, y_m}(\epsilon) > 1 - \lambda$ for all $m, n \in \mathbb{Z}^+$ with $m > n \ge n_0$. By Lemma 2.12, $\{y_n\}$ is a *Cauchy* sequence in X. □

Lemma 3.4. *Let (X, G^*, Δ) be a Menger PGM-space and $x, y \in X$. If there exists $\phi \in \Phi_0$, such that*

$$G^*_{x,y,y}(\phi(t) + o) \ge G^*_{x,y,y}(t/2) \tag{3.6}$$

for all $t > 0$. Then $x = y$.

Proof. Let $\lambda \in (0,1]$ and we put $a/2 = G^*_\lambda(x, y, y)$. Since $\phi(\cdot)$ is upper semi-continuous from the right at the point a, for given $\epsilon > 0$, there exists $s > a$ such that $\phi(s) < \phi(a) + \varepsilon$. By Lemma 2.10, $s/2 > G^*_\lambda(x, y, y)$ implies that $G^*_{x,y,y}(s/2) > 1 - \lambda$. So, it follows from (3.6) that

$$G^*_{x,y,y}(\phi(s) + \epsilon) \ge G^*_{x,y,y}(\phi(s) + o) \ge G^*_{x,y,y}(s/2) > 1 - \lambda$$

which implies that $G^*_\lambda(x, y, y) < \phi(s) + \epsilon < \phi(a) + 2\epsilon$. By the arbitrariness of ϵ, we get $a/2 = G^*_\lambda(x, y, y) \le \phi(a)$, thus $a = 0$, i.e., $G^*_\lambda(x, y, y) = 0$. By (2) of Lemma 2.10, we conclude that $x = y$. □

Lemma 3.5. *Let $(X, G^*, \Delta_{\min})$ be a Menger PGM-space. Suppose that there exists a function $\phi \in \Phi$, such that*

$$G^*_{x,y,y}(\phi(t) + o) \ge \min\{G^*_{x,y,y}(t), G^*_{y,x,x}(t)\}. \tag{3.7}$$

Then $x = y$.

Proof. We know that

$$G^*_{y,x,x}(t) = G^*_{x,y,x}(t) \ge \Delta(G^*_{x,y,y}(t/2), G^*_{y,y,x}(t/2)) \ge G^*_{x,y,y}(t/2).$$

Since ϕ is upper-continuous from the right, it follows from (3.7) that

$$G^*_{x,y,y}(\phi(t) + o) \ge \min\{G^*_{x,y,y}(t), G^*_{x,y,y}(t/2)\} = G^*_{x,y,y}(t/2).$$

Then by Lemma 3.4, we can conclude that $x = y$. □

We are now ready to give our main results.

Theorem 3.6. *Let L, M, P and Q be self-maps on a complete Menger PGM-space $(X, G^*, \Delta_{\min})$, If the following conditions are satisfied:*

(i) $L(X) \subseteq Q(X)$, $M(X) \subseteq P(X)$;
(ii) either P or L is continuous;
(iii) (L, P) is compatible and (M, Q) is weakly compatible;
(iv) there exists $\phi \in \Phi_0$, such that

$$\begin{aligned} G^*_{Lx,My,My}(\phi(t)) \ge \min\{&G^*_{Px,Lx,Lx}(t), G^*_{Qy,My,My}(t), G^*_{Px,Qy,Qy}(t), \\ &G^*_{Qy,Lx,Lx}(\beta t), [G^*_{Px,\xi,\xi} \oplus G^*_{\xi,My,My}]((2-\beta)t)\} \end{aligned} \tag{3.8}$$

for all $x, y, \xi \in X$, $\beta \in (0,2)$ and $t > 0$. Then L, M, P and Q have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From condition (i), there exist $x_1, x_2 \in X$, such that $Lx_0 = Qx_1 = y_0$ and $Mx_1 = Px_2 = y_1$. Inductively, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X, such that

$$Lx_{2n} = Qx_{2n+1} = y_{2n}, \quad Mx_{2n+1} = Px_{2n+2} = y_{2n+1}, \quad n = 0, 1, 2, \ldots.$$

• Assume that there exists $\phi \in \Phi_0$, such that (3.8) holds. Putting $x = x_{2n}$, $y = y_{2n+1}$, $\xi = y_{2n}$ in (3.8), we get

$$\begin{aligned} G^*_{y_{2n},y_{2n+1},y_{2n+1}}(\phi(t)) &= G^*_{Lx_{2n},Mx_{2n+1},Mx_{2n+1}}(\phi(t)) \\ &\geq \min\{G^*_{Px_{2n},Lx_{2n},Lx_{2n}}(t), G^*_{Qx_{2n+1},Mx_{2n+1},Mx_{2n+1}}(t), G^*_{Px_{2n},Qx_{2n+1},Qx_{2n+1}}(t), \\ &\quad G^*_{Qx_{2n+1},Lx_{2n},Lx_{2n}}(\beta t), [G^*_{Px_{2n},y_{2n},y_{2n}} \oplus G^*_{y_{2n},Mx_{2n+1},Mx_{2n+1}}]((2-\beta)t)\} \\ &\geq \min\{G^*_{y_{2n-1},y_{2n},y_{2n}}(t), G^*_{y_{2n},y_{2n+1},y_{2n+1}}(t), G^*_{y_{2n-1},y_{2n},y_{2n}}((2-\beta)/2), \\ &\quad G^*_{y_{2n},y_{2n+1},y_{2n+1}}((2-\beta)/2)\}. \end{aligned}$$

Letting $\beta \to 0$, we obtain

$$G^*_{y_{2n},y_{2n+1},y_{2n+1}}(\phi(t)) \geq \min\{G^*_{y_{2n-1},y_{2n},y_{2n}}(t), G^*_{y_{2n},y_{2n+1},y_{2n+1}}(t)\}. \tag{3.9}$$

Similarly, we can prove that

$$G^*_{y_{2n+1},y_{2n+2},y_{2n+2}}(\phi(t)) \geq \min\{G^*_{y_{2n},y_{2n+1},y_{2n+1}}(t), G^*_{y_{2n+1},y_{2n+2},y_{2n+2}}(t)\}. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$G^*_{y_n,y_{n+1},y_{n+1}}(\phi(t)) \geq \min\{G^*_{y_{n-1},y_n,y_n}(t), G^*_{y_n,y_{n+1},y_{n+1}}(t)\}, \quad n = 1, 2, \ldots.$$

By Lemma 3.3, we know that $\{y_n\}$ is a *Cauchy* sequence in X. Since (X, G^*, Δ) is complete, we can assume that $y_n \to z \in X$, and so

$$\lim_{n\to\infty} Lx_{2n} = \lim_{n\to\infty} Px_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Mx_{2n+1} = z. \tag{3.11}$$

Now we prove z is a common fixed point of L, M, P and Q.

Case I. Suppose that P is continuous. By (3.11) we have $PLx_{2n} \to Pz$ and $PPx_{2n} \to Pz$. Noting that (L, P) is compatible, we get $G^*_{LPx_{2n},PLx_{2n},PLx_{2n}}(t) \to 1$ for all $t > 0$, and thus

$$G^*_{LPx_{2n},Pz,Pz}(t) \geq \Delta(G^*_{LPx_{2n},PLx_{2n},PLx_{2n}}(t/2), G^*_{PLx_{2n},Pz,Pz}(t/2)) \to 1, \qquad (n \to \infty)$$

which shows that $LPx_{2n} \to Pz (n \to \infty)$.

We first prove that z is a fixed point of P and L. Putting $x = Px_{2n}$, $y = x_{2n+1}$, $\xi = LPx_{2n}$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} &G^*_{LPx_{2n},Mx_{2n+1},Mx_{2n+1}}(\phi(t)) \\ &\quad \geq \min\{G^*_{PPx_{2n},LPx_{2n},LPx_{2n}}(t), G^*_{Qx_{2n+1},Mx_{2n+1},Mx_{2n+1}}(t), G^*_{PPx_{2n},Qx_{2n+1},Qx_{2n+1}}(t), \\ &\qquad G^*_{Qx_{2n+1},LPx_{2n},LPx_{2n}}(t), [G^*_{PPx_{2n},LPx_{2n},LPx_{2n}} \oplus G^*_{LPx_{2n},Mx_{2n+1},Mx_{2n+1}}](t)\} \\ &\quad \geq \min\{G^*_{PPx_{2n},LPx_{2n},LPx_{2n}}(t), G^*_{Qx_{2n+1},Mx_{2n+1},Mx_{2n+1}}(t), G^*_{PPx_{2n},Qx_{2n+1},Qx_{2n+1}}(t), \\ &\qquad G^*_{Qx_{2n+1},LPx_{2n},LPx_{2n}}(t), G^*_{PPx_{2n},LPx_{2n},LPx_{2n}}(\epsilon), G^*_{LPx_{2n},Mx_{2n+1},Mx_{2n+1}}(t-\epsilon)\}, \end{aligned}$$

where $\epsilon \in (0, t)$. Letting $n \to \infty$, by Lemma 2.9, we get

$$G^*_{Pz,z,z}(\phi(t) + o) \geq \min\{1, 1, G^*_{Pz,z,z}(t), G^*_{z,Pz,Pz}(t), 1, G^*_{Pz,z,z}(t-\epsilon)\}.$$

Letting $\epsilon \to 0$, we get

$$G^*_{Pz,z,z}(\phi(t)+o) \geq \min\{G^*_{Pz,z,z}(t), G^*_{z,Pz,Pz}(t)\},$$

which implies that $Pz = z$ by Lemma 3.3.

Putting $x = \xi = z$, $y = x_{2n+1}$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} G^*_{Lz,Mx_{2n+1},Mx_{2n+1}}(\phi(t)) &\geq \min\{G^*_{Pz,Lz,Lz}(t), G^*_{Qx_{2n+1},Mx_{2n+1},Mx_{2n+1}}(t), G^*_{Pz,Qx_{2n+1},Qx_{2n+1}}(t), \\ &\qquad G^*_{Qx_{2n+1},Lz,Lz}(t), [G^*_{Pz,z,z} \oplus G^*_{z,Mx_{2n+1},Mx_{2n+1}}](t)\} \\ &= \min\{G^*_{Pz,Lz,Lz}(t), G^*_{Qx_{2n+1},Mx_{2n+1},Mx_{2n+1}}(t), G^*_{Pz,Qx_{2n+1},Qx_{2n+1}}(t), \\ &\qquad G^*_{Qx_{2n+1},Lz,Lz}(t), G^*_{z,Mx_{2n+1},Mx_{2n+1}}(t)\}. \end{aligned}$$

Letting $n \to \infty$, by Lemma 2.9, we get

$$G^*_{Lz,z,z}(\phi(t)+o) \geq \min\{G^*_{z,Lz,Lz}(t), 1, 1, G^*_{z,Lz,Lz}(t), 1\} = G^*_{z,Lz,Lz}(t) \geq G^*_{Lz,z,z}(t/2)$$

for all $t > 0$. By Lemma 3.4, we conclude that $Lz = z$. Therefore, z is a common fixed point of P and L.

Next, from $Lz = z$ and (3.11), we can prove that z is also a common fixed point of M and Q, i.e., $Mz = Qz = z$.

In fact, since $L(X) \subseteq Q(X)$, there exists $v \in X$, such that $z = Lz = Qv$. Putting $x = x_{2n}$, $y = v$, $\xi = z$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} G^*_{Lx_{2n},Mv,Mv}(\phi(t)) &\geq \min\{G^*_{Px_{2n},Lx_{2n},Lx_{2n}}(t), G^*_{Qv,Mv,Mv}(t), G^*_{Px_{2n},Qv,Qv}(t), \\ &\qquad G^*_{Qv,Lx_{2n},Lx_{2n}}(t), [G^*_{Px_{2n},z,z} \oplus G^*_{z,Mv,Mv}](t)\} \\ &\geq \min\{G^*_{Px_{2n},Lx_{2n},Lx_{2n}}(t), G^*_{Qv,Mv,Mv}(t), G^*_{Px_{2n},Qv,Qv}(t), \\ &\qquad G^*_{Qv,Lx_{2n},Lx_{2n}}(t), G^*_{Px_{2n},z,z}(\epsilon), G^*_{z,Mv,Mv}(t-\epsilon)\}. \end{aligned}$$

Letting $n \to \infty$, by Lemma 2.9, we get

$$G^*_{z,Mv,Mv}(\phi(t)+o) \geq \min\{1, G^*_{z,Mv,Mv}(t), 1, 1, 1, G^*_{z,Mv,Mv}(t-\epsilon)\} = G^*_{z,Mv,Mv}(t-\epsilon)$$

for all $t > 0$ and $\epsilon \in (0,t)$. Letting $\epsilon \to 0$, we obtain $Mv = z$ by Lemma 3.4. So, we have $Qv = z = Mv$, i.e., v is a coincidence point of Q and M. Since (M,Q) is weakly compatible, we have $MQv = QMv$, and thus $Mz = Qz = z$. Therefore, z is a common fixed point of L, M, P and Q.

Case II. Suppose that L is continuous. Noting that $Lx_{2n} \to z$ and $Px_{2n} \to z$. We have $LLx_{2n} \to Lz$ and $LPx_{2n} \to Lz$. Since (L,P) is compatible, we have $G^*_{PLx_{2n},LPx_{2n},LPx_{2n}}(t) \to 1$ for all $t > 0$. From this fact, it is easy to prove that $PLx_{2n} \to Lz$. Putting $x = Lx_{2n}$, $y = x_{2n+1}$, $\xi = Lz$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} &G^*_{LLx_{2n},Mx_{2n+1},Mx_{2n+1}}(\phi(t)) \\ &\quad\geq \min\{G^*_{PLx_{2n},LLx_{2n},LLx_{2n}}(t), G^*_{Qx_{2n+1},Mx_{2n+1},Mx_{2n+1}}(t), G^*_{PLx_{2n},Qx_{2n+1},Qx_{2n+1}}(t), \\ &\qquad G^*_{Qx_{2n+1},LLx_{2n},LLx_{2n}}(t), [G^*_{PLx_{2n},Lz,Lz} \oplus G^*_{Lz,Mx_{2n+1},Mx_{2n+1}}](t)\} \\ &\quad\geq \min\{G^*_{PLx_{2n},LLx_{2n},LLx_{2n}}(t), G^*_{Qx_{2n+1},Mx_{2n+1},Mx_{2n+1}}(t), G^*_{PLx_{2n},Qx_{2n+1},Qx_{2n+1}}(t), \\ &\qquad G^*_{Qx_{2n+1},LLx_{2n},LLx_{2n}}(t), G^*_{PLx_{2n},Lz,Lz}(\epsilon), G^*_{Lz,Mx_{2n+1},Mx_{2n+1}}(t-\epsilon)\} \end{aligned}$$

for all $t > 0$ and $\epsilon \in (0,t)$. Letting $n \to \infty$, by Lemma 2.9, we get

$$G^*_{Lz,z,z}(\phi(t)+o) \geq \min\{1, 1, G^*_{Lz,z,z}(t), G^*_{z,Lz,Lz}(t), 1, G^*_{Lz,z,z}(t-\epsilon)\}.$$

Letting $\epsilon \to 0$, it follows that

$$G^*_{Lz,z,z}(\phi(t)+o) \geq \min\{G^*_{Lz,z,z}(t), G^*_{z,Lz,Lz}(t)\}$$

for all $t > 0$, which implies that $Lz = z$.

In the same way as in Case I, from $Lz = z$ and (3.11), it is not difficult to prove that $Mz = Qz = z$. Next, we only need to show that $Pz = z$.

Since $M(X) \subseteq P(X)$, there exists $w \in X$, such that $z = Mz = Pw$. Putting $x = w$, $y = x_{2n+1}$, $\xi = z$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} G^*_{Lw,Mx_{2n+1},Mx_{2n+1}}(\phi(t)) \geq \min\{&G^*_{Pw,Lw,Lw}(t), G^*_{Qx_{2n+1},Mx_{2n+1},Mx_{2n+1}}(t), G^*_{Pw,Qx_{2n+1},Qx_{2n+1}}(t), \\ &G^*_{Qx_{2n+1},Lw,Lw}(t), [G^*_{Pw,z,z} \oplus G^*_{z,Mx_{2n+1},Mx_{2n+1}}](t)\} \\ \geq \min\{&G^*_{Pw,Lw,Lw}(t), G^*_{Qx_{2n+1},Mx_{2n+1},Mx_{2n+1}}(t), G^*_{Pw,Qx_{2n+1},Qx_{2n+1}}(t), \\ &G^*_{Qx_{2n+1},Lw,Lw}(t), G^*_{z,Mx_{2n+1},Mx_{2n+1}}(t)\}. \end{aligned}$$

Letting $n \to \infty$, by Lemma 2.9, we get

$$G^*_{Lw,z,z}(\phi(t) + o) \geq \min\{G^*_{z,Lw,Lw}(t), 1, 1, G^*_{z,Lw,Lw}(t), 1\} = G^*_{z,Lw,Lw}(t) \geq G^*_{Lw,z,z}(t/2)$$

for all $t > 0$, which implies that $Lw = z = Pw$. Noting that (L, P) is compatible and so it is also weakly compatible. Hence $Pz = PLw = LPw = Lz = z$. This shows that z is a common fixed point of L, M, P, and Q.

Finally, we show the uniqueness. Let u be another common fixed point of L, M, P, and Q. Then $Lu = Mu = Pu = Qu = u$. Putting $x = \xi = z$, $y = u$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} G^*_{z,u,u}(\phi(t)) &= G^*_{Lz,Mu,Mu}(\phi(t)) \\ &\geq \min\{G^*_{Pz,Lz,Lz}(t), G^*_{Qu,Mu,Mu}(t), G^*_{Pz,Qu,Qu}(t), \\ &\quad G^*_{Qu,Lz,Lz}(t), [G^*_{Pz,z,z} \oplus G^*_{z,Mu,Mu}](t)\} \\ &\geq \min\{1, 1, G^*_{z,u,u}(t), G^*_{u,z,z}(t), G^*_{z,u,u}(t)\} \\ &= \min\{G^*_{z,u,u}(t), G^*_{u,z,z}(t)\}, \end{aligned}$$

which implies that $z = u$. Therefore, z is a unique common fixed point of L, M, P, and Q. □

Taking $\phi(t) = kt$ in Theorem 3.6, where $k \in (0, 1/2)$ is a constant, we get the following consequence.

Corollary 3.7. *Let L, M, P and Q be self-maps on a complete Menger PGM-space $(X, G^*, \Delta_{\min})$, satisfying conditions (i)-(iii) in* Theorem 3.6 *and the following*

(iv)′ there exists a constant $k \in (0, 1/2)$, such that

$$\begin{aligned} G^*_{Lx,My,My}(kt) \geq \min\{&G^*_{Px,Lx,Lx}(t), G^*_{Qy,My,My}(t), G^*_{Px,Qy,Qy}(t), \\ &G^*_{Qy,Lx,Lx}(\beta t), [G^*_{Px,\xi,\xi} \oplus G^*_{\xi,My,My}]((2-\beta)t)\} \end{aligned} \tag{3.12}$$

for all $x, y, \xi \in X$, $\beta \in (0, 2)$ and $t > 0$. Then L, M, P and Q have a unique common fixed point in X.

Corollary 3.8. *Let L, M, P and Q be self-maps on a complete Menger PGM-space $(X, G^*, \Delta_{\min})$, satisfying conditions (i)-(iii) in* Theorem 3.6 *and the following:*

(iv)′′ there exists $k \in (0, 1/2)$ such that

$$\begin{aligned} G^*_{Lp,Mq,Mq}(kx) \geq \min\{&G^*_{Pp,Lp,Lp}(x), G^*_{Qq,Mq,Mq}(x), G^*_{Pp,Qq,Qq}(x), \\ &G^*_{Qq,Lp,Lp}(\beta x), G^*_{Pp,Mq,Mq}((2-\beta)x)\} \end{aligned} \tag{3.13}$$

for all $p, q \in X$, $\beta \in (0, 2)$ and $x > 0$. Then L, M, P and Q have a unique common fixed point in X.

Proof. By (PGM-4), we have

$$G^*_{Px,My,My}(t) \geq \sup_{0<s<t} \min\{G^*_{Px,\xi,\xi}(s), G^*_{\xi,My,My}(t-s)\} = [G^*_{Px,\xi,\xi} \oplus G^*_{\xi,My,My}](t)$$

for all $x, y, \xi \in X$, $t > 0$. Hence, it is not difficult to see that (3.13) implies (3.12). So, the conclusion of Corollary 3.8 follows from Corollary 3.7 immediately. □

Taking $P = Q = I$ (the identity mapping on X), $L = A$, $M = B$ in Theorem 3.6, we have the following corollary.

Corollary 3.9. *Let A and B be self-maps on a complete Menger PGM-space $(X, G^*, \Delta_{\min})$. If there exists a function $\phi \in \Phi_0$, such that*

$$G^*_{Ax,By,By}(\phi(t)) \geq \min\{G^*_{x,Ax,Ax}(t), G^*_{y,By,By}(t), G^*_{x,y,y}(t), G^*_{y,Ax,Ax}(t), [G^*_{x,\xi,\xi} \oplus G^*_{\xi,By,By}]((2-\beta)t)\}$$

for all $x, y, \xi \in X$, $\beta \in (0, 2)$, $t > 0$. Then A and B have a unique common fixed point in X.

Theorem 3.10. *Let A, B, S, T, L and M be self-maps on a complete Menger PGM-space $(X, G^*, \Delta_{\min})$, satisfying the following conditions:*

(i) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$;

(ii) $AB = BA$, $ST = TS$, $LB = BL$, $MT = TM$;

(iii) either AB or L is continuous;

(iv) (L, AB) is compatible and (M, ST) is weakly compatible;

(v) there exists $\phi \in \Phi_0$, such that

$$\begin{aligned} G^*_{Lx,My,My}(\phi(t)) \geq \min\{&G^*_{ABx,Lx,Lx}(t), G^*_{STy,My,My}(t), G^*_{ABx,STy,STy}(t), \\ &G^*_{STy,Lx,Lx}(\beta t), [G^*_{ABx,\xi,\xi} \oplus G^*_{\xi,My,My}]((2-\beta)t)\} \end{aligned} \tag{3.14}$$

for all $x, y, \xi \in X$, $\beta \in (0, 2)$, $t > 0$. Then A, B, S, T, L and M have a unique common fixed point in X.

Proof. Putting $P = AB$ and $Q = ST$, it is easy to see that conditions (i) and (iii)-(v) of the theorem imply conditions (i)-(iv) of Theorem 3.6. Therefore, by Theorem 3.6, L, M, P and Q have a unique common fixed point z in X, i.e.,

$$Lz = Mz = Pz = Qz = z. \tag{3.15}$$

Now, we prove that z is a common fixed point of A and B. By (3.15) and condition (ii), we have $LBz = BLz = Bz$ and $PBz = (AB)Bz = (BA)Bz = B(AB)z = BPz = Bz$. thus, by condition (iv) of Theorem 3.6, putting $x = \xi = Bz$, $y = z$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} G^*_{Bz,z,z}(\phi(t)) &= G^*_{LBz,Mz,Mz}(\phi(t)) \\ &\geq \min\{G^*_{PBz,LBz,LBz}(t), G^*_{Qz,Mz,Mz}(t), G^*_{PBz,Qz,Qz}(t), \\ &\quad G^*_{Qz,LBz,LBz}(t), [G^*_{PBz,Bz,Bz} \oplus G^*_{Bz,Mz,Mz}](t)\} \\ &\geq \min\{1, 1, G^*_{Bz,z,z}(t), G^*_{z,Bz,Bz}(t), G^*_{Bz,z,z}(t)\} \\ &= \min\{G^*_{Bz,z,z}(t), G^*_{z,Bz,Bz}(t)\}, \end{aligned}$$

which implies that $Bz = z$, and so $z = Pz = ABz = Az$. Therefore, z is a common fixed point of A and B.

We next prove that z is also a fixed point of T and S. In fact, by (3.15) and condition (ii), we have $MTz = TMz = Tz$ and $QTz = (ST)Tz = (TS)Tz = TQz = Tz$. Putting $x = \xi = z$, $y = Tz$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} G^*_{z,Tz,Tz}(\phi(t)) &= G^*_{Lz,MTz,MTz}(\phi(t)) \\ &\geq \min\{G^*_{Pz,Lz,Lz}(t), G^*_{QTz,MTz,MTz}(t), G^*_{Pz,QTz,QTz}(t), \\ &\quad G^*_{QTz,Lz,Lz}(t), [G^*_{Pz,z,z} \oplus G^*_{z,MTz,MTz}](t)\} \\ &\geq \min\{1, 1, G^*_{z,Tz,Tz}(t), G^*_{Tz,z,z}(t), G^*_{z,Tz,Tz}(t)\} \\ &= \min\{G^*_{z,Tz,Tz}(t), G^*_{Tz,z,z}(t)\}, \end{aligned}$$

which implies that $Tz = z$, and so $Sz = STz = Qz = z$. This shows that z is also a common fixed point of T and S. Therefore, z is a common fixed point of A, B, S, T, L and M. Since z is a unique common fixed point of P, Q, L and M, it is easy to see that z is also a unique common fixed point of A, B, S, T, L and M. This completes the proof. □

Remark 3.11. We can also obtain Theorem 3.6 by putting $B = T = I$ and $S = Q$ and $A = P$ in Theorem 3.10. Therefore, Theorem 3.6 and Theorem 3.10 are equivalent.

Taking $\phi(t) = kt$ in Theorem 3.10, where $k \in (0, 1/2)$ is a constant, we get the following consequence.

Corollary 3.12. *Let A, B, S, T, L and M be self-maps on a complete Menger PGM-space* $(X, G^*, \Delta_{\min})$*, satisfying the conditions (i)-(iv) of* Theorem 3.10 *and the following:*

(v)′ there exists $k \in (0, 1/2)$*, such that*

$$G^*_{Lx,My,My}(kt) \geq \min\{G^*_{ABx,Lx,Lx}(t), G^*_{STy,My,My}(t), G^*_{ABx,STy,STy}(t), G^*_{STy,Lx,Lx}(\beta t), [G^*_{ABx,\xi,\xi} \oplus G^*_{\xi,My,My}]((2-\beta)t)\} \tag{3.16}$$

for all $x, y, \xi \in X$*,* $\beta \in (0, 2)$ *and* $t > 0$*. Then A, B, S, T, L and M have a unique common fixed point in X.*

Corollary 3.13. *Let A, B, S, T, L and M be self-maps on a complete Menger PGM-space* (X, G^*, Δ)*, satisfying the conditions (i)-(iv) of* Theorem 3.10 *and the following:*

(v)″ there exists $k \in (0, 1/2)$*, such that*

$$G^*_{Lx,My,My}(kt) \geq \min\{G^*_{ABx,Lx,Lx}(t), G^*_{STy,My,My}(t), G^*_{ABx,STy,STy}(t), G^*_{STy,Lx,Lx}(\beta t), G^*_{ABx,My,My}((2-\beta)t)\} \tag{3.17}$$

for all $x, y, \xi \in X$*,* $\beta \in (0, 2)$ *and* $t > 0$*. Then A, B, S, T, L and M have a unique common fixed point in X.*

Proof. We know that $G^*_{ABx,My,My}(t) \geq [G^*_{ABx,\xi,\xi} \oplus G^*_{\xi,My,My}](t)$ for all $x, y, \xi \in X$ and $t > 0$. Hence, it is not difficult to see that (3.17) in Corollary 3.13 implies (3.16) in Corollary 3.12, and so the conclusion of Corollary 3.13 follows from Corollary 3.12 immediately. □

Theorem 3.14. *Let* $P_1, P_2, \ldots, P_{2n}, Q_0$ *and* Q_1 *be self-maps on a complete Menger PGM-space* $(X, G^*, \Delta_{\min})$*, satisfying conditions:*

$$
\begin{aligned}
&(i) && Q_0(X) \subseteq P_1P_3\cdots P_{2n-1}(X), Q_1(X) \subseteq P_2P_4\cdots P_{2n}(X);\\
&(ii) && P_2(P_4\cdots P_{2n}) = (P_4\cdots P_{2n})P_2,\\
& && P_2P_4(P_6\cdots P_{2n}) = (P_6\cdots P_{2n})P_2P_4,\\
& && \vdots\\
& && P_2\cdots P_{2n-1}(P_{2n}) = (P_{2n})P_2\cdots P_{2n-1},\\
& && Q_0(P_4\cdots P_{2n}) = (P_4\cdots P_{2n})Q_0,\\
& && Q_0(P_6\cdots P_{2n}) = (P_6\cdots P_{2n})Q_0,\\
& && \vdots\\
& && Q_0(P_{2n}) = (P_{2n})Q_0,\\
& && P_1(P_3\cdots P_{2n-1}) = (P_3\cdots P_{2n-1})P_1,\\
& && P_1P_3(P_5\cdots P_{2n-1}) = (P_5\cdots P_{2n-1})P_1P_3,\\
& && \vdots\\
& && P_1\cdots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1\cdots P_{2n-3},\\
& && Q_1(P_3\cdots P_{2n-1}) = (P_3\cdots P_{2n-1})Q_1,\\
& && Q_1(P_5\cdots P_{2n-1}) = (P_5\cdots P_{2n-1})Q_1,\\
& && \vdots\\
& && Q_1P_{2n-1} = P_{2n-1}Q_1;
\end{aligned}
$$

(iii) *either* $P_2 \cdots P_{2n}$ *or* Q_0 *is continuous*;

(iv) $(Q_0, P_2 \cdots P_{2n})$ *is compatible and* $(Q_1, P_1 \cdots P_{2n-1})$ *is weakly compatible*;

(v) *there exists* $\phi \in \Phi_0$, *such that*

$$\begin{aligned} G^*_{Q_0x,Q_1y,Q_1y}(\phi(t)) \geq\ & \min\{G^*_{P_2P_4\cdots P_{2n}x,Q_0x,Q_0x}(t), G^*_{P_1P_3\cdots P_{2n-1}y,Q_1y,Q_1y}(t), \\ & G^*_{P_2P_4\cdots P_{2n}x,P_1P_3\cdots P_{2n-1}y,P_1P_3\cdots P_{2n-1}y}(t), G^*_{P_1P_3\cdots P_{2n-1}y,Q_0x,Q_0x}(\beta t), \\ & [G^*_{P_2P_4\cdots P_{2n}x,\xi,\xi} \oplus G^*_{\xi,Q_1y,Q_1y}]((2-\beta)t)\} \end{aligned} \tag{3.18}$$

for all $x, y, \xi \in X$, $\beta \in (0, 2)$ *and* $t > 0$. *Then* $P_1, P_2, \ldots, P_{2n}, Q_0$ *and* Q_1 *have a unique common fixed point in* X.

Proof. The proof is similar to that of Theorem 3.10. □

4. Common fixed point theorems in G-metric spaces

In this section, we shall use the obtained results in Section 3 to get some corresponding fixed point theorems for compatible and weakly compatible maps in G-metric spaces.

Theorem 4.1. *Let* L, M, P *and* Q *be self-maps on a complete* G*-metric space* (X, G) *satisfying the following conditions:*

(i) $L(X) \subseteq Q(X)$, $M(X) \subseteq P(X)$;

(ii) either P *or* L *is continuous;*

(iii) (L, P) *is compatible and* (M, Q) *is weakly compatible;*

(iv) there exists $\phi \in \Phi_1$, *such that for all* $x, y, \in X$,

$$G(Lx, My, My) \leq \phi(G(x, y, y)),$$

where

$$G(x, y, y) = \max\{G(Px, Lx, Lx), G(Qy, My, My), G(Px, Qy, Qy), [G(Qy, Lx, Lx) + G(Px, My, My)]/2\}.$$

Then L, M, P *and* Q *have a unique common fixed point in* X.

Proof. Let $(X, G^*, \Delta_{\min})$ be the induced Menger PGM-space by (X, G), where G^* is defined by (2.1). It is easy to see that conditions (i)-(iii) of Theorem 4.1 imply conditions (i)-(iii) of Theorem 3.6, respectively. It remains to prove that condition (iv) of Theorem 4.1 implies condition (iv) of Theorem 3.10.

By (2.1), we know that the value of each function $G^*_{u,v,v}(\cdot)$ $(u, v \in X)$ in the induced Menger PGM-space only can equal 0 or 1. Hence, without loss of generality, we may assume that

$$\min\{G^*_{Px,Lx,Lx}(t), G^*_{Qy,My,My}(t), G^*_{Px,Qy,Qy}(t), G^*_{Qy,Lx,Lx}(\beta t), [G^*_{Px,\xi,\xi} \oplus G^*_{\xi,My,My}]((2-\beta)t)\} = 1.$$

This implies that

$$G(Px, Lx, Lx) < t, \qquad G(Qy, My, My) < t, \qquad G(Px, Qy, Qy) < t$$

and

$$G(Qy, Lx, Lx) < \beta t, \qquad G(Px, My, My) < (2-\beta)t. \tag{4.1}$$

It follows from (4.1) that implies that $[G(Qy, Lx, Lx)+G(Px, My, My)]/2 < t$. Thus, we have $G(x, y, y) < t$. Noting that ϕ is strictly increasing, by condition (iv) we get $G(Lx, My, My) \leq \phi(G(x, y, y)) < \phi(t)$, which implies that $G^*_{Lx,My,My}(\phi(t)) = 1$. Hence inequality (3.8) holds, i.e., condition (iv) of Theorem 3.6 is satisfied. Therefore, the conclusion follows from Theorem 3.6 immediately. □

In the same way, by Theorem 3.10, we can prove the following theorem.

Theorem 4.2. *Let A, B, S, T, L and M be self-maps on a complete G-metric space (X, G) satisfying the following conditions:*
(i) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$;
(ii) $AB = BA$, $ST = TS$, $LB = BL$, $MT = TM$;
(iii) either AB or L is continuous;
(iv) (L, AB) is compatible and (M, ST) is weakly compatible;
(v) there exists $\phi \in \Phi_1$, such that for all $x, y \in X$,

$$G(Lx, My, My) \leq \phi(G_1(x, y, y)),$$

where

$$\begin{aligned} G_1(x,y,y) &= \min\{G(ABx, Lx, Lx), G(STy, My, My), G(ABx, STy, STy), \\ &\quad [G(STy, Lx, Lx) + G(ABx, My, My)]/2\}. \end{aligned}$$

Then A, B, S, T, L and M have a unique common fixed point in X.

Taking $P = Q = I$, $L = A$, $M = B$ in Theorem 4.1, we get the following consequence.

Corollary 4.3. *Let A and B be self-maps on a complete G-metric space (X, G), if that there exists $\phi \in \Phi_1$, such that*

$$G(Ax, By, By) \leq \phi(\max\{G(x, Ax, Ax), G(y, By, By), G(x, y, y), [G(y, Ax, Ax) + G(x, By, By)]/2\})$$

for all $x, y \in X$. Then A and B have a unique common fixed point in X.

5. An application

In this section, we provide an example to illustrate the validity of Theorem 3.6.

Example 5.1. Let $X = [0, 1]$. Define a function $G^* : X^3 \times [0, 1] \to [0, 1]$ by $G^*_{x,y,z}(t) = \frac{t}{t+G(x,y,z)}$, where $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$, and $t > 0$. It is easy to verify that $(X, G^*, \Delta_{\min})$ is a Menger PGM-space. Define L, M, P and Q: $X \to X$ as follows

$$Lx = \frac{1}{8}x, \qquad Mx = \begin{cases} 0, & x \in [0, \frac{1}{2}), \\ \frac{1}{7}, & x \in [\frac{1}{2}, 1]. \end{cases}$$

$$Px = \frac{1}{2}x, \qquad Qx = \begin{cases} \frac{1}{3}x, & x \in [0, \frac{1}{2}), \\ 1, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Let $\phi(t) = \frac{3}{7}t$. Then it is obvious that $\phi \in \Phi_0$. Consider the sequence $\{x_n = \frac{1}{n}\}$ in X, then

$$\lim_{n\to\infty} Lx_n = \lim_{n\to\infty} Px_n = 0.$$

We can verify that $G^*_{LPx_n,PLx_n,PLx_n}(t) \to 1$ and $G^*_{PLx_n,LPx_n,LPx_n}(t) \to 1$ for all $t > 0$. So (L, P) is compatible. Also, (M, Q) is weakly compatible.

On the other hand, if $x \in [0, 1]$ and $y \in [0, \frac{1}{2})$, then for any $t > 0$, we have

$$\begin{aligned} G^*_{Lx,My,My}(\frac{3}{7}t) &= \frac{t}{t + \frac{7}{12}x} > \frac{t}{t + \frac{3}{4}x} = G^*_{Px,Lx,Lx}(t) \\ &\geq \min\{G^*_{Px,Lx,Lx}(t), G^*_{Qy,My,My}(t), G^*_{Px,Qy,Qy}(t), G^*_{Qy,Lx,Lx}(\beta t), \\ &\quad [G^*_{Px,\xi,\xi} \oplus G^*_{\xi,My,My}]((2 - \beta)t)\}. \end{aligned}$$

If $x \in [0,1]$ and $y \in [\frac{1}{2},1]$, then for any $t > 0$, we have

$$\begin{aligned} G^*_{Lx,My,My}(\frac{3}{7}t) &= \frac{t}{t+\frac{2}{3}-\frac{7}{12}x} > \frac{t}{t+\frac{3}{4}x} = G^*_{Px,Lx,Lx}(t) \\ &\geq \min\{G^*_{Px,Lx,Lx}(t), G^*_{Qy,My,My}(t), G^*_{Px,Qy,Qy}(t), G^*_{Qy,Lx,Lx}(\beta t), \\ &\quad [G^*_{Px,\xi,\xi} \oplus G^*_{\xi,My,My}]((2-\beta)t)\}. \end{aligned}$$

Thus, all the conditions of Theorem 3.6 are satisfied and 0 is the unique common fixed point in X.

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