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Polynomiography via an iterative method corresponding to Simpson's $\frac{1}{3}$ rule

Shin Min Kang[a], Shahid M. Ramay[b], Muhmmad Tanveer[c], Waqas Nazeer[d,*]

[a] *Department of Mathematics and the RINS, Gyeongsang National University, Jinju 52828, Korea.*
[b] *College of Science, Physics and Astronomy Department, King Saud University Riyadh, Saudi Arabia.*
[c] *Department of Mathematics and Statistics, University of Lahore, Lahore 54000, Pakistan.*
[d] *Division of Science and Technology, University of Education, Lahore 54000, Pakistan.*

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Abstract

The aim of this paper is to present some artwork produced via polynomiography of a few complex polynomials and a few special polynomials arising in science as well as a few considered to arrive at beautiful but anticipated designs. In this paper an iterative method corresponding to Simpson's $\frac{1}{3}$ rule is used instead of Newton's method. The word "polynomiography" coined by Kalantari for that visualization process. The images obtained are called polynomiographs. Polynomiographs have importance for both the art and science aspects. By using an iterative method corresponding to Simpson's $\frac{1}{3}$ rule, we obtain quite new nicely looking polynomiographs that are different from Newton's method. Presented examples show that we obtain very interesting patterns for complex polynomial equations, permutation matrices, doubly stochastic matrices, Chebyshev polynomial, polynomial arising in physics and Alexander polynomial in knot theory. We believe that the results of this paper enrich the functionality of the existing polynomiography software.

Keywords: Polynomiography, Newton's method, Simpson's $\frac{1}{3}$ rule.
2010 MSC: 30C10, 65H05.

1. Introduction

Polynomials are one of the most significant objects in many fields of mathematics. Polynomial root-finding has played a key role in the history of mathematics. It is one of the oldest and most deeply studied mathematical problems. In 2000 BC Babylonians solved quadratic equation (quadratics). The problem of

*Corresponding author
Email addresses: smkang@gnu.ac.kr (Shin Min Kang), schaudhry@ksu.edu.sa (Shahid M. Ramay), tanveer.7955180@yahoo.com (Muhmmad Tanveer), nazeer.waqas@ue.edu.pk (Waqas Nazeer)

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polynomial roots finding was known since the Sumerians 3000 years B.C. Seventeen centuries later Euclid solved quadratics with geometrical construction. In 1539 Cardan gave complete solution to cubics. In 1670 Isaac Newton proposed the method for finding the roots of polynomial. About seventy years later Lagrange showed that polynomial of degree 5 or higher cannot be solved by the methods used for quadratics, cubics, and quartics. In 1799 Gauss proved the Fundamental Theorem of Algebra. 27 years later Abel proved the impossibility of generally solving equations of degree higher than 4. General root-finding method has to be iterative and can only be done approximately. The strange and unpredictable chaotic behavior of Newton's method in the complex plane initially investigated by Cayley in 1879 while applying Newton's method to the equation $z^3 - 1 = 0$, which is known as Cayley's problem [2]. This problem was solved by Julia [4] in 1918 and then Mandelbrot [11]. The last interesting contribution to the polynomials root finding history was made by Kalantari [7], who introduced the polynomiography. As a method which generates nice looking graphics, it was patented by Kalantari [5] in 2005. Polynomiography is defined to be "the art and science of visualization in approximation of the zeros of complex polynomials, via fractal and non fractal images created using the mathematical convergence properties of iteration functions" [7]. An individual image is called a "polynomiographs". Polynomiography combines both art and science aspects.

Polynomiography gives a new way to solve the ancient problem by using new algorithms and computer technology. Polynomiography is based on the use of one or an infinite number of iteration methods formulated for the purpose of approximation of the root of polynomials, for example, Newton's method, Halley's method, etc. The word "fractal", which partially appeared in the definition of polynomiography, was coined by the famous mathematician Mandelbrot [11]. Both fractal images and polynomiographs can be obtained via different iterative schemes. Fractals are self-similar and have typical structure and independent of scale. On the other hand, polynomiographs are quite different. The "polynomiograph" can be controlled by the shape and designed in a more predictable way by using different iteration methods to the infinite variety of complex polynomials. Generally, fractals and polynomiographs belong to different classes of graphical objects. Polynomiography has diverse applications in math, science, education, art and design. According to Fundamental Theorem of Algebra, any complex polynomial with complex coefficients $\{a_n, a_{n-1}, ..., a_1, a_0\}$:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \tag{1.1}$$

or by its zeros (roots) $\{r_1, r_2, ..., r_{n-1}, r_n\}$:

$$p(z) = (z - r_1)(z - r_2) \cdots (z - r_n) \tag{1.2}$$

of degree n has n roots (zeros) which may or may not be distinct. The degree of polynomial describes the number of basins of attraction and placing roots on the complex plane manually localization of basins can be controlled.

Usually, polynomiographs are colored based on the number of iterations needed to obtain the approximation of some polynomial roots with a given accuracy and a chosen iteration method. The description of polynomiography, its theoretical background and artistic applications are described in [5, 6, 7, 9, 10, 12, 13, 15, 16].

In [10] Kotarski et al. used Mann and Ishikawa iterations instead of the standard Picard iteration to obtain some generalization of Kalantari's polynomiography and presented some polynomiographs for the cubic equation $z^3 - 1 = 0$, permutation, and double stochastic matrices. Kang et al. [9], using the ideas from [10], have used the S-iteration in polynomiography. Earlier, the other types of iterations have been used in [15] for superfractals and in [13] for fractals generated by iterative function systems. Julia sets and Mandelbrot sets [1] and the antifractals [14] have been also investigated using Noor iteration instead of the standard Picard iteration. Gdawiec et al. [3] have used different iterations, different convergence tests, and different coloring to obtain a great variety of polynomiographs.

In this paper we use an iterative method corresponding to Simpson's $\frac{1}{3}$ rule instead of Newton method and iteration methods from Basic Family of iterations [8] to obtain nice looking polynomiographs that are quite new and interesting from the artistic point of view. Real parts of the parameters alter symmetry, whereas imaginary ones cause asymmetric twisting of polynomiographs.

The paper is organized as follows. In section 2 the theory of Newton's method and Basic Family of iterations is presented. Section 3 presents an iterative method corresponding to Simpson's $\frac{1}{3}$ rule. In section 4 convergence test is defined. Section 5 presents many applications of polynomiographs obtained experimentally as the result of the proposed iterative method corresponding to Simpson's $\frac{1}{3}$ rule for complex equations, permutation matrices, doubly stochastic matrices and for some special polynomials. The last section, section 6 concludes the paper and shows the future directions.

2. Newton's method and basic family of iterations

It is of great importance to solve equations of the form

$$f(x) = 0, \tag{2.1}$$

in many applications in mathematics, physics, chemistry, and of course in the computation of some important mathematical constants or functions like square roots. Newton's method for finding the roots of a complex polynomial p is given by the formula:

$$z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)}, \quad n = 0, 1, 2, ..., \tag{2.2}$$

where $z_o \in \mathbb{C}$ is a starting point.

Fractals by Newton method can be seen at boundaries of basins. In [12, 16] basin boundaries of Newton method (2.2) are discussed. Generalizations procedures for approximation of all the roots of complex polynomials, making use of a fundamental family of iteration functions given in [5, 7]. This family is called "Basic Family". It is represented as $\{B_m(z)\}_{m=2}^{\infty}$. Let $p(z)$ be a polynomial of degree $n \geq 2$ with complex coefficients. Define $D_o(z) = 1$, and for each natural number $m \geq 1$, Let

$$D_m(z) = \det \begin{bmatrix} p'(z) & \frac{p''(z)}{2!} & \cdots & \frac{p^{(m-1)}(z)}{(m-1)!} & \frac{p^{(m)}(z)}{(m)!} \\ p(z) & p'(z) & \ddots & \ddots & \frac{p^{(m-1)}(z)}{(m-1)!} \\ 0 & p(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & p'(z) & \frac{p''(z)}{2!} \\ 0 & 0 & \cdots & p(z) & p'(z) \end{bmatrix}. \tag{2.3}$$

Then Basic Family of iterations is defined as:

$$B_m(z) = z - p(z)\frac{D_{m-2}(z)}{D_{m-1}(z)}, \quad m = 2, 3, 4, ... \tag{2.4}$$

It is clear that the first member of the sequence, $B_2(z)$ is the Newton method and $B_3(z)$ is the Halley method. In this paper, we are only interested in one method to generate polynomiographs, "an iterative method corresponding to Simpson's $\frac{1}{3}$ rule", which is demonstrated as follows:

3. Iterative methods corresponding to Simpson's $\frac{1}{3}$ rule

During the last century, the different numerical techniques for solving nonlinear equation $f(x) = 0$ have been successfully applied. Now we define

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'(\frac{x_n+y_n}{2}) + f'(y_n)},$$
$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, ...$$

This is so-called an iterative method corresponding to Simpson's $\frac{1}{3}$ rule for solving nonlinear equations. Let $p(z)$ be the complex polynomial. Then

$$z_{n+1} = z_n - \frac{6p(z_n)}{p'(z_n) + 4p'(\frac{z_n+y_n}{2}) + p'(y_n)},$$
$$y_n = z_n - \frac{p(z_n)}{p'(z_n)}, \quad n = 0, 1, 2, ..., \tag{3.1}$$

where $z_o \in \mathbb{C}$ is a starting point, is so-called an iterative method corresponding to Simpson's $\frac{1}{3}$ rule for solving nonlinear complex equations. The sequence $\{z_n\}_{n=0}^{\infty}$ is called the orbit of the point z_o converges to a root z^* of p then, we say that z_o is attracted to z^*. A set of all such starting points for which $\{z_n\}_{n=0}^{\infty}$ converges to root z^* is called the basin of attraction of z^*.

4. Convergence test

In the numerical algorithms that are based on iterative processes we need a stop criterion for the process, that is, a test that tells us that the process has converged or it is very near to the solution. This type of test is called a convergence test. Usually, in the iterative processes that use a feedback, like the root finding methods, the standard convergence test has the following form:

$$|z_{n+1} - z_n| < \varepsilon, \tag{4.1}$$

where z_{n+1} and z_n are two successive points in the iteration process and $\varepsilon > 0$ is a given accuracy. In this paper we also use the stop criterion (4.1).

5. Applications of an iterative method corresponding to Simpson's $\frac{1}{3}$ rule in polynomiography

The applications of an iterative method corresponding to Simpson's $\frac{1}{3}$ rule for solving nonlinear complex equations perturbs the shape of polynomial basins and makes the polynomiographs look more "fractal". The aim of using an iterative method corresponding to Simpson's $\frac{1}{3}$ rule for solving nonlinear complex equations is to create images that are quite new, different from images by Newton's method and interesting from the aesthetic point of view.

In this section we present some examples of polynomiographs for different complex polynomials equation $p(z) = 0$, permutation matrices, doubly stochastic matrices and some special polynomials. The different colors of a images depend upon number of iterations to reach a root with given accuracy $\varepsilon = 0.00001$. One can obtain infinitely many nice looking polynomiographs by changing parameter k,where k is the upper bound of the number of iterations, but we fixed k as $k = 15$ in this paper.

5.1. Polynomiograph for $z^3 - 1 = 0$

Complex polynomial equation $z^3 - 1 = 0$, has three roots: $1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$. The polynomiograph is presented in the following figure with three distinct basins of attraction to the three roots of the polynomial $z^3 - 1 = 0$.

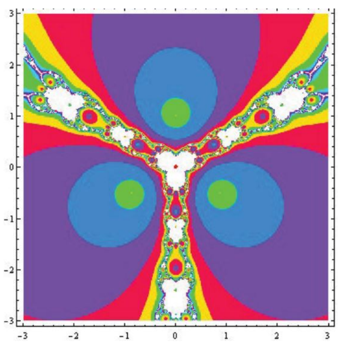

Figure 1. Polynomiography for $z^3 - 1 = 0$

5.2. Polynomiograph for $z^3 + 3 = 0$

The polynomiograph is presented in the following figure with three distinct basins of attraction to the three roots of the polynomial $z^3 + 3 = 0$.

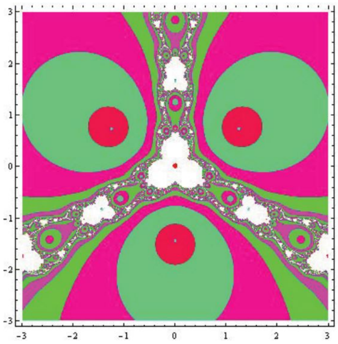

Figure 2. Polynomiography for $z^3 + 3 = 0$

5.3. Polynomiograph for $z^4 - 1 = 0$

The polynomiographs is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 - 1 = 0$.

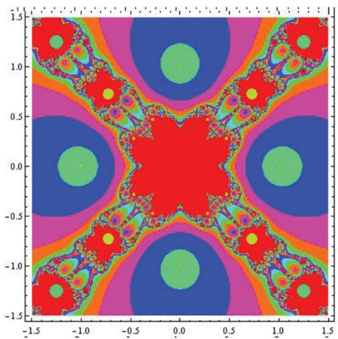

Figure 3. Polynomiograph for $z^4 - 1 = 0$

5.4. Polynomiograph for $z^4 + 4 = 0$

The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 + 4 = 0$.

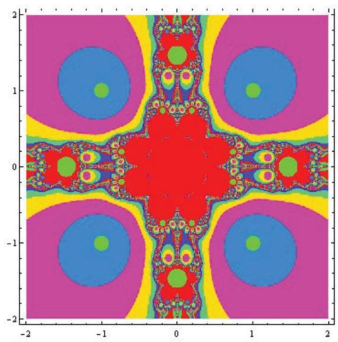

Figure 4. Polynomiograph for $z^4 + 4 = 0$

5.5. Polynomiograph for $z^5 - 1 = 0$

The polynomiograph is presented in the following figure with five distinct basins of attraction to the five roots of the polynomial $z^5 - 1 = 0$.

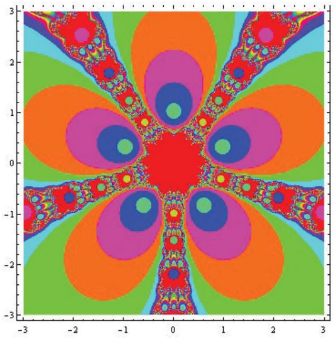

Figure 5. Polynomiograph for $z^5 - 1 = 0$ (Flower with five padels)

5.6. Polynomiograph of a Chebyshev polynomial

Polynomiograph here is that of a fifth degree Chebyshev polynomial using an iterative method corresponding to Simpson's $\frac{1}{3}$ rule. The polynomiograph is presented in the following figure with five distinct basins of attraction to the five roots of the polynomial $16z^5 - 20z^3 + 5z = 0$.

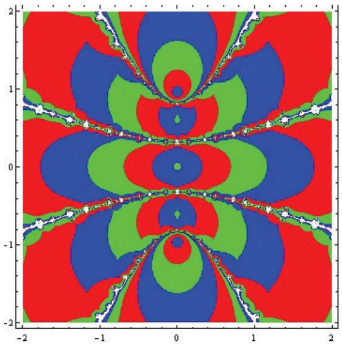

Figure 6. Polynomiography for Chebyshev polynomial (Sparrow)

5.7. Polynomiograph with a Polynomial in Knot Theory

The next image is based on, Alexander polynomial $z^6 - 3z^5 + 4z^4 - 5z^3 + 4z^2 - 3z + 1 = 0$ arising in knot theory.

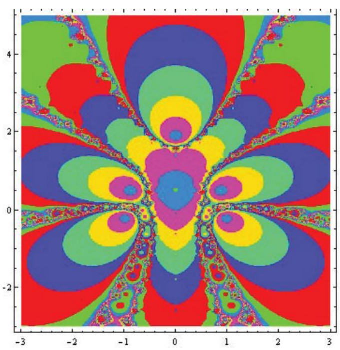

Figure 7. Polynomiograph for Alexander polynomial (Butterfly)

5.8. Polynomiograph for $z^7 + z^2 - 1 = 0$

The polynomiograph is presented in the following figure with seven distinct basins of attraction to the seven roots of the polynomial $z^7 + z^2 - 1 = 0$.

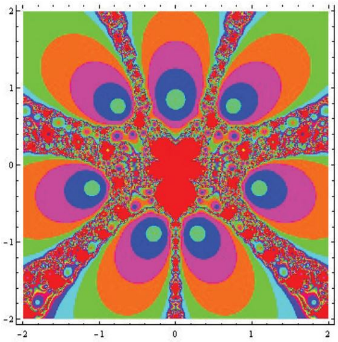

Figure 8. Polynomiograph for $z^7 + z^2 - 1 = 0$ (Carpet)

5.9. Polynomiographs for permutation matrices

An $n \times n$ matrix $\Pi = (\pi_{ij})$ is a matrix where rows and columns form a permutation of the identity matrix. To each permutation matrix Π, we associate a complex polynomial defined as follows. First, we set θ_{ij} to be the complex number associated to the location (i, j).

$$\theta_{ij} = i + j\mathbf{i}, \quad \mathbf{i} = \sqrt{-1}.$$

Now given the matrix Π we define a corresponding matrix

$$\overline{\Pi} = (\overline{\pi}_{ij}), \quad \overline{\pi}_{ij} = \pi_{j,(n+1-i)}.$$

This matrix is analogues to the transpose, except that the i-th row of Π corresponds to the i-th column of $\overline{\Pi}$ corresponds to the bottom up. Finally, the matrix $\Pi = (\pi_{ij})$ can be associated to a complex polynomial $p_{\Pi}(z)$ and can be defined as

$$p_{\Pi}(z) = \prod_{\overline{\pi}_{ij}=1} (z - \theta_{ij}),$$

a polynomial of degree n associated with the complex permutation polynomial corresponding to Π. We create polynomiography with permutations polynomials. As an example, for $n = 2$ the permutation matrices Π_1, $\Pi_2, \overline{\Pi}_1$ and $\overline{\Pi}_2$ and their corresponding polynomials are

$$\Pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \overline{\Pi}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \overline{\Pi}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$p_{\Pi_1}(z) = (z - (1 + 2\mathbf{i}))(z - (2 + \mathbf{i})), \tag{5.1}$$

$$p_{\Pi_2}(z) = (z - (1 + \mathbf{i}))(z - (2 + \mathbf{i})). \tag{5.2}$$

Many polynomiographs can be associated with these permutation polynomials. The polynomiographs of above mentioned polynomials are presented in the following figures. Both images are obtained with accuracy $\varepsilon = 0.00001$ and $k = 15$.

- **Polynomiographs for equation** (5.1)**:**

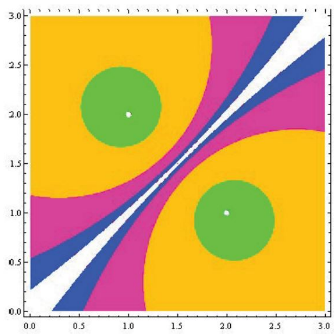

Figure 9. Polynomiograph for permutation matrix (5.1)

- **Polynomiographs for equation** (5.2)**:**

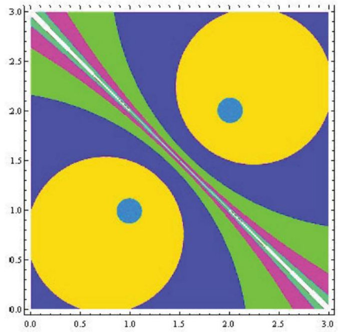

Figure 10. Polynomiograph for permutation matrix (5.2)

5.10. Polynomiograph for doubly stochastic matrix

An $n \times n$ matrix is doubly stochastic matrix if all elements are non negative reals and the sum of the entries in each row and each column equals 1. We will associate a corresponding polynomial with each doubly stochastic matrix. According to Birkhoff-von Neumann theorem [12] any double stochastic matrix A can be represented as a convex combination of permutation matrices

$$A = \sum_{i=1}^{k} \alpha_i \Pi_i,$$

where $\sum_{i=1}^{k} \alpha_i = 1$ and $\alpha_i \geq 0$ for $i = 1, 2, 3, ..., k$. The corresponding complex polynomial p_A to a doubly stochastic matrix $A = (a_{ij})$ is defined as follows:

$$p_A(z) = \prod_{\overline{a}_{ij} > 0} (z - \overline{a}_{ij}\theta_{ij}),$$

where θ_{ij} be as defined previously and $\overline{A}$ to A is constructed as matrix $\overline{\Pi}$ to Π. As an example, for $n = 2$ take the following doubly stochastic matrix A :

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The corresponding complex polynomial p_A to the matrix A has the following form:

$$p_A(z) = \left(z - \frac{1+\mathbf{i}}{2}\right)\left(z - \frac{1+2\mathbf{i}}{2}\right)\left(z - \frac{2+\mathbf{i}}{2}\right)\left(z - \frac{2+2\mathbf{i}}{2}\right). \tag{5.3}$$

Polynomiograph for a doubly stochastic matrix A are presented in following figure.

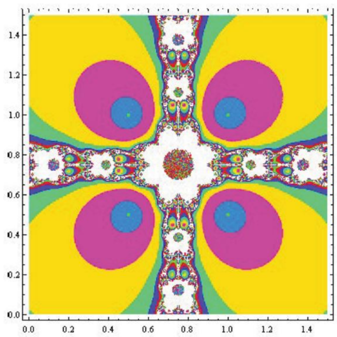

Figure 11. Polynomiograph for doubly stochastic matrix

5.11. Polynomiograph for $z^{19} - 1 = 0$

The polynomiograph is presented in the following figure with nineteen distinct basins of attraction to the nineteen roots of the polynomial $z^{19} - 1 = 0$.

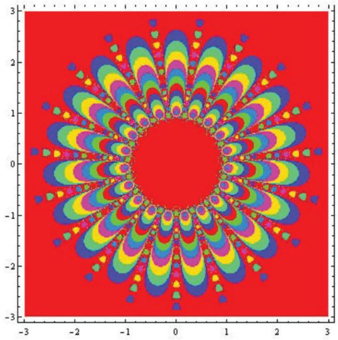

Figure 12. Polynomiograph for $z^{19} - 1 = 0$ (Carpet)

5.12. Polynomiograph for $[(2z)^{10} - 1](z^{20} - 1) = 0$

The polynomiograph is presented in the following figure for the polynomial $[(2z)^{10} - 1](z^{20} - 1) = 0$.

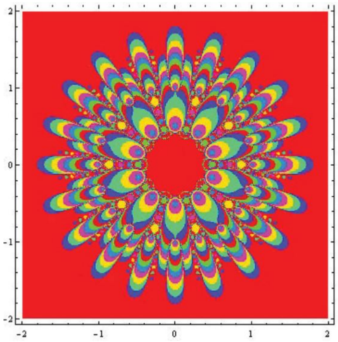

Figure 13. Polynomiograph for $[(2z)^{10} - 1] \times (z^{20} - 1) = 0$ (Flower)

5.13. Polynomiograph for a Polynomial in Physics

The next set of image come from a polynomial $z^{21} + 5z^{14} - \frac{22}{15}z^7 - \frac{11}{675} = 0$ arising in physics. Nature has its own beautiful polynomials.

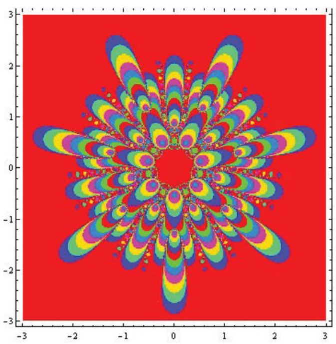

Figure 14. Polynomiography for a polynomial in Physics (Flower)

6. Conclusions

In this paper we presented some artwork produced via polynomiography of a few complex polynomials and a few special polynomials arising in science as well as a few considered to arrive at beautiful but anticipated designs obtained by using an iterative method corresponding to Simpson's $\frac{1}{3}$ rule instead of

Newton's method. By using an iterative method corresponding to Simpson's $\frac{1}{3}$ rule, we created the images of different flowers, carpets, sparrow and butterflies. Polynomiographs obtained for different complex polynomial equations, permutation matrices, doubly stochastic matrices and some special polynomials are quite new and different in comparison to Newton's method. We think that the results of this paper can inspire those who are interested in creating automatically aesthetic patterns. We believe that polynomiography can be used to teach about polynomials and polynomial roots finding, also the functionality of the existing polynomiography software can be increased using an iterative method corresponding to Simpson's $\frac{1}{3}$ rule.

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