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The sufficient conditions for dynamical systems of semigroup actions to have some stronger forms of sensitivities

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Abstract

In this paper, we introduce several stronger forms of sensitivities in the dynamical systems of semigroup actions, such as thick sensitivity and thickly syndetical sensitivity, and obtain some sufficient conditions for a dynamical system to have such sensitivities. We prove that a weakly mixing system of semigroup actions is thickly sensitive and a minimal weakly mixing system as well as a nonminimal M-system of semigroup actions is thickly syndetically sensitive.

Keywords: Semigroup action, thick sensitivity, thickly syndetical sensitivity, weakly mixing.
2010 MSC: 54H20, 37D45.

1. Introduction

The dynamical system $(K(X), T_K)$ on the hyperspace $K(X)$ consisting of all nonempty compact subsets of X with the Hausdorff metric is one of natural systems induced by the classical dynamical system (X, T). A natural problem arises: are there any connections of dynamical properties between (X, T) and $(K(X), T_K)$? The problem has aroused extensive concern of a growing number of scholars (see [1, 4, 8]). It should be noted that Banks [1] derived the following related result regarding the transitivity of dynamical systems.

Theorem 1.1 ([1]). *Let (X, T) be a dynamical system. Then the following statements are equivalent.*
(1) (X, T) is weakly mixing.
(2) $(K(X), T_K)$ is weakly mixing.
(3) $(K(X), T_K)$ is transitive.

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In 2014, Liu and Wang [5] introduced some stronger versions of sensitivities, including thick sensitivity and thickly syndetical sensitivity. Besides, some sufficient conditions for a dynamical system to have these sensitivities were obtained. Naturally, one will ask whether the result of Theorem 1.1 remains true in the setting of dynamical systems of semigroup actions. In Section 3, we give an affirmative answer to this problem (see Theorem 3.9). Then for the dynamical systems of semigroup actions, we introduce several stronger forms of sensitivities in Section 4, such as thick sensitivity and thickly syndetical sensitivity. Besides, by Theorem 3.9, we present some sufficient conditions for a dynamical system of semigroup actions to have these sensitivities (see Theorem 4.10– 4.13). The gotten results in this article generalize the corresponding results of [5]. Furthermore, as a byproduct of this article, we show that if both of two dynamical systems of semigroup actions are syndetically transitive and weakly mixing, then the product system of them is also syndetically transitive and weakly mixing.

2. Preliminaries

Throughout the paper, by a dynamical system we mean a triple (S, X, π), where X is a compact metric space with the metric d and S is a semigroup which acts on X, such that the action

$$\pi : S \times X \to X, (s, x) \to sx$$

is continuous. Then $t(sx) = t(sx)$ for all $x \in X$ and $t, s \in S$ and the map $X \to X, x \to sx$ is continuous. Just as Ellis [2] had mentioned in the literature exploring the fine structure of recurrence for semigroup actions, the topology on S is irrelevant for most of the problems with which we are concerned. For convenience, we assume that the topology on the semigroup S in the present article is generated by the following metric

$$\rho(s, t) = \max_{x\in X} d(sx, tx)$$

for all $s, t \in S$.

Sometimes we write the dynamical system as a pair (S, X) or even as X, when S is understood. A semigroup is called a monoid if there is an identity element e for the operation. Besides, we also require that $ex = x$ for all $x \in X$. A semigroup is abelian if $st = ts$ for all $s, t \in S$. Let $S = \{f^n : n \in \mathbb{N}\}$. If $f : X \to X$ is a continuous map, then the classical dynamical system (S, X) is called a cascade. Notation: (X, f).

Below we briefly state some necessary notions in the dynamical systems of semigroup actions. For a dynamical system (S, X), the orbit of a point $x \in X$ is the set $Sx = \{sx|s \in S\}$. By $\overline{A}$ we will denote the closure of $A \subset X$. Given $k \in S$ (resp.$K \subset S$) and $T \subset S$, we write $k^{-1}T = \{s \in S : ks \in T\}$ (resp. $K^{-1}T = \bigcup_{k\in K} k^{-1}T$).

For nonempty open subsets U, V of X and $\epsilon > 0$, we let

$$\begin{aligned}
&N(U, V) = \{s \in S|U \cap s^{-1}V \neq \emptyset\},\\
&N(x, U) = \{s \in S|sx \in U\},\\
&N(U, \epsilon) = \{s \in S| \text{ there are } x, y \in U \text{ such that } d(sx, sy) > \epsilon\}.
\end{aligned}$$

A dynamical system (S, X) is called:

(1) point transitive if there exists a point x with a dense orbit, namely, $\overline{Sx} = X$. Such a point is called a transitive point;
(2) transitive if $N(U, V) \neq \emptyset$ for every pair of nonempty open subsets U, V of X;
(3) weakly mixing if $(S \otimes S, X \times X)$ is transitive, where $(S \otimes S) = \{(s, s) : s \in S\}$.

It has been shown that a dynamical system (S, X) is transitive if and only if it has dense transitive points (see [3] for more details). A dynamical system (S, X) is minimal if there is no proper subset Y of X which is nonempty, closed and invariant. Equivalently, $\overline{Sx} = X$ for all $x \in X$. A subset $Y \subset X$ is called a minimal set if it is nonempty closed invariant and there is no proper subset of Y with these properties. Equivalently,

$\overline{Sx} = Y$ for all $x \in Y$. A point $x \in X$ is called a minimal point of (S, X) if it belongs to some minimal set of (S, X). A dynamical system (S, X) is an M-system if it is transitive and the set of minimal points is dense in X.

Definition 2.1 ([3])**.** A subset T of a semigroup S is said to be syndetic if there is a compact subset K of S such that for every $s \in S$, there is $k \in K$ with $ks \in T$. Equivalently, $S = K^{-1}T$.

Definition 2.2 ([3])**.** Let (S, X) be a dynamical system. A point $x \in X$ is said to be an almost periodic point of (S, X) if $N(x, U)$ is syndetic in S for every neighborhood U of x. The set of all the almost periodic points of (S, X) is denoted by $AP(S)$.

It has also been shown in [3] that $x \in X$ is a minimal point if and only if it is an almost periodic point.

Let X be a compact metric space with a metric d. Let $K(X)$ be the hyperspace on X, that is, the space of nonempty compact subsets of X equipped with the Hausdorff metric d_H defined by

$$d_H(A, B) = \max\{\sup\{d(x, B) : x \in A\}, \sup\{d(y, A) : y \in B\}\},$$

where $A, B \in K(X)$. The space $K(X)$ that we consider here is equipped with the Vietoris topology (see [7]). A basis for this topology on $K(X)$ is given by the collection of sets of the form

$$\langle U_1, \cdots, U_n\rangle = \left\{K \in K(X) : K \subset \bigcup_{i=1}^{n} U_i \text{ and } K \cap U_i \neq \emptyset \text{ for each } i = 1, \cdots, n\right\},$$

where $U_1, \cdots, U_n$, $n \in \mathbb{N}$, are nonempty open subsets of X. It is not difficult to verify that if $U = U_i$ for some $i = 1, \cdots, n$, then for any $m \geq 1$, we have

$$\langle U_1, \cdots, U_n\rangle = \langle U_1, \cdots, U_n, \underbrace{U, \cdots, U}_{m}\rangle.$$

For any $s \in S$, we define $s : K(X) \to K(X)$ as follows:

$$sK = \{sx | x \in K\} \text{ for every } K \in K(X).$$

It is easy to check that $(S, K(X))$ is also a dynamical system.

3. Weakly mixing properties of semigroup actions

Definition 3.1. The dynamical system (S, X) is called syndetically transitive if $N(U, V)$ is syndetic for every pair of nonempty subsets U, V of X.

Corresponding to the definition of syndetic sets in semigroups as above, we introduce a new concept of thick sets in semigroups.

Definition 3.2. A subset T of a semigroup S is said to be thick if for any compact subset K of S, there is $s \in S$ such that $Ks \subset T$.

Remark 3.3. What the readers may be concerned about is whether a syndetic set intersects with any thick set given by Definition 3.2. Just as usual, the following statements hold trivially in semigroups:
(1) The intersection of any syndetic set and any thick set is nonempty.
(2) A set which has a nonempty intersection with any syndetic set must be thick.

The following lemma is important to describe the weakly mixing property of a dynamical system.

Lemma 3.4 ([9]). *Let (S,X) be a dynamical system. If (S,X) is weakly mixing, then for every positive integer n, the system*

$$(\underbrace{S \otimes S \otimes \cdots \otimes S}_{n}, X^n)$$

is transitive, where X^n denotes the n-fold Cartesian product of X.

Before presenting the main results of this section, we firstly prove a proposition which gives some equivalent conditions for a dynamical system to be weakly mixing.

Proposition 3.5. *Let (S,X) be a dynamical system, where S is an abelian semigroup. Assume that every $s \in S$ is a surjective map from X onto itself. Then the following statements are equivalent.*
(1) (S,X) is weakly mixing.
(2) For any nonempty open subsets U,V of X, $N(U,V) \cap N(V,V) \neq \emptyset$.
(3) For any nonempty open subsets U,V,W of X, $N(U,V) \cap N(U,W) \neq \emptyset$.
(4) For any nonempty open subsets U,V of X, $N(U,V)$ is a thick set of S.

Proof. (1)⇒(2) and (1)⇒(3) are trivial, so we only prove the left situations.

(2)⇒(1): Let U_1, U_2, V_1, V_2 be nonempty open subsets of X. From (2) it follows that there are $s_1, s_2 \in S$ such that $A = V_2 \cap s_1^{-1}V_1$ and $B = s_1^{-1}U_1 \cap s_2^{-1}A$ are two nonempty open subsets of X. Furthermore, there is $s \in S$ such that $U_2 \cap s^{-1}B \neq \emptyset$ and $B \cap s^{-1}B \neq \emptyset$. Moreover, we have

$$U_2 \cap s^{-1}s_2^{-1}V_2 \supset U_2 \cap s^{-1}B$$

and

$$s_1^{-1}(U_1 \cap s^{-1}s_2^{-1}V_1) \supset B \cap s^{-1}B.$$

Accordingly, $s_2 s \in N(U_1,V_1) \cap N(U_2,V_2)$, which implies that (S,X) is weakly mixing.

(3)⇒(1): Let U_1, U_2, V_1, V_2 be nonempty open subsets of X. It follows from (3) that there is $s_1 \in S$ such that $U = U_2 \cap s_1^{-1}U_1$ is a nonempty open subset of X. Let $V = s_1^{-1}V_1$ and $W = V_2$. Then there is $s \in S$ such that $U \cap s^{-1}V \neq \emptyset$ and $U \cap s^{-1}W \neq \emptyset$. Also, we have

$$s_1^{-1}(U_1 \cap s^{-1}V_1) \supset U \cap s^{-1}V$$

and

$$U_2 \cap s^{-1}V_2 \supset U \cap s^{-1}W.$$

Consequently, $s \in N(U_1,V_1) \cap N(U_2,V_2)$, which yields that (S,X) is weakly mixing.

(1)⇒(4): Let $U,V \subset X$ be any two nonempty open sets, then there are $x,y \in X$ and $\epsilon > 0$ such that $B(x,\epsilon) \subset U$, $B(y,\epsilon) \subset V$, where $B(x,\epsilon)$ denotes the ϵ-neighborhood of x(similarly hereinafter). For any compact subset K of S, there is $n > 0$ such that $K \subset \bigcup_{i=1}^n B(k_i, \epsilon/2)$. It follows from Lemma 3.4 that there is $s_0 \in S$ such that $B(x,\epsilon) \cap s_0^{-1}k_i^{-1}B(y,\epsilon/2) \neq \emptyset$ for any $1 \leq i \leq n$. In the sequel, we verify that for any $k \in K$, $B(x,\epsilon) \cap s_0^{-1}k^{-1}B(y,\epsilon) \neq \emptyset$. In fact, for any $k \in K$, there is $1 \leq j \leq n$ such that $k \in B(k_j, \epsilon/2)$. Thus there is $z \in X$ such that $z \in B(x,\epsilon) \cap s_0^{-1}k_j^{-1}B(y,\epsilon/2)$. So $d(k_j s_0 z, y) < \frac{\epsilon}{2}$, which together with $d(k_j s_0 z, k s_0 z) < \frac{\epsilon}{2}$ implies that $d(k s_0 z, y) < \epsilon$. Hence, $z \in B(x,\epsilon) \cap s_0^{-1}k^{-1}B(y,\epsilon)$. Namely, $K s_0 \subset N(U,V)$.

(4)⇒(2): Let $U,V \subset X$ be any two nonempty open sets, then there is $s_1 \in S$ such that $W = U \cap s_1^{-1}V$ is a nonempty open subset of X. Pick $s_2 \in S$, it follows from (4) that $N(s_2^{-1}s_1^{-1}W, W)$ is thick. Take $K = \{s_2, s_1 s_2\}$, then there is $s \in S$ such that $s_2^{-1}s_1^{-1}W \cap s^{-1}s_2^{-1}W \neq \emptyset$ and $s_2^{-1}s_1^{-1}W \cap s^{-1}s_2^{-1}s_1^{-1}W \neq \emptyset$, which together with the following

$$s_2^{-1}s_1^{-1}(U \cap s^{-1}V) \supset s_2^{-1}s_1^{-1}W \cap s^{-1}s_2^{-1}W$$

and

$$s_2^{-1}s_1^{-1}s_1^{-1}(V \cap s^{-1}V) \supset s_2^{-1}s_1^{-1}W \cap s^{-1}s_2^{-1}s_1^{-1}W$$

enable us to see that $s \in N(U,V) \cap N(V,V)$. □

The following lemma plays a key role in the proof of Theorem 3.7.

Lemma 3.6. *Let (S, X) be a dynamical system, where S is an abelian semigroup. Assume that every $s \in S$ is a surjective map from X onto itself. If (S, X) is weakly mixing and syndetically transitive, then for any compact subset K of S and any two nonempty open subsets U, V of X, there is a syndetic set $T \subset S$ such that $TK \subset N(U, V)$.*

Proof. We can choose $a, b \in X$ and $\epsilon > 0$ such that $B(a, \epsilon) \subset U$ and $B(b, \epsilon) \subset V$. It follows from Proposition 3.5 and the weakly mixing property of (S, X) that $N(B(a, \epsilon), B(b, \epsilon/2))$ is thick. Accordingly, for any compact subset K of S, there is $s \in S$ such that $Ks \subset N(B(a, \epsilon), B(b, \epsilon/2))$. For any $k \in K$, write $E_k = B(a, \epsilon) \cap (ks)^{-1}B(b, \epsilon/2)$ and $F_k = B(a, \epsilon) \cap (ks)^{-1}B(b, \epsilon)$. Next, we are going to prove the following crucial property $(\star)$.
$(\star)$ For any $k \in K$, there is $s_k \in S$ such that $G = \bigcap_{k\in K} s_k^{-1}F_k \neq \emptyset$.

Let $K \subset \bigcup_{i=1}^{n} B(k_i, \epsilon/2)$. For $i = 1, 2, \cdots, n$, there are $s_{k_i} \in S$ such that $H = \bigcap_{i=1}^{n} s_{k_i}^{-1}E_{k_i} \neq \emptyset$. If $x \in H$, we claim that $x \in G$. In fact, for any $k \in K$, there is $1 \leq j \leq n$ such that $k \in B(k_j, \epsilon/2)$. Take $s_k = s_{k_j}$. Since $x \in s_{k_j}^{-1}E_{k_j}$, we have $s_{k_j}x \in B(a, \epsilon)$ and $d(k_j s s_{k_j} x, b) < \frac{\epsilon}{2}$, which together with $d(k s s_{k_j} x, k_j s s_{k_j} x) < \frac{\epsilon}{2}$ yield that $s_{k_j}x \in (ks)^{-1}B(b, \epsilon)$. It is not difficult to check that $x \in G$.

From the proof of the property $(\star)$, we know that $H \subset G$, which together with the fact that H is a nonempty open subset of X enables us to see that $N(G, G) \neq \emptyset$. If $t \in N(G, G)$, there is $x \in X$ such that $x \in G \cap t^{-1}G$. Write $y_k = s_k x$ for any $k \in K$, then $y_k \in F_k \subset U$ and $ty_k \in F_k \subset (ks)^{-1}B(b, \epsilon)$ for any $k \in K$. Furthermore, it is clear that $kst \in N(U, V)$. This proves that $sN(G, G)K \subset N(U, V)$. Since $T = sN(G, G)$ is syndetic, the result is proved. □

Theorem 3.7. *Let (S, X) and (S, Y) be two dynamical systems, where S is an abelian semigroup. Assume that every $s \in S$ is a surjective map from X onto itself. If both (S, X) and (S, Y) are weakly mixing and syndetically transitive, then $(S \otimes S, X \times Y)$ is weakly mixing and syndetically transitive.*

Proof. Let $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ be nonempty open sets, and let $E = N(U_1, U_2)$, $F = N(V_1, V_2)$. By Proposition 3.5 it suffices to show that $E \cap F$ is syndetic and thick. Since E is syndetic, there is a compact subset K_1 of S such that $K_1^{-1}E = S$. It follows from Lemma 3.6 that there is a syndetic set T_1 such that $T_1K_1 \subset F$. In this case, there also is a compact subset K_2 of S such that $K_2^{-1}T_1 = S$. For any $s \in S$, there is $k_2 \in K_2$ such that $k_2s \in T_1$. Furthermore, there is $k_1 \in K_1$ such that $k_1k_2s \in E$. Consequently, $k_1k_2s \in E \cap T_1K_1$. Note that the set K_1K_2 is also a compact subset of S, it is easy to check that $E \cap F$ is syndetic. On the other hand, by Lemma 3.6, for any compact subset K of S, there is a syndetic set T_2 satisfying $T_2K \subset F$. Denote by K_3 a corresponding compact set of the syndetic set T_2(namely, $K_3^{-1}T_2 = S$). As E is thick, for $KK_3 \subset S$, there is $s \in S$ such that $KK_3s \subset E$. For this s, there is $k_3 \in K_3$ such that $k_3s \in T_2$, which means that $k_3sK \subset T_2K \cap E$. Hence, $E \cap F$ is thick. □

Remark 3.8. It is universally acknowledged that if both of two discrete dynamical systems are syndetically transitive and weakly mixing, then the product dynamical system of them is also syndetically transitive and weakly mixing (see [6] for details). In the above result, we generalize the corresponding classical result to the dynamical systems of semigroup actions by making use of the topology on the semigroup S defined at the beginning of the paper.

Next, we present another main result of this section, which is a generalization of the main result of [1] and an auxiliary tool for further investigations in the following section.

Theorem 3.9. *Let (S, X) be a dynamical system, where S is an abelian semigroup. Assume that every $s \in S$ is a surjective map from X onto itself. Then the following statements are equivalent.*
(1) (S, X) *is weakly mixing.*
(2) $(S, K(X))$ *is weakly mixing.*
(3) $(S, K(X))$ *is transitive.*

Proof. (1)⇒(2): Let U, V be any two nonempty open subsets of $K(X)$, then there are nonempty subsets $U_1, \cdots, U_n, V_1, \cdots, V_n$ of X such that $\langle U_1, \cdots, U_n\rangle \subset U$ and $\langle V_1, \cdots, V_n\rangle \subset V$ (it is not difficult to see that the number of U_i can be identical with that of V_i). By Proposition 3.5 it suffices to show that there is $s \in S$ such that $U \cap s^{-1}V \neq \emptyset$ and $V \cap s^{-1}V \neq \emptyset$. From Lemma 3.4, there is $s \in S$ such that $U_i \cap s^{-1}V_i \neq \emptyset$ and $V_i \cap s^{-1}V_i \neq \emptyset$ for all $1 \leq i \leq n$. Let

$$E = \langle U_1 \cap s^{-1}V_1, \cdots, U_n \cap s^{-1}V_n\rangle$$

and

$$F = \langle V_1 \cap s^{-1}V_1, \cdots, V_n \cap s^{-1}V_n\rangle.$$

Then both E and F are nonempty open subsets of $K(X)$. So there are $A, B \in K(X)$ such that $A \in E$ and $B \in F$. Noticing that $sE \subset V$ and $sF \subset V$, we have $A \in U \cap s^{-1}V$ and $B \in V \cap s^{-1}V$.

(2)⇒(3): Follows directly from the definitions.

(3)⇒(1): From Proposition 3.5, it suffices to show that for any nonempty open subsets U, V, W of X, there is $s \in S$ such that $U \cap s^{-1}V \neq \emptyset$ and $U \cap s^{-1}W \neq \emptyset$. Since $(S, K(X))$ is transitive, there is $s \in S$ such that $E = \langle U\rangle \cap s^{-1}\langle V, W\rangle \neq \emptyset$. Pick $A \in E$, then we have $A \subset U$, $sA \cap V \neq \emptyset$ and $sA \cap W \neq \emptyset$. It is not difficult to verify that $U \cap s^{-1}V \neq \emptyset$ and $U \cap s^{-1}W \neq \emptyset$. □

4. Several sensitivities of semigroup actions

Definition 4.1. Let (S, X) be a dynamical system. A point $x \in X$ is called a fixed point of (S, X) if $sx = x$ for every $s \in S$.

Next, we introduce the definition of thickly syndetic sets in semigroups.

Definition 4.2. A subset T of S is said to be thickly syndetic if for any compact subset K of S, there is a syndetic set $A \subset S$ such that $AK \subset T$.

Proposition 4.3. *The intersection of any two thickly syndetic sets is syndetic.*

Proof. Let $A_1, A_2 \subset S$ be any two thickly syndetic sets. Pick a compact set $K_1 \subset S$, then there is a syndetic set T_1 satisfying $T_1K_1 \subset A_1$. Denote by M_1 a corresponding compact set of the syndetic set T_1(namely, $M_1^{-1}T_1 = S$). Let $K_2 = K_1M_1$, then there is a syndetic set T_2 such that $T_2K_1M_1 \subset A_2$. As T_2 is syndetic, there is a compact set M_2 satisfying $M_2^{-1}T_2 = S$. For any $s \in S$, we have $m_2s \in T_2$ for some $m_2 \in M_2$ and $m_1m_2s \in T_1$ for some $m_1 \in M_1$. Hence, $m_1m_2sK_1 \subset A_1 \cap A_2$. Since $M_1M_2K_1$ is also a compact subset of S, we obtain that $A_1 \cap A_2$ is syndetic. □

Before presenting the definition involving sensitivities we are concerned about, we first recall the following definition and give a simple equivalent condition of it.

Definition 4.4 ([3])**.** Let (S, X) be a dynamical system. Then (S, X) is called sensitive if there is a positive ϵ (sensitivity constant) such that for every $x \in X$ and every open neighborhood U of x, there is $y \in U$ and $s \in S$ with $d(sx, sy) > \epsilon$.

Proposition 4.5. (S, X) *is sensitive if and only if* $N(U, \epsilon) \neq \emptyset$ *for some* $\epsilon > 0$ *and every nonempty open set* $U \subset X$*.*

Proof. The necessity is obvious, so we only prove the sufficiency.

For every $x \in X$ and every open neighborhood U of x, there are $y, z \in U$ and $s \in S$ such that $d(sy, sz) > \epsilon$. Note that $d(sy, sz) \leq d(sx, sy) + d(sx, sz)$, then at least one of $d(sx, sy)$ and $d(sx, sz)$ is larger than $\frac{\epsilon}{2}$. □

Now we investigate several sensitivities of semigroup actions. To this end, we first introduce the following definition regarding some stronger forms of sensitivities.

Definition 4.6. Let (S, X) be a dynamical system. (S, X) is called

(1) syndetically sensitive if $N(U, \epsilon)$ is syndetic for some $\epsilon > 0$ and every nonempty open set $U \subset X$;
(2) thickly sensitive if $N(U, \epsilon)$ is thick for some $\epsilon > 0$ and every nonempty open set $U \subset X$;
(3) thickly syndetically sensitive if $N(U, \epsilon)$ is thickly syndetic for some $\epsilon > 0$ and every nonempty open set $U \subset X$.

Here ϵ is called the sensitivity constant.

Lemma 4.7. *Let (S, X) be a dynamical system, where S is an abelian semigroup. Then $sAP(S) \subset AP(S)$ for any $s \in S$.*

Proof. The proof is easy and we omit it here. □

As a prelude to our main results in this section, we first prove the following two useful lemmas.

Lemma 4.8. *Let (S, X) be a dynamical system and Y be a minimal set of X. If (S, X) is point transitive and x is a transitive point, then $N(x, B(Y, \epsilon))$ is thick for any $\epsilon > 0$, where*

$$B(Y, \epsilon) = \{x \in X | d(x, Y) = \inf_{y \in Y} d(x, y) < \epsilon\}.$$

Proof. For any compact subset K of S and $\epsilon > 0$, we have $K \subset \bigcup_{i=1}^n B(k_i, \epsilon/2)$ for some $n > 0$. By the continuities of $k_1, \cdots, k_n$, there is $\delta > 0$ such that for any $a, b \in X$ with $d(a, b) < \delta$, we have $d(k_i a, k_i b) < \frac{\epsilon}{2}$ for all $1 \le i \le n$. Since x is a transitive point, there is $s \in S$ such that $d(sx, Y) < \delta$, which implies that $d(k_i sx, Y) < \frac{\epsilon}{2}$ for all $1 \le i \le n$. For any $k \in K$, there is $1 \le j \le n$ satisfying $k \in B(k_j, \epsilon/2)$, which together with $d(k_j sx, Y) < \frac{\epsilon}{2}$ yields that $d(ksx, Y) < \epsilon$. So $Ks \subset N(x, B(Y, \epsilon))$. □

Lemma 4.9. *Let (S, X) be a dynamical system and Y be a minimal set of X, where S is an abelian semigroup. If (S, X) is an M-system and U is a nonempty open subset of X, then $N(U, B(Y, \epsilon))$ is thickly syndetic for any $\epsilon > 0$.*

Proof. Let K be a compact subset of S and let $\epsilon > 0$. Then $K \subset \bigcup_{i=1}^n B(k_i, \epsilon/2)$ for some $n > 0$ and some $k_i \in K$. By the uniform continuities of $k_1, \cdots, k_n$, there is $\delta > 0$ such that for any $a, b \in X$ with $d(a, b) < \delta$, we have $d(k_i a, k_i b) < \frac{\epsilon}{2}$ for all $1 \le i \le n$. Let $x \in U$ be a transitive point and Y be a minimal set. Then there is $s \in S$ such that $d(sx, Y) < \frac{\delta}{2}$. It is not difficult to see that, from the continuity of s and the density of the set of minimal points, there is a minimal point $y \in U$ satisfying $d(sx, sy) < \frac{\delta}{2}$, which gives $d(sy, Y) < \delta$. By Lemma 4.7, sy is also a minimal point. Then there is a syndetic set $T \subset S$ such that $d(tsy, Y) < \delta$ for any $t \in T$. For any $k \in K$, there is $1 \le j \le n$ satisfying $k \in B(k_j, \epsilon/2)$, which together with $d(k_j tsy, Y) < \frac{\epsilon}{2}$ implies that $d(ktsy, Y) < \epsilon$. Consequently, we have $KTs \subset N(U, B(Y, \epsilon))$. Noting that Ts is syndetic, we obtain that $N(U, B(Y, \epsilon))$ is thickly syndetic. □

Of concern are stronger forms of sensitivities dependence on initial conditions of dynamical systems of semigroup actions. What amounts to the same thing, the center problem we consider in this part is to acquire some sufficient conditions for a dynamical system to have those stronger forms of sensitivities.

Theorem 4.10. *Let (S, X) be a dynamical system, where S is an abelian semigroup. If (S, X) is minimal and sensitive, then (S, X) is syndetically sensitive.*

Proof. Let $\epsilon > 0$ be a sensitivity constant of (S, X) and let U be a nonempty open subset of X, then there are $x, y \in U$ and $s \in S$ such that $d(sx, sy) > \epsilon$. Furthermore, there is an open subset $V \subset U$ such that $d(sx, sV) > \epsilon$ due to the continuity of s. We will prove that $N(U, \epsilon/2)$ is syndetic. Since (S, X) is minimal, there is $s_1 \in S$ such that $s_1 x \in V$. Accordingly, $d(sx, ss_1 x) \ge d(sx, sV) > \epsilon$. Besides, by the uniform continuity of s_1, there is $\delta \in (0, \epsilon/4)$ such that for any $a, b \in X$ with $d(a, b) < \delta$, we have $d(s_1 a, s_1 b) < \frac{\epsilon}{4}$. It is clear that $N(sx, B(sx,\delta))$ is syndetic. For any $t \in N(sx, B(sx,\delta))$, we have $d(sx, stx) < \delta$, which enables us to obtain that $d(s_1sx, s_1stx) < \frac{\epsilon}{4}$. Therefore,

$$d(stx, sts_1x) \geq d(sx, s_1sx) - d(sx, stx) - d(s_1sx, s_1stx) > \frac{\epsilon}{2}.$$

That is to say, $st \in N(U, \epsilon/2)$. So $sN(sx, B(sx,\delta)) \subset N(U, \epsilon/2)$. □

Theorem 4.11. *Let (S, X) be a dynamical system, where S is an abelian semigroup. Assume that every $s \in S$ is a surjective map from X onto itself. If (S, X) is weakly mixing, then it is thickly sensitive.*

Proof. Let $D = diam(X)$, $\epsilon \in (0, D/4)$ and U be a nonempty open subset of X. It follows from Theorem 3.9 that $(S, K(X))$ is transitive. Besides, it is obvious that X is a fixed point of $(S, K(X))$. Then there is a transitive point V of $(S, K(X))$ such that $V \in \langle U\rangle$. It follows from Lemma 4.8 that $N(V, B_{d_H}(X, \epsilon))$ is thick. Then $N(U, \epsilon)$ is thick as $N(V, B_{d_H}(X, \epsilon)) \subset N(U, \epsilon)$. □

Theorem 4.12. *Let (S, X) be a dynamical system, where S is an abelian semigroup. Assume that every $s \in S$ is a surjective map from X onto itself. If (S, X) is minimal and weakly mixing, then it is thickly syndetically sensitive.*

Proof. At the beginning of the proof, we claim that the following property $(\star)$ holds.
$(\star)$ $\{t|t \in S, d_H(tU, X) < \epsilon\}$ is syndetic for any nonempty open subset U of X and any $\epsilon > 0$.

It follows from Theorem 3.9 that $(S, K(X))$ is transitive. Let V be a transitive point of $(S, K(X))$ such that $V \in \langle U\rangle$. Then there is $s \in S$ satisfying $d_H(sV, X) < \epsilon/6$. By the continuity of s, there is $\delta > 0$ such that $B(V, \delta) \subset U$ and for any $a, b \in X$ with $d(a, b) < \delta$, we have $d(sa, sb) < \frac{\epsilon}{6}$. Assume that $x, z_i \in V$ for $1 \leq i \leq l$. Then $d(y_i, z_i) < \delta$ for $1 \leq i \leq l$ implies $\{y_1, \cdots, y_l\} \subset U$ and $d_H(s(\{x, y_1, \cdots, y_l\}), s(\{x, z_1, \cdots, z_l\})) < \frac{\epsilon}{6}$. Since $\overline{Sx} = X$, there are $s_1, \cdots, s_l \in S$ such that $d(s_ix, z_i) < \delta$ for $1 \leq i \leq l$. Write $X_l = \{x, s_1x, \cdots, s_lx\}$, then $X_l \subset U$ and $d_H(sX_l, sV) < \frac{\epsilon}{6}$. Therefore $d_H(sX_l, X) < \frac{\epsilon}{2}$. By the continuities of $s_1, \cdots, s_l$, there is $\delta' \in (0, \epsilon/2)$ such that for any $a, b \in X$ with $d(a, b) < \delta'$, we have $d(s_ia, s_ib) < \frac{\epsilon}{2}$ for all $1 \leq i \leq l$. It is clear that $N(sx, B(sx, \delta'))$ is syndetic due to $sx \in AP(S)$. If $t \in N(sx, B(sx, \delta'))$, namely, $d(tsx, sx) < \delta'$, then we have $d(s_itsx, s_isx) < \frac{\epsilon}{2}$ for all $1 \leq i \leq l$, which enables us to conclude that $d_H(tsX_l, sX_l) < \frac{\epsilon}{2}$. It follows that

$$\begin{aligned} d_H(tsU, X) &\leq d_H(tsX_l, X) \\ &\leq d_H(tsX_l, sX_l) + d_H(sX_l, X) \\ &< \epsilon. \end{aligned}$$

Hence, one can obtain the inclusion: $N(sx, B(sx, \delta'))s \subset \{t|t \in S, d_H(tU, X) < \epsilon\}$, which implies the result as desired.

Let $D = diam(X)$, $\epsilon \in (0, D/4)$ and U be a nonempty open subset of X. For any compact subset K of S, we have $K \subset \bigcup_{i=1}^{n} B(k_i, \epsilon/3)$ for some $n > 0$. By the continuities of $k_1, \cdots, k_n$, there is $\delta \in (0, \epsilon/2)$ such that for any $A, B \in K(X)$ with $d_H(A, B) < \delta$, we have $d_H(k_iA, k_iB) < \frac{\epsilon}{2}$ for all $1 \leq i \leq n$. Put $T = \{t|t \in S, d_H(tU, X) < \delta\}$, it follows from the property $(\star)$ that T is syndetic. For any $t \in T$, we have $d_H(tU, X) < \delta$, then $d_H(k_itU, X) < \frac{\epsilon}{2}, i = 1, \cdots, n$. For any $k \in K$, there is $1 \leq j \leq n$ such that $k \in B(k_j, \epsilon/3)$, which implies $d(k_jtx, ktx) < \frac{\epsilon}{3}$ for any $x \in U$, so $d_H(k_jtU, ktU) < \frac{\epsilon}{2}$. Accordingly, we get that $d_H(ktU, X) < \epsilon$ for any $k \in K$ and any $t \in T$, which means that

$$KT \subset \{s|s \in S, d_H(sU, X) < \epsilon\}.$$

Consequently, it is obvious that $N(U, \epsilon)$ is thickly syndetic as $\{s|s \in S, d_H(sU, X) < \epsilon\} \subset N(U, \epsilon)$. □

Theorem 4.13. *Let (S, X) be a dynamical system, where S is an abelian semigroup. If (S, X) is a non-minimal M-system, then it is thickly syndetically sensitive.*

Proof. Let Y, Z be minimal sets of X with $d(Y, Z) = \epsilon$, and let U be a nonempty open subset of X. For any compact subset K of S, we have $K \subset \bigcup_{i=1}^{n} B(k_i, \epsilon/8)$ for some $n > 0$ and some $k_i \in K$. By the uniform continuities of $k_1, \cdots, k_n$, there is $\delta > 0$ such that for any $a, b \in X$ with $d(a, b) < \delta$, we have $d(k_i a, k_i b) < \frac{\epsilon}{8}$ for all $1 \leq i \leq n$. By Lemma 4.9, we know that $N(U, B(Y, \delta))$ and $N(U, B(Z, \delta))$ are thickly syndetic. It follows from Proposition 4.3 that $N(U, B(Y, \delta)) \cap N(U, B(Z, \delta))$ is syndetic. Accordingly, for any $s \in T = N(U, B(Y, \delta)) \cap N(U, B(Z, \delta))$, there are $x, y \in U$ such that $d(sx, Y) < \delta$ and $d(sy, Z) < \delta$. For any $k \in K$, there is $1 \leq j \leq n$ such that $k \in B(k_j, \epsilon/8)$. So it is obvious to get that $d(k_j sx, Y) < \frac{\epsilon}{8}$, which together with $d(k_j sx, ksx) < \frac{\epsilon}{8}$ yields that $d(ksx, Y) < \frac{\epsilon}{4}$. Hence, $ks \in N(U, B(Y, \epsilon/4))$. Furthermore, we can obtain

$$ks \in N(U, B(Y, \epsilon/4)) \cap N(U, B(Z, \epsilon/4))$$

for any $k \in K$ and any $s \in T$. It is not difficult to verify the following

$$N(U, B(Y, \epsilon/4)) \cap N(U, B(Z, \epsilon/4)) \subset N(U, \epsilon/2).$$

Hence, we have $KT \subset N(U, \epsilon/2)$, a result as desired. □

Conclusions:

Up to now, a growing number of results are obtained in dynamical systems of semigroup actions. This paper is following the tendency. Several stronger forms of sensitivities, such as thick sensitivity and thickly syndetical sensitivity, are introduced in the dynamical systems of semigroup actions, and some sufficient conditions for a dynamical system to have such sensitivities are obtained. Furthermore, it is proven that a weakly mixing system of semigroup actions is thickly sensitive and a minimal weakly mixing system as well as a nonminimal M-system of semigroup actions is thickly syndetically sensitive. The main results in this article will contribute to introducing some chaotic concepts and studying the complexity in the setting of dynamical systems of semigroup actions.

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