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Existence of periodic solutions for second-order nonlinear difference equations

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Abstract

By using the critical point method, the existence of periodic solutions for second-order nonlinear difference equations is obtained. The proof is based on the Saddle Point Theorem in combination with variational technique. The problem is to solve the existence of periodic solutions of second-order nonlinear difference equations. One of our results obtained complements the result in the literature.

Keywords: Existence, periodic solutions, second-order, nonlinear difference equations, discrete variational theory.
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1. Introduction

Recently, the theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural networks, ecology, cybernetics, etc. For the general background of difference equations, one can refer to the monographs [1, 2, 3]. For the past twenty years, there has been much progress on the qualitative properties of difference equations, which included result in stability and attractive [13, 15] and result in oscillation and other topics, see [1, 2, 3, 8, 9, 10, 12, 21, 22, 23, 24, 25]. Therefore, it is worthwhile to explore this topic.

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For any $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \cdots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. Let the symbol $*$ denote the transpose of a vector.

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The present paper considers the following second-order nonlinear difference equation

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) + q_n u_n^{\delta} + f(n, u_{n+1}, u_n, u_{n-1}) = 0,\ n \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $\delta > 0$ is the ratio of odd positive integers, $\{p_n\}$ and $\{q_n\}$ are real sequences, $f \in C(\mathbb{Z} \times \mathbb{R}^3, \mathbb{R})$, T is a given positive integer, $p_{n+T} = p_n > 0$, $q_{n+T} = q_n < 0$, $f(n+T, v_1, v_2, v_3) = f(n, v_1, v_2, v_3)$.

Eq. (1.1) can be considered as a discrete analogue of a special case of the following second-order nonlinear functional differential equation

$$\big(p(t)\varphi(u')\big)' + f(t, u(t+1), u(t), u(t-1)) = 0,\ t \in \mathbb{R}. \tag{1.2}$$

Eq. (1.2) includes the following equation

$$\big(p(t)\varphi(u')\big)' + f(t, u(t)) = 0,\ t \in \mathbb{R},$$

which has arisen in the study of fluid dynamics, combustion theory, gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, and adiabatic reactor [5, 7, 11]. Equations similar in structure to (1.2) arise in the study of the existence of solitary waves of lattice differential equations, see Smets and Willem [14].

When $\delta = 1$, and $f(n, u_{n+1}, u_n, u_{n-1}) = 0$, (1.1) becomes

$$\Delta\left(p_n \Delta u_{n-1}\right) + q_n u_n = 0, \tag{1.3}$$

which has been extensively investigated by many authors [1, 3, 6], for results on oscillation, asymptotic behavior, boundary value problems, disconjugacy and disfocality.

In [21], the periodic solutions of second-order self-adjoint difference equation

$$\Delta\left(p_n \Delta u_{n-1}\right) + q_n u_n = f(n, u_n) \tag{1.4}$$

has been considered.

When $f(n, u_{n+1}, u_n, u_{n-1}) = 0, n \in \mathbb{Z}(0)$, (1.1) reduces to the following equation

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) + q_n u_n^{\delta} = 0, \tag{1.5}$$

which has been studied in [1, 6, 22] for results on oscillation, asymptotic behavior and the existence of positive solutions.

Moreover, if $q_n u_n^{\delta} + f(n, u_{n+1}, u_n, u_{n-1}) = q_n g(u_n) + r_n$, (1.1) has been considered in [16] for oscillatory properties of its all solutions.

When $\beta > \delta+1$, in Theorem 3.2, Cai and Yu [4] have obtained some sufficient conditions for the existence of periodic solutions of the following nonlinear difference equation

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) + q_n u_n^{\delta} = f(n, u_n),\ n \in \mathbb{Z}. \tag{1.6}$$

Furthermore, [4] is the only paper we found which deals with the problem of periodic solutions to second-order difference equation (1.6). When $\beta < \delta + 1$, can we still find the periodic solutions of (1.6)?

By using various methods and techniques, such as Schauder fixed point theorem, the cone theoretic fixed point theorem, the method of upper and lower solutions, coincidence degree theory, a series of existence results of nontrivial solutions for differential equations have been obtained in [14, 16, 19]. Critical point theory is also an important tool to deal with problems on differential equations [14, 19]. Because of applications in many areas of difference equations [1, 2, 3], recently, a few authors have gradually paid attention to applying critical point theory to deal with periodic solutions of discrete systems, see [8, 9, 10, 17, 21, 23]. Particularly, Guo and Yu [8, 9, 10] and Shi *et al.* [17] studied the existence of periodic solutions of second-order nonlinear difference equations by using the critical point theory. However, to the best of our knowledge, when $\delta \neq 1$ the results on periodic solutions of second-order nonlinear difference equation (1.1) are very scarce in the literature (see[4]), because there are few known methods for considering the existence of periodic solutions of discrete systems. Furthermore, since f in (1.1) depends on u_{n+1} and u_{n-1}, the traditional ways of establishing the functional in [8, 9, 10, 21, 23] are inapplicable to our case. The main purpose of this paper is to give some sufficient conditions for the existence of periodic solutions to second-order nonlinear difference equations. The main approach used in our paper are variational techniques and the Saddle Point Theorem. In particular, one of our results obtained complements the result in the literature [4]. In fact, one can see the Remark 1.4 for details. The motivation for the present work stems from the recent papers in [4, 23].

For basic knowledge on variational methods, we refer the reader to [14].

Let

$$\underline{p} = \min_{n\in\mathbf{Z}(1,T)}\{p_n\},\ \bar{p} = \max_{n\in\mathbb{Z}(1,T)}\{p_n\},\ \underline{q} = \min_{n\in\mathbf{Z}(1,T)}\{q_n\},\ \bar{q} = \max_{n\in\mathbb{Z}(1,T)}\{q_n\}.$$

Now we state the main results of this paper.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*
(F_1) *there exists a functional* $F(n,v_1,v_2) \in C^1(\mathbb{Z}\times\mathbb{R}^2,\mathbb{R})$ *such that*

$$F(n+T,v_1,v_2) = F(n,v_1,v_2),$$

$$\frac{\partial F(n-1,v_2,v_3)}{\partial v_2} + \frac{\partial F(n,v_1,v_2)}{\partial v_2} = f(n,v_1,v_2,v_3);$$

(F_2) *there exist constants* $R_1>0$ *and* $1<\alpha<2$ *such that for* $n\in\mathbb{Z}$ *and* $\sqrt{v_1^2+v_2^2} \geq R_1$,

$$0 < \frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1 + \frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2 \leq \frac{\alpha}{2}(\delta+1)F(n,v_1,v_2);$$

(F_3) *there exist constants* $a_1>0$, $a_2>0$ *and* $1<\gamma\leq\alpha$ *such that*

$$F(n,v_1,v_2) \geq a_1\left(\sqrt{v_1^2+v_2^2}\right)^{\frac{\gamma}{2}(\delta+1)} - a_2,\ \forall (n,v_1,v_2)\in\mathbb{Z}\times\mathbb{R}^2.$$

Then for any given positive integer $m>0$, (1.1) *has at least one* mT*-periodic solution.*

Remark 1.2. Assumption (F_2) implies that for each $n\in\mathbb{Z}$ there exist constants $a_3>0$ and $a_4>0$ such that (F_2') $F(n,v_1,v_2) \leq a_3\left(\sqrt{v_1^2+v_2^2}\right)^{\frac{\alpha}{2}(\delta+1)} + a_4,\ \forall (n,v_1,v_2)\in\mathbb{Z}\times\mathbb{R}^2.$

In fact, let $v=(v_1,v_2)$ and $\nabla_v F(n,v)$ be the gradient of $F(n,v)$ in v. From (F_2), we have

$$\frac{v}{|v|}\cdot\frac{\nabla_v F(n,v)}{F(n,v)} \leq \frac{\frac{\alpha}{2}(\delta+1)}{|v|},\ \text{for } n\in\mathbb{Z} \text{ and } |v|\geq R_1.$$

Thus,

$$\frac{d\ln F(n,v)}{d|v|} \leq \frac{\frac{\alpha}{2}(\delta+1)}{|v|},$$

which implies

$$\frac{d}{d|v|}(\ln F(n,v) - \frac{\alpha}{2}(\delta+1)\ln|v|) \leq 0, \tag{1.7}$$

for $n\in\mathbb{Z}$ and $|v|\geq R_1$.

Denote $G = \max\{\ln F(n,v) - \frac{\alpha}{2}(\delta+1)\ln|v| :\ n\in\mathbb{Z},\ |v| = R_1\}$. By (1.7),

$$\ln F(n,v) - \frac{\alpha}{2}(\delta+1)\ln|v| \leq G,\ \text{for } n\in\mathbb{Z} \text{ and } |v|\geq R_1.$$

That is,

$$F(n,v) \leq a_3|v|^{\frac{\alpha}{2}(\delta+1)},\ \text{for } n\in\mathbb{Z} \text{ and } |v|\geq R_1,$$

where $a_3 = e^G$.

Let $a_4 = \max\{|F(n,v)| :\ n\in\mathbb{Z},\ |v|\leq R_1\}$. Then (F_2') holds. If $f(n,u_{n+1},u_n,u_{n-1}) = -f(n,u_n)$, (1.1) reduces to (1.6). Then, we have the following results.

Theorem 1.3. *Assume that the following hypotheses are satisfied:*
(F_4) *there exists a functional* $F(n,v) \in C^1(\mathbb{Z}\times\mathbb{R},\mathbb{R}),\ F(n+T,v) = F(n,v)$ *such that*

$$\frac{\partial F(n,v)}{\partial v} = f(n,v);$$

(F_5) *there exist constants* $R_2 > 0$ *and* $1 < \alpha < 2$ *such that for* $n \in \mathbb{Z}$ *and* $\sqrt{v_1^2 + v_2^2} \geq R_2$,

$$\frac{\alpha}{2}(\delta+1)F(n,v) \leq vf(n,v) < 0;$$

(F_6) *there exist constants* $a_5 > 0,\ a_6 > 0$ *and* $1 < \gamma \leq \alpha$ *such that*

$$F(n,v) \leq -a_5|v|^{\frac{\gamma}{2}(\delta+1)} + a_6,\ \forall (n,v) \in \mathbb{Z}\times\mathbb{R}.$$

Then for any given positive integer $m > 0$, (1.6) *has at least one* mT*-periodic solution.*

Remark 1.4. When $\beta > \delta+1$, in Theorem 3.2, Cai and Yu [4] have obtained some criteria for the existence of periodic solutions of (1.6). When $\beta < \delta+1$, we can still find the periodic solutions of (1.6). Hence, Theorem 1.3 complements the existing one.

The rest of the paper is organized as follows. First, in Section 2, we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of periodic solutions of (1.1) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give an example to illustrate the main result.

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for (1.1) and give some lemmas which will be of fundamental importance in proving our main results. We start by some basic notations.

Let S be the set of sequences $u = (\cdots, u_{-n}, \cdots, u_{-1}, u_0, u_1, \cdots, u_n, \cdots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S = \{\{u_n\} | u_n \in \mathbb{R},\ n \in \mathbb{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbb{R}$, $au + bv$ is defined by

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, E_{mT} is defined as a subspace of S by

$$E_{mT} = \{u \in S | u_{n+mT} = u_n,\ \forall n \in \mathbb{Z}\}.$$

Clearly, E_{mT} is isomorphic to $\mathbb{R}^{mT}$. E_{mT} can be equipped with the inner product

$$\langle u, v\rangle = \sum_{j=1}^{mT} u_j v_j,\ \forall u, v \in E_{mT}, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{mT} u_j^2\right)^{\frac{1}{2}},\ \forall u \in E_{mT}. \tag{2.2}$$

It is obvious that E_{mT} with the inner product (2.1) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbb{R}^{mT}$.

On the other hand, we define the norm $\|\cdot\|_s$ on E_{mT} as follows:

$$\|u\|_s = \left(\sum_{j=1}^{mT} |u_j|^s\right)^{\frac{1}{s}}, \tag{2.3}$$

for all $u \in E_{mT}$ and $s > 1$.

Since $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants $c_1,\ c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_s \leq c_2\|u\|_2,\ \forall u \in E_{mT}. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_{mT}$, define the functional J on E_{mT} as follows:

$$J(u) = -\frac{1}{\delta+1}\sum_{n=1}^{mT} p_{n+1}\left(\Delta u_n\right)^{\delta+1} + \frac{1}{\delta+1}\sum_{n=1}^{mT} q_n u_n^{\delta+1} + \sum_{n=1}^{mT} F(n, u_{n+1}, u_n),$$

$$J(u) := -H(u) + \frac{1}{\delta+1}\sum_{n=1}^{mT} q_n u_n^{\delta+1} + \sum_{n=1}^{mT} F(n, u_{n+1}, u_n), \tag{2.5}$$

where

$$H(u) = \frac{1}{\delta+1}\sum_{n=1}^{mT} p_{n+1}\left(\Delta u_n\right)^{\delta+1},\ \frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3).$$

Clearly, $J \in C^1(E_{mT}, \mathbb{R})$ and for any $u = \{u_n\}_{n\in\mathbb{Z}} \in E_{mT}$, by using $u_0 = u_{mT},\ u_1 = u_{mT+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_n} = \Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) + q_n u_n^{\delta} + f(n, u_{n+1}, u_n, u_{n-1}).$$

Thus, u is a critical point of J on E_{mT} if and only if

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) + q_n u_n^{\delta} + f(n, u_{n+1}, u_n, u_{n-1}) = 0,\ \forall n \in \mathbb{Z}(1, mT).$$

Due to the periodicity of $u = \{u_n\}_{n\in\mathbb{Z}} \in E_{mT}$ and $f(n, v_1, v_2, v_3)$ in the first variable n, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J on E_{mT}. That is, the functional J is just the variational framework of (1.1).

Let

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

be a $mT \times mT$ matrix. By matrix theory, we see that the eigenvalues of P are

$$\lambda_k = 2\left(1 - \cos\frac{2k}{mT}\pi\right), k = 0, 1, 2, \cdots, mT-1. \tag{2.6}$$

Thus, $\lambda_0 = 0, \lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_{mT-1} > 0$. Therefore,

$$\begin{cases} \lambda_{\min} = \min\{\lambda_1, \lambda_2, \cdots, \lambda_{mT-1}\} = 2\left(1 - \cos\frac{2}{mT}\pi\right), \\ \lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_{mT-1}\} = \begin{cases} 4, & \text{when mT is even,} \\ 2\left(1 + \cos\frac{1}{mT}\pi\right), & \text{when mT is odd.} \end{cases} \end{cases} \tag{2.7}$$

Let

$$W = \ker P = \{u \in E_{mT} | Pu = 0 \in \mathbb{R}^{mT}\}.$$

Then

$$W = \{u \in E_{mT} | u = \{c\},\ c \in \mathbb{R}\}.$$

Let V be the direct orthogonal complement of E_{mT} to W, i.e., $E_{mT} = V \oplus W$. For convenience, we identify $u \in E_{mT}$ with $u = (u_1, u_2, \cdots, u_{mT})^*$.

Let E be a real Banach space, $J \in C^1(E, \mathbb{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{u^{(k)}\} \subset E$ for which $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)}) \to 0 (k \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Saddle Point Theorem [14])**.** *Let E be a real Banach space, $E = E_1 \oplus E_2$, where $E_1 \neq \{0\}$ and is finite dimensional. Suppose that $J \in C^1(E, \mathbb{R})$ satisfies the P.S. condition and*
(J_1) there exist constants $\sigma,\ \rho > 0$ such that $J|_{\partial B_\rho \cap E_1} \leq \sigma$;
(J_2) there exists $e \in B_\rho \cap E_1$ and a constant $\omega \geq \sigma$ such that $J_{e+E_2} \geq \omega$.

Then J possesses a critical value $c \geq \omega$, where

$$c = \inf_{h \in \Gamma} \max_{u \in B_\rho \cap E_1} J(h(u)),\ \Gamma = \{h \in C(\bar{B}_\rho \cap E_1, E) \mid h|_{\partial B_\rho \cap E_1} = id\}$$

and id denotes the identity operator.

Lemma 2.2. *Assume that $(F_1) - (F_3)$ are satisfied. Then J satisfies the P.S. condition.*

Proof. Let $\{u^{(k)}\} \subset E_{mT}$ be such that $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)}) \to 0$ as $k \to \infty$. Then there exists a positive constant M_1 such that $|J(u^{(k)})| \leq M_1$.

For k large enough, we have

$$\left|\left\langle J'\left(u^{(k)}\right), u^{(k)}\right\rangle\right| \leq \left\|u^{(k)}\right\|_2.$$

So

$$\begin{aligned} & M_1 + \frac{1}{\delta+1}\left\|u^{(k)}\right\|_2 \\ \geq\ & J\left(u^{(k)}\right) - \frac{1}{\delta+1}\left\langle J'\left(u^{(k)}\right), u^{(k)}\right\rangle \\ =\ & \sum_{n=1}^{mT}\left[F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) - \frac{1}{\delta+1}\left(\frac{\partial F\left(n-1, u_n^{(k)}, u_{n-1}^{(k)}\right)}{\partial v_2}\cdot u_n^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_2}\cdot u_n^{(k)}\right)\right] \\ =\ & \sum_{n=1}^{mT}\left[F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) - \frac{1}{\delta+1}\left(\frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_1}\cdot u_{n+1}^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_2}\cdot u_n^{(k)}\right)\right]. \end{aligned}$$

Take

$$I_1 = \left\{n \in \mathbb{Z}(1, mT) | \sqrt{\left(u_{n+1}^{(k)}\right)^2 + \left(u_n^{(k)}\right)^2} \geq R_1\right\},\ I_2 = \left\{n \in \mathbb{Z}(1, mT) | \sqrt{\left(u_{n+1}^{(k)}\right)^2 + \left(u_n^{(k)}\right)^2} < R_1\right\}.$$

By (F_2), we have

$$M_1 + \frac{1}{\delta+1}\left\|u^{(k)}\right\|_2$$

$$
\begin{aligned}
&\geq \sum_{n=1}^{mT} F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) - \frac{1}{\delta+1} \sum_{n \in I_1} \left[\frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_1} \cdot u_{n+1}^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_2} \cdot u_n^{(k)} \right] \\
&\quad - \frac{1}{\delta+1} \sum_{n \in I_2} \left[\frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_1} \cdot u_{n+1}^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_2} \cdot u_n^{(k)} \right] \\
&\geq \sum_{n=1}^{mT} F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) - \frac{\alpha}{2} \sum_{n \in I_1} F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) \\
&\quad - \frac{1}{\delta+1} \sum_{n \in I_2} \left[\frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_1} \cdot u_{n+1}^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_2} \cdot u_n^{(k)} \right] \\
&= \left(1 - \frac{\alpha}{2}\right) \sum_{n=1}^{mT} F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) \\
&\quad + \frac{1}{\delta+1} \sum_{n \in I_2} \left[\frac{\alpha}{2}(\delta+1) F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) - \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_1} \cdot u_{n+1}^{(k)} - \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_2} \cdot u_n^{(k)} \right].
\end{aligned}
$$

The continuity of $\frac{\alpha}{2}(\delta+1)F(n, v_1, v_2) - \frac{\partial F(n,v_1,v_2)}{\partial v_1} v_1 - \frac{\partial F(n,v_1,v_2)}{\partial v_2} v_2$ with respect to the second and third variables implies that there exists a constant $M_2 > 0$ such that

$$
\frac{\alpha}{2}(\delta+1)F(n, v_1, v_2) - \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 - \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 \geq -M_2,
$$

for $n \in \mathbb{Z}(1, mT)$ and $\sqrt{v_1^2 + v_2^2} \leq R_1$. Therefore,

$$
M_1 + \frac{1}{\delta+1} \left\| u^{(k)} \right\|_2 \geq \left(1 - \frac{\alpha}{2}\right) \sum_{n=1}^{mT} F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) - \frac{1}{\delta+1} mTM_2.
$$

By (F_3), we get

$$
\begin{aligned}
M_1 + \frac{1}{\delta+1} \left\| u^{(k)} \right\|_2 &\geq \left(1 - \frac{\alpha}{2}\right) a_1 \sum_{n=1}^{mT} \left[\sqrt{\left(u_{n+1}^{(k)}\right)^2 + \left(u_n^{(k)}\right)^2} \right]^{\frac{\gamma}{2}(\delta+1)} - \left(1 - \frac{\alpha}{2}\right) a_2 mT - \frac{1}{\delta+1} mTM_2 \\
&\geq \left(1 - \frac{\alpha}{2}\right) a_1 \sum_{n=1}^{mT} \left| u_n^{(k)} \right|^{\frac{\gamma}{2}(\delta+1)} - M_3,
\end{aligned}
$$

where $M_3 = \left(1 - \frac{\alpha}{2}\right) a_2 mT + \frac{1}{\delta+1} mTM_2$. Combining with (2.4), we have

$$
M_1 + \frac{1}{\delta+1} \left\| u^{(k)} \right\|_2 \geq \left(1 - \frac{\alpha}{2}\right) a_1 c_1^{\frac{\gamma}{2}(\delta+1)} \left\| u^{(k)} \right\|_2^{\frac{\gamma}{2}(\delta+1)} - M_3.
$$

Thus,

$$
\left(1 - \frac{\alpha}{2}\right) a_1 c_1^{\frac{\gamma}{2}(\delta+1)} \left\| u^{(k)} \right\|_2^{\frac{\gamma}{2}(\delta+1)} - \frac{1}{\delta+1} \left\| u^{(k)} \right\|_2 \leq M_1 + M_3.
$$

This implies that $\left\{ \left\| u^{(k)} \right\|_2 \right\}$ is bounded on the finite dimensional space E_{mT}. As a consequence, it has a convergent subsequence. □

3. Proof of the main results

In this Section, we shall prove our main results by using the critical point theory.

Proof. By Lemma 2.2, J satisfies the P.S. condition. To apply the Saddle Point Theorem, it suffices to prove that J satisfies the conditions (J_1) and (J_2).

For any $w \in W$, since $H(w) = 0$, we have

$$J(w) = \frac{1}{\delta+1}\sum_{n=1}^{mT} q_n w_n^{\delta+1} + \sum_{n=1}^{mT} F(n, w_{n+1}, w_n).$$

By (F_3),

$$J(w) \geq a_1 \sum_{n=1}^{mT}\left(\sqrt{w_{n+1}^2 + w_n^2}\right)^{\frac{\gamma}{2}(\delta+1)} - a_2 mT \geq -a_2 mT.$$

Since

$$\frac{\underline{p}}{\delta+1} c_1^{\delta+1}\left[\left(\sum_{n=1}^{mT}(\Delta v_n)^2\right)^{\frac{1}{2}}\right]^{\delta+1} \leq H(v) \leq \frac{\bar{p}}{\delta+1} c_2^{\delta+1}\left[\left(\sum_{n=1}^{mT}(\Delta v_n)^2\right)^{\frac{1}{2}}\right]^{\delta+1},$$

and

$$\lambda_{\min}\|v\|_2^2 \leq \sum_{n=1}^{mT}(\Delta v_n)^2 = v^* P v \leq \lambda_{\max}\|v\|_2^2,$$

we get

$$\frac{\underline{p}}{\delta+1} c_1^{\delta+1}\lambda_{\min}^{\frac{\delta+1}{2}}\|v\|_2^{\delta+1} \leq H(v) \leq \frac{\bar{p}}{\delta+1} c_2^{\delta+1}\lambda_{\max}^{\frac{\delta+1}{2}}\|v\|_2^{\delta+1}. \tag{3.1}$$

Besides,

$$\underline{q} c_2^{\delta+1}\|v\|_2^{\delta+1} \leq \underline{q}\sum_{n=1}^{mT} v_n^{\delta+1} \leq \sum_{n=1}^{mT} q_n v_n^{\delta+1} \leq \bar{q}\sum_{n=1}^{mT} v_n^{\delta+1} \leq \bar{q} c_1^{\delta+1}\|v\|_2^{\delta+1}. \tag{3.2}$$

Combining with (F_2'), (2.4), (3.1) and (3.2), for any $v \in V$, we have

$$\begin{aligned}
J(v) &= -H(v) + \frac{1}{\delta+1}\sum_{n=1}^{mT} q_n v_n^{\delta+1} + \sum_{n=1}^{mT} F(n, v_{n+1}, v_n),\\
&\leq -\frac{\underline{p}}{\delta+1} c_1^{\delta+1}\lambda_{\min}^{\frac{\delta+1}{2}}\|v\|_2^{\delta+1} + \frac{\bar{q}}{\delta+1} c_1^{\delta+1}\|v\|_2^{\delta+1} + a_3\sum_{n=1}^{mT}\left(\sqrt{v_{n+1}^2 + v_n^2}\right)^{\frac{\alpha}{2}(\delta+1)} + a_4 mT\\
&\leq -\frac{\underline{p}}{\delta+1} c_1^{\delta+1}\lambda_{\min}^{\frac{\delta+1}{2}}\|v\|_2^{\delta+1} + \frac{\bar{q}}{\delta+1} c_1^{\delta+1}\|v\|_2^{\delta+1} + a_3 c_2^{\frac{\alpha}{2}(\delta+1)}\left[\sum_{n=1}^{mT}\left(v_{n+1}^2 + v_n^2\right)\right]^{\frac{\alpha}{4}(\delta+1)} + a_4 mT\\
&\leq -\frac{\underline{p}}{\delta+1} c_1^{\delta+1}\lambda_{\min}^{\frac{\delta+1}{2}}\|v\|_2^{\delta+1} + \frac{\bar{q}}{\delta+1} c_1^{\delta+1}\|v\|_2^{\delta+1} + 2^{\frac{\alpha}{4}(\delta+1)} a_3 c_2^{\frac{\alpha}{2}(\delta+1)}\|v\|_2^{\frac{\alpha}{2}(\delta+1)} + a_4 mT.
\end{aligned}$$

Let $\mu = -a_2 mT$, since $1 < \alpha < 2$, there exists a constant $\rho > 0$ large enough such that

$$J(v) \leq \mu - 1 < \mu, \ \forall v \in V, \ \|v\|_2 = \rho.$$

Thus, by Lemma 2.1, Eq. (1.1) has at least one mT-periodic solution. □

Remark 3.1. Due to Theorem 1.1, the conclusion of Theorem 1.3 is obviously true.

4. Example

As an application of the main theorem, we give an example to illustrate our result.

Example 4.1. For all $n \in \mathbb{Z}$, assume that

$$\Delta\left(\sin^2\left(\frac{\pi n}{T}\right)(\Delta u_{n-1})^3\right) + \cos^2\left(\frac{\pi n}{T}\right)u_n^3 + 6u_n\left[\psi(n)\left(u_{n+1}^2 + u_n^2\right)^2 + \psi(n-1)\left(u_n^2 + u_{n-1}^2\right)^2\right] = 0, \quad (4.1)$$

where ψ is continuously differentiable and $\psi(n) > 0$, T is a given positive integer, $\psi(n+T) = \psi(n)$. We have

$$f(n, v_1, v_2, v_3) = 6v_2\left[\psi(n)\left(v_1^2 + v_2^2\right)^2 + \psi(n-1)\left(v_2^2 + v_3^2\right)^2\right]$$

and

$$F(n, v_1, v_2) = \psi(n)\left(v_1^2 + v_2^2\right)^3.$$

Then

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = 6v_2\left[\psi(n)\left(v_1^2 + v_2^2\right)^2 + \psi(n-1)\left(v_2^2 + v_3^2\right)^2\right].$$

It is easy to verify all the assumptions of Theorem 1.1 are satisfied. Consequently, for any given positive integer $m > 0$, (4.1) has at least one mT-periodic solution.

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