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Some new fixed point results in partial ordered metric spaces via admissible mappings and two new functions

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Abstract

The purpose of this paper is to discuss the existence of fixed points for new classes of mappings defined on an ordered metric space. The obtained results generalize and improve some fixed point results in the literature. Some examples show the usefulness of our results.

Keywords: Common fixed point, generalized weakly contraction, generalized metric spaces, upper class, C-class function.
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1. Introduction and preliminries

In 1922, Banach proposed the well known Banach's contraction principle in [10]. From then on, many researchers focused on the fixed point theory and made much contribution to nonlinear analysis. It plays an

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important role in mathematics and has various applications, such as integro-differential equation, economic equilibrium theory, etc. Later on, in the past half century, it has been extensively studied and generalized to many settings; see [2, 3, 5, 6, 7, 11, 12, 14, 15, 18, 20, 21, 24, 25, 26, 27, 28, 29, 34, 35, 36, 37, 39, 41, 42, 44, 45].

We recall some definitions in the following:

Definition 1.1. The function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

i) ψ is continuous and non-decreasing,

ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.2. Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be weakly contractive if

$$d(fx, fy) \le d(x, y) - \varphi(d(x, y)) \ \forall\ x, y \in X,$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is an altering distance function.

In 2008, the weak contractive mapping was generalized by Dutta and Choudhury [17] and they proved the following theorem.

Theorem 1.3 ([17])**.** *Let (X, d) be a complete metric space and $f : X \to X$ satisfy*

$$\psi(d(fx, fy)) \le \psi(d(x, y)) - \varphi(d(x, y)) \ \forall\ x, y \in X,$$

where $\psi, \varphi : [0, +\infty) \to: [0, +\infty)$ are altering distance functions. Then f has a unique fixed point in X.

It is obvious that if $\psi(t) = t$, then the contraction in Theorem 1.3 reduces to a weakly contraction. Thus, Theorem 1.3 holds if $\psi(t) = t$ and φ is weakly contractive. Namely, the case that $\psi(t) = t$ and φ is weakly contractive, is taken as a particular case of Theorem 1.3. Furthermore, Theorem 1.3 holds when ϕ is lower semi-continuous and $\phi(t) = 0$ if and only if $t = 0$ (see [1, 16]). Subsequently, the concept of (ψ, α, β)-weak contraction was introduced by Eslamian and Abkar [18]. They improved Theorem 1.3 by a more weak contraction.

Theorem 1.4. *Let (X, d) be a complete metric space and $f : X \to X$ satisfies*

$$\psi(d(fx, fy)) \le \alpha\,(d(x, y)) - \beta(d(x, y))$$

for all $x, y \in X$, where $\psi, \alpha, \beta : [0, +\infty) \to [0, +\infty)$ are such that ψ is an altering distance function, α is continuous, β is lower semi-continuous and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall\ t > 0$$

and $\alpha(0) = \beta(0) = 0$. Then f has a unique fixed point.

In 2012, Aydi et al. [8] defined $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \psi(t) - \alpha(t) + \beta(t)$ for all $t \ge 0$ and proved that actually Theorem 1.3 implied Theorem 1.4.

Simultaneously, firstly in 2004, Ran and Reurings [38] introduced a partial order relation in the metric spaces. Then the researchers turned from metric space into partially ordered metric space. Therefore, the scope of space is expanded. In 2010, Harjani and Sadarangani [22] extended Theorem 1.3 in the setting of partially ordered metric spaces.

Theorem 1.5. *Let $(X, d \preceq)$ be a partially ordered complete metric space and $f : X \to X$ be continuous non-decreasing such that*

$$\psi(d(fx, fy)) \le \psi(d(x, y)) - \varphi(d(x, y)) \quad \forall x, y\ (x \preceq y),$$

where $\psi, \varphi : [0, +\infty) \to: [0, +\infty)$ are altering distance functions. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point $x^* \in X$.

Choudhury and Kundu [13] generalized Theorems 1.4 and 1.5 as follows.

Theorem 1.6. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $f : X \to X$ be a non-decreasing mapping such that*

$$\psi(d(fx, fy)) \leq \alpha\,(d(x, y)) - \beta(d(x, y)) \;\; \forall\, x \preceq y,$$

where $\psi, \alpha, \beta : [0, +\infty) \to [0, +\infty)$ *are functions such that* ψ *is an altering distance function,* α *is continuous,* β *is lower semi-continuous,*

$$\psi(t) - \alpha(t) + \beta(t) > 0 \;\; \forall\, t > 0$$

and $\alpha(0) = \beta(0) = 0$. *If there exists* $x_0 \in X$ *such that* $x_0 \preceq fx_0$, *then* f *has a unique fixed point* $x^* \in X$.

Aydi et al. [8] proved that Theorem 1.5 deduced Theorem 1.6. Karapinar and Salimi ([31]) proved the following theorem which shows that Theorems 1.4 and 1.5 can be generalized and the methods of Aydi et al. [8] cannot be used for the proof.

Theorem 1.7. *Let* $(X, d, \preceq)$ *be an ordered metric space such that* (X, d) *is complete and let* $f : X \to X$ *be a non-decreasing self mapping. Assume that there exist* $\psi \in \Psi$, $\alpha \in \Phi_\alpha$ *and* $\beta \in \Phi_\beta$ *such that*

$$\psi(t) - \alpha(s) + \beta(s) > 0 \text{ for all } t > 0 \text{ and } s = t \text{ or } s = 0$$

and

$$\psi(d(fx, fy)) \leq \alpha\,(d(x, y)) - \beta(d(x, y)) \text{ for all comparable } x, y \in X,$$

where $\Psi = \{\psi : [0, \infty) \to [0, \infty) \mid \psi \text{ is non-decreasing and lower semicontinuous}\}$, $\Phi_\alpha = \{\alpha : [0, \infty) \to [0, \infty) \mid \alpha \text{ is upper semicontinuous}\}$ *and* $\Phi_\beta = \{\beta : [0, \infty) \to [0, \infty) \mid \beta \text{ is lower semicontinuous}\}$.

Suppose that either

(a) f *is continuous, or*

(b) if a non-decreasing sequence $\{x_n\}$ *is such that* $x_n \to x$, *then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

Then f *has a fixed point.*

We list some basic definitions in the following.

Definition 1.8 ([43])**.** Let $T : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$. We say that T is an $\alpha-$admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$, $x, y \in X$.

Definition 1.9 ([40])**.** Let $T : X \to X$, $\eta : X \times X \to \mathbb{R}^+$. We say that T is an $\eta-$subadmissible mapping if $\eta(x, y) \leq 1$ implies $\eta(Tx, Ty) \leq 1, x, y \in X$.

We recollect the following auxiliary result which will be used efficiently in the proofs of main results.

Definition 1.10 ([30])**.** An α-admissible mapping $T : X \to X$ is called triangular α-admissible if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$.

Lemma 1.11 ([30])**.** *Let* $T : X \to X$ *be a triangular* α*-admissible mapping. Assume that there exists* $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$. *Define a sequence* $\{x_n\}$ *by* $x_{n+1} = Tx_n$. *Then we have* $\alpha(x_n, x_m) \geq 1$ *for all* $m, n \in \mathbb{N}$ *with* $n < m$.

Definition 1.12 ([30])**.** An η-subadmissible mapping $T : X \to X$ is called triangular η-subadmissible if $\eta(x, z) \leq 1$ and $\eta(z, y) \leq 1$ imply $\eta(x, y) \leq 1$.

Lemma 1.13 ([30])**.** *Let* $T : X \to X$ *be a triangular* η*-subadmissible map. Assume that there exists* $x_1 \in X$ *such that* $\eta(x_1, Tx_1) \leq 1$. *Define a sequence* $\{x_n\}$ *by* $x_{n+1} = Tx_n$. *Then we have* $\eta(x_n, x_m) \leq 1$ *for all* $m, n \in \mathbb{N}$ *with* $n < m$.

In [33], Long et al. considered the family of non-decreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ and proved the following theorem.

Theorem 1.14. *Let* (X, d) *be a complete metric space and* T *be an* α*-admissible mapping. Assume that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y) \;\; \forall\, x, y \in X,$$

where $\psi \in \Psi$. *Also, suppose that the following assertions hold* *(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(ii) either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Definition 1.15 ([3])**.** A function $\varphi : [0, \infty) \to [0, \infty)$ is called an ultra-altering distance function if the following conditions hold:

1. φ is continuous,
2. $\varphi(0) \geq 0$ and $\varphi(t) \neq 0, t \neq 0$.

Lemma 1.16 ([9])**.** *Suppose that (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$, $d(x_{m(k)-1}, x_{n(k)}) \leq \epsilon$ and*
(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon$;
(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$;
(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$.

Remark 1.17. We can get $\lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon$.

2. Main results

The concept of C-class functions (see Definition 2.1) was introduced by Ansari [3]. See Example 2.2 and [5, 19, 23, 32].

Definition 2.1 ([3])**.** A function $f : [0, \infty)^2 \to \mathbb{R}$ is called a *C-class* function if it is continuous and satisfies the following axioms:

(1) $f(s, t) \leq s$;
(2) $f(s, t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

Note that we have $f(0, 0) = 0$.
We denote C-class functions as $\mathcal{C}$.

Example 2.2 ([3])**.** The following functions $f : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:

(1) $f(s, t) = s - t$, $f(s, t) = s \Rightarrow t = 0$.

(2) $f(s, t) = ms$, $0<m<1$, $f(s, t) = s \Rightarrow s = 0$.

(3) $f(s, t) = \frac{s}{(1+t)^r}$; $r \in (0, \infty)$, $f(s, t) = s \Rightarrow s = 0$ or $t = 0$.

(4) $f(s, t) = \log(t + a^s)/(1 + t)$, $a > 1$, $f(s, t) = s \Rightarrow s = 0$ or $t = 0$.

(5) $f(s, t) = \ln(1 + a^s)/2$, $a > e$, $f(s, t) = s \Rightarrow s = 0$.

(6) $f(s, t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0, \infty)$, $f(s, t) = s \Rightarrow t = 0$.

(7) $f(s, t) = s \log_{t+a} a$, $a > 1$, $f(s, t) = s \Rightarrow s = 0$ or $t = 0$.

(8) $f(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $f(s, t) = s \Rightarrow t = 0$.

(9) $f(s, t) = s\beta(s)$, $\beta : [0, \infty) \to [0, 1)$ and is continuous, $f(s, t) = s \Rightarrow s = 0$.

(10) $f(s, t) = s - \frac{t}{k+t}$, where $k > 0$, $f(s, t) = s \Rightarrow t = 0$.

(11) $f(s, t) = s - \varphi(s), f(s, t) = s \Rightarrow s = 0$, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$.

(12) $f(s, t) = sh(s, t)$, $f(s, t) = s \Rightarrow s = 0$, where $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0$.

(13) $f(s,t) = s - (\frac{2+t}{1+t})t$, $f(s,t) = s \Rightarrow t = 0$.

(14) $f(s,t) = \sqrt[n]{\ln(1+s^n)}$, $f(s,t) = s \Rightarrow s = 0$.

(15) $f(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$, where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.

(16) $f(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$, $f(s,t) = s \Rightarrow s = 0$.

Remark 2.3. We assume that F is increasing with respect to the first variable and decreasing with respect the second variable.

The concept of a *upper class* (F, h) was introduced by Ansari [4, 6].

Definition 2.4 ([4, 6])**.** We say that $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type II if it satisfies

$$x, y \geq 1 \Longrightarrow h(1,1,z) \leq h(x,y,z) \quad \forall x,y,z \in \mathbb{R}^+.$$

Example 2.5.

(1) $h(x,y,z) = (z+l)^{xy}, l > 1$.
(2) $h(x,y,z) = (xy+l)^z, l > 0$.
(3) $h(x,y,z) = xyz$.
(4) $h(x,y,z) = xz$.
(5) $h(x,y,z) = z$.
(6) $h(x,y,z) = (\frac{x+y}{2})z$.
(7) $h(x,y,z) = \frac{x+xy+y}{3}z$.
(8) $h(x,y,z) = (\frac{\sum_{i=0}^{n} x^{n-i}y^i}{n+1})z$.
(9) $h(x,y,z) = (\frac{\sum_{i=0}^{n} x^{n-i}y^i}{n+1} + l)^z, l > 1$.
(10) $h(x,y,z) = x^m y^n z^p, m,n,p \in \mathbb{N}$.
(11) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m,n,p,q,k \in \mathbb{N}$.

Note that $z \leq h(1,1,z)$.

Definition 2.6 ([4, 6])**.** Let $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. We say the pair (F, h) is a upper class of type II if F is a function, h is a subclass of type II and

$$0 \leq s \leq 1 \Longrightarrow F(s,t) \leq F(1,t),$$

$$h(1,1,z) \leq F(s,t) \Longrightarrow z \leq st \quad \forall z,s,t \in \mathbb{R}^+.$$

Example 2.7.

(1) $h(x,y,z) = (z+l)^{xy}, l > 1$, $F(s,t) = st + l$.
(2) $h(x,y,z) = (xy+l)^z, l > 0$, $F(s,t) = (1+l)^{st}$.
(3) $h(x,y,z) = xyz$, $F(s,t) = st$.
(4) $h(x,y,z) = xz$, $F(s,t) = st$.
(5) $h(x,y,z) = z$, $F(s,t) = st$.
(6) $h(x,y,z) = \frac{x+xy+y}{3}z$, $F(s,t) = st$.
(7) $h(x,y,z) = (\frac{x+y}{2})z$, $F(s,t) = st$.
(8) $h(x,y,z) = (\frac{\sum_{i=0}^{n} x^{n-i}y^i}{n+1})z$, $F(s,t) = st$.
(9) $h(x,y,z) = (\frac{\sum_{i=0}^{n} x^{n-i}y^i}{n+1} + l)^z, l > 1$, $F(s,t) = (1+l)^{st}$.
(10) $h(x,y,z) = x^m y^n z^p, m,n,p \in \mathbb{N}$, $F(s,t) = s^p t^p$.
(11) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m,n,p,q,k \in \mathbb{N}$, $F(s,t) = (st)^k$.
(12) $h(x,y,z) = (z+l)^{xy}, l > 1$, $F(s,t) = st + \frac{l}{k}, k \geq 1$.

Definition 2.8. Let $F : R^+ \times R^+ \to R$. We say the pair (F, h) is a special upper class of type II if F is a function, h is a subclass of type II and

$$0 \leq s \leq 1 \Longrightarrow F(s,t) \leq F(1,t),$$

$$h(1,1,z) \leq F(1,t) \Longrightarrow z \leq t \quad \forall x,y,z,s,t \in \mathbb{R}^+.$$

Example 2.9.

(1) $h(x,y,z) = (z^k + l)^{xy},\ F(s,t) = st^k + l,\ l > 1,\ k > 0.$
(2) $h(x,y,z) = (xy + l)^z,\ F(s,t) = (1 + l)^{st}, l, k > 0.$
(3) $h(x,y,z) = xyz,\ F(s,t) = s^m t.$
(4) $h(x,y,z) = xz^k,\ F(s,t) = s^m t^k,\ k > 0.$
(5) $h(x,y,z) = z,\ F(s,t) = st.$
(6) $h(x,y,z) = \frac{x+xy+y}{3} z, F(s,t) = st.$
(7) $h(x,y,z) = (\frac{x+y}{2}) z, F(s,t) = st.$
(8) $h(x,y,z) = (\frac{\sum_{i=0}^{n} x^{n-i} y^i}{n+1}) z^k, F(s,t) = s^m t^k, k > 0.$
(9) $h(x,y,z) = (\frac{\sum_{i=0}^{n} x^{n-i} y^i}{n+1} + l)^z, l > 1, F(s,t) = (1 + l)^{st}.$
(10) $h(x,y,z) = x^m y^n z^p, m, n, p \in \mathbb{N}, F(s,t) = s^q t^p.$
(11) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}, F(s,t) = st^k, k > 0.$

Definition 2.10 ([4, 6])**.** We say that $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I if it satisfies

$$x \geq 1 \Longrightarrow h(1,y) \leq h(x,y) \quad \forall x,y \in \mathbb{R}^+.$$

Example 2.11.

(1) $h(x,y) = (y + l)^x, l > 1.$
(2) $h(x,y) = (x + l)^y, l > 0.$
(3) $h(x,y) = xy.$
(4) $h(x,y) = ((x + 1)/2)y.$
(5) $h(x,y) = \frac{2x+1}{3} y.$
(6) $h(x,y) = \frac{x^n + x^{n-1} + \ldots + x + 1}{n+1} y.$
(7) $h(x,y) = (\frac{x^n + x^{n-1} + \ldots + x + 1}{n+1} + l)^y, l > 1.$

Note that $y \leq h(1,y)$.

Definition 2.12 ([4, 6])**.** Let $h, F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. We say pair (F, h) is a upper class of type I if F is a function, h is a subclass of type I and

$$0 \leq s \leq 1 \Longrightarrow F(s,t) \leq F(1,t),$$

$$h(1,y) \leq F(s,t) \Longrightarrow y \leq st \quad \forall x,y,s,t \in \mathbb{R}^+.$$

Example 2.13.

(1) $h(x,y) = (y + l)^x, l > 1, F(s,t) = st + l.$
(2) $h(x,y) = (x + l)^y, l > 0, F(s,t) = (1 + l)^{st}.$
(3) $h(x,y) = xy, F(s,t) = st.$
(4) $h(x,y) = ((x + 1)/2)y, F(s,t) = st.$
(5) $h(x,y) = \frac{2x+1}{3} y, F(s,t) = st.$
(6) $h(x,y) = \frac{x^n + x^{n-1} + \ldots + x + 1}{n+1} y, F(s,t) = st.$
(7) $h(x,y) = (\frac{x^n + x^{n-1} + \ldots + x + 1}{n+1} + l)^y, l > 1, F(s,t) = (1 + l)^{st}.$
(8) $h(x,y) = (y + l)^x, l > 1, F(s,t) = st + \frac{l}{k}, k \geq 1.$

Definition 2.14. Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. We say the pair (F, h) is a special upper class of type I if F is a function, h is a subclass of type I and

$$0 \le s \le 1 \Longrightarrow F(s,t) \le F(1,t),$$

$$h(1,y) \le F(1,t) \Longrightarrow y \le t \quad \forall x,y,s,t \in \mathbb{R}^+.$$

Example 2.15.

(1) $h(x,y) = (y^k + l)^x, l > 1, F(s,t) = s^m t^k + l, k > 0.$
(2) $h(x,y) = (x^n + l)^{y^k}, l > 1, F(s,t) = (1 + l)^{s^m t^k}, k > 0.$
(3) $h(x,y) = x^n y^k, F(s,t) = s^m t^k, k > 0.$
(4) $h(x,y) = ((x + 1)/2)y, F(s,t) = st.$
(5) $h(x,y) = \frac{2x+1}{3} y, F(s,t) = st.$
(6) $h(x,y) = (\frac{x^n + x^{n-1} + ... + x + 1}{n+1}) y^k, F(s,t) = st^k, k > 0.$
(7) $h(x,y) = (\frac{x^n + x^{n-1} + ... + x + 1}{n+1} + l)^y, l > 1, F(s,t) = (1 + l)^{st}.$

Theorem 2.16. *Let $(X, d, \preceq)$ be a partially ordered metric space on X such that (X, d) is complete, $T : X \to X$ a self mapping and let $\gamma : X \times X \longrightarrow [0,\infty)$ be a function such that T is a non-decreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$ and $\beta \in \Phi_\beta$ which are defined in* Theorem 1.7 *such that*

$$\psi(t) - f(\alpha(s), \beta(s)) > 0 \text{ for all } \quad t > 0 \text{ and } s = t \text{ or } s = 0, \tag{2.1}$$

where f is an element of $\mathcal{C}$ and for all $x, y \in X$, we have

$$\gamma(x,Tx)\gamma(y,Ty) \ge 1 \Longrightarrow h(\gamma(x,x), \gamma(y,y), \psi(d(Tx,Ty))) \le F(1, f(\alpha(d(x,y)), \beta(d(x,y)))), \tag{2.2}$$

where the pair (F, h) is an upper class of type II. Suppose that either

(i) T is continuous or

(ii) if a non-decreasing sequence $\{x_n\} \subset X$ is such that $x_n \to x$ as $n \to \infty$ and $\gamma(x_n, Tx_n) \ge 1$ and $\gamma(x_n, x_n) \ge 1$ for all n, then $\gamma(x, Tx) \ge 1$ and $\gamma(x, x) \ge 1$ and $x_n \preceq x$ for all $n \in \mathbb{N}$.

If there exists $x_0 \in X$ such that $\gamma(x_0, x_0) \ge 1, \gamma(x_0, Tx_0) \ge 1$ and $x_0 \preceq Tx_0$, then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\gamma(x_0, Tx_0) \ge 1$.

Define a sequence $\{x_n\} \subset X$ by

$$x_n = T^n(x_0) = Tx_{n-1} \text{ for } n \in \mathbb{N}.$$

Suppose that $x_{n_0} = x_{n_0 - 1}$ for some $n_0 \in N$. Then it is clear that x_{n_0} is a fixed point of T and hence the proof is completed.

From now on, we suppose that $x_n \ne x_{n-1}$ for all $n \in \mathbb{N}$.

Since T is a γ-admissible mapping and $\gamma(x_0, Tx_0) \ge 1$, we deduce that $\gamma(x_1, Tx_1) = \gamma(Tx_0, T^2 x_0) \ge 1$. By continuing this process, we get that $\gamma(x_n, Tx_n) = \gamma(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Also from $\gamma(x_0, x_0) \ge 1$ we obtain $\gamma(x_1, x_1) = \gamma(Tx_0, Tx_0) \ge 1$, by continuing this process, we get $\gamma(x_n, x_n) \ge 1$ for $n \in \mathbb{N} \cup \{0\}$

Since T is non-decreasing and $x_0 \preceq Tx_0$, we have

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_{n-1} \preceq x_n \preceq \cdots . \tag{2.3}$$

We will show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $\gamma(x_n, x_n) \ge 1$, $\gamma(x_n, Tx_n) \ge 1$ for each $n \in N$, by (2.2), we have

$$h(1, 1, \psi(d(x_n, x_{n+1}))) \le h(\gamma(x_{n-1}, x_{n-1}), \gamma(x_n, x_n), \psi(d(x_n, x_{n+1})))$$

$$\begin{aligned}&= h(\gamma(x_{n-1}, x_{n-1}), \gamma(x_n, x_n), \psi(d(Tx_{n-1}, Tx_n)))\\&\leq F(1, f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n)))).\end{aligned}$$

Hence

$$\psi(d(x_n, x_{n+1})) \leq f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n))). \tag{2.4}$$

If $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$, since ψ is non-decreasing, then

$$\psi(d(x_{n-1}, x_n)) \leq \psi(d(x_n, x_{n+1})).$$

Combining with (2.4), we obtain that

$$\begin{aligned}\psi(d(x_{n-1}, x_n)) &\leq \psi(d(x_n, x_{n+1}))\\&\leq f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n)).\end{aligned}$$

It yields that

$$\psi(d(x_{n-1}, x_n)) \leq f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n)).$$

Since $x_{n-1} \neq x_n$, $d(x_{n-1}, x_n) > 0$, then the above inequality contradicts (2.1). Therefore, our assumption $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$ is wrong. Thus we have that

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

Set $d_n = d(x_n, x_{n+1})$. Then the sequence $\{d_n\}$ is non-increasing. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d_n = r$. Suppose that $r > 0$. By (2.4), we have

$$\begin{aligned}\psi(r) &\leq \lim_{n\to\infty} \inf \psi(d_n)\\&\leq \lim_{n\to\infty} \sup \psi(d_n)\\&\leq \lim_{n\to\infty} \sup f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n)))\\&\leq f(\alpha(r), \beta(r)),\end{aligned}$$

which is a contradiction to (2.1). Therefore, $r = 0$. Hence

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

We shall show that the sequence $\{x_n\}$ is a Cauchy sequence.

Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 1.16 there exist an $\varepsilon > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{n\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{2.6}$$

Now, from (2.2) with $x = x_{m(k)}$ and $y = x_{n(k)}$, we have

$$\begin{aligned}h(1, 1, \psi(d(x_{m(k)+1}, x_{n(k)+1}))) &\leq h(\gamma(x_{m(k)}, x_{m(k)}), \gamma(x_{n(k)}, x_{n(k)}), \psi(d(x_{m(k)+1}, x_{n(k)+1})))\\&= h(\gamma(x_{m(k)}, x_{m(k)}), \gamma(x_{n(k)}, x_{n(k)}), \psi(d(Tx_{m(k)}, Tx_{n(k)})))\\&\leq F(1, f(\alpha(d(x_{m(k)}, x_{n(k)})), \beta(d(x_{m(k)}, x_{n(k)})))).\end{aligned}$$

Therefore,

$$\psi(d(x_{m(k)+1}, x_{n(k)+1})) \leq f(\alpha(d(x_{m(k)}, x_{n(k)})), \beta(d(x_{m(k)}, x_{n(k)}))).$$

Taking the $\lim\inf$ as $k \to +\infty$ in the above inequality, we have

$$\psi(\varepsilon) \leq \lim_{n\to\infty} \inf \psi(d(x_{m(k)+1}, x_{n(k)+1}))$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty} \sup \psi(d(x_{m(k)+1}, x_{n(k)+1}))\\
&\leq \lim_{n\to\infty} \sup f(\alpha(d(x_{m(k)}, x_{n(k)})), \beta(d(x_{m(k)}, x_{n(k)})))\\
&\leq f(\lim_{n\to\infty} \sup \alpha(d(x_{m(k)}, x_{n(k)})), \lim_{n\to\infty} \inf \beta(d(x_{m(k)}, x_{n(k)})))\\
&\leq f(\alpha(\varepsilon), \beta(\varepsilon)),
\end{aligned}
$$

where the fourth inequality is because of continuity of f. So we have

$$\psi(\varepsilon) \leq f(\alpha(\varepsilon), \beta(\varepsilon)),$$

which contradicts the fact that $\psi(t) - f(\alpha(t), \beta(t)) > 0$ for all $t > 0$. Hence

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = 0,$$

that is, the sequence $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists an $x^* \in X$ such that $x_n \to x^* \in X$.

Suppose that (i) holds. Then

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*.$$

Thus x^* is a fixed point of T.

Suppose that (ii) holds, that is, $\gamma(x_0, x_0) \geq 1, \gamma(x_0, Tx_0) \geq 1$ and $x_n \preceq x_0$. Now from (2.2), we have that

$$
\begin{aligned}
h(1, 1, \psi(d(Tx_0, x_{n+1}))) &\leq h(\gamma(x_0, x_0), \gamma(x_n, x_n), \psi(d(x_1, x_{n+1})))\\
&= h(\gamma(x_0, x_0), \gamma(x_n, x_n), \psi(d(Tx_0, Tx_n)))\\
&\leq F(1, f(\alpha(d(x_0, x_n)), \beta(d(x_0, x_n)))),
\end{aligned}
$$

which implies that

$$\psi(d(Tx_0, x_{n+1})) \leq f(\alpha(d(x_0, x_n)), \beta(d(x_0, x_n))).$$

Taking the $\liminf$ as $n \to \infty$ in the above inequality, we obtain

$$
\begin{aligned}
\psi(d(Tx_0, x_0)) &\leq \lim_{n\to\infty} \inf \psi(d(Tx_0, x_{n+1}))\\
&= \lim_{n\to\infty} \inf \psi(d(Tx_0, Tx_n))\\
&\leq \lim_{n\to\infty} \sup f(\alpha(d(x_0, x_n)), \beta(d(x_0, x_n)))\\
&\leq f(\alpha(0), \beta(0)).
\end{aligned}
$$

If $d(Tx_0, x_0) \neq 0$, then the above inequality contradicts to (2.1). Hence

$$\lim_{n\to\infty} d(Tx_0, x_{n+1}) = \lim_{n\to\infty} d(Tx_0, x_0) = 0$$

and so $Tx_0 = x_0$. Then x_0 is the fixed point of T. □

Corollary 2.17. *Let $(X, d, \preceq)$ be a partially ordered metric space on X such that (X, d) is complete, $T : X \to X$ a self mapping and let $\gamma : X \times X \to [0, \infty)$ be a function such that T is a non-decreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$ and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - f(\alpha(s), \beta(s)) > 0 \text{ for all } \quad t > 0 \text{ and } s = t \text{ or } s = 0,$$

where f is an element of $\mathcal{C}$ and for all $x, y \in X$, we have

$$\gamma(x, Tx)\gamma(y, Ty)h(\gamma(x, x), \gamma(y, y), \psi(d(Tx, Ty))) \leq F(1, f(\alpha(d(x, y)), \beta(d(x, y)))),$$

where the pair (F, h) is upper class of type II. Suppose that either *(i) T is continuous or*

(ii) if a non-decreasing sequence $\{x_n\} \subset X$ is such that $x_n \to x$ as $n \to \infty$ and $\gamma(x_n, Tx_n) \geq 1$ and $\gamma(x_n, x_n) \geq 1$ for all n, then $\gamma(x, Tx) \geq 1$ and $\gamma(x, x) \geq 1$ and $x_n \preceq x$.

If there exists $x_0 \in X$ such that $\gamma(x_0, x_0) \geq 1, \gamma(x_0, Tx_0) \geq 1$ and $x_0 \preceq Tx_0$, then T has a fixed point.

Taking $h(x, y, z) = (z + l)^{xy}, l > 1$, $F(s, t) = st + l$ and $f(s, t) = s - t$ in Theorem 1.14, we obtain the following corollary.

Corollary 2.18. *$(X, d, \preceq)$ be a partially ordered metric space on X such that (X, d) is complete, $T : X \to X$ a self mapping and let $\gamma : X \times X \to [0, \infty)$ be a function such that T is a non-decreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$ and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - f(\alpha(s), \beta(s)) > 0 \text{ for all } \quad t > 0 \text{ and } s = t \text{ or } s = 0$$

and

$$\gamma(x, Tx)\gamma(y, Ty) \geq 1 \Rightarrow (\psi(d(Tx, Ty)) + l)^{\gamma(x,x)\gamma(y,y)} \leq \alpha(d(x, y)) - \beta(d(x, y)) + l$$

for all comparable $x, y \in X$ where $l \geq 1$. Suppose that either

(i) T is continuous or

(ii) if a non-decreasing sequence $\{x_n\} \subset X$ is such that $x_n \to x$ as $n \to \infty$ and $\gamma(x_n, Tx_n) \geq 1$ and $\gamma(x_n, x_n) \geq 1$ for all n, then $\gamma(x, Tx) \geq 1$ and $\gamma(x, x) \geq 1$ and $x_n \preceq x$ for all $n \in \mathbb{N}$.

If there exists $x_0 \in X$ such that $\gamma(x_0, x_0) \geq 1, \gamma(x_0, Tx_0) \geq 1$ and $x_0 \preceq Tx_0$, then T has a fixed point.

Remark 2.19. Corollary 2.18 is actually [33, Theorem 2.1]. So we generalize the results of [33]. Our contraction can be reduced to that in [33]. Indeed, [33, Theorem 2.1] is taken as a particular case of Theorem 1.14. When $h(x, y, z) = (z + l)^{xy}$, $l > 1$, $F(s, t) = st + l$ and $f(s, t) = s - t$, Theorem 1.14 can reduce to [33, Theorem 2.1].

We will illustrate the example to show that our contractions is weaker than that in [33]. The condition can be applied to Theorem 1.14, but not applied to [33, Theorem 2.1].

Example 2.20. Let $X = [0, +\infty)$ be endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $T : X \to X$ be defined by

$$Tx = \begin{cases} x, & \text{if } x \in [0, 1], \\ x^2 + 1, & \text{if } x \in (1, \infty). \end{cases}$$

Let $h(x, y, z) = (z + \frac{3}{2})^{xy}$, $F(s, t) = st + \frac{3}{2}$ and $f(s, t) = \frac{2}{3}s$. Define also $\gamma : X \times X \to [0, +\infty)$,

$$\gamma(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

Let $\psi(t) = 2t + 1$, $\alpha(t) = 3t + 2$ and $\beta(t) = 3t/2 + 1$.

We prove that Theorem 1.14 can be applied to T. But [33, Theorem 2.1] cannot be applied. Clearly, (X, d) is a complete metric space. We show that T is a γ-admissible mapping. Let $x, y \in X$. If $\gamma(x, y) \geq 1$, then $x, y \in [0, 1]$. On the other hand for all $x \in [0, 1]$ and $y \in [0, 1]$, we have $Tx \leq 1$ and $Ty \leq 1$. It follows that $\gamma(Tx, Ty) \geq 1$. Thus the assertion holds. Because of the above arguments, $\gamma(0, 0) \geq 1$. Now, if $\{x_n\}$ is a sequence in X such that $\gamma(x_n, x_n) \geq 1$ for all $n \in \mathbb{N} \bigcup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subset [0, 1]$ and hence $x \in [0, 1]$. This implies that $\gamma(x, x) \geq 1$. Also $\psi(t) = 2t + 1 > 3t/2 + 1 = \alpha(t) - \beta(t)$ and $\psi(t) = 2t + 1 > 1 = \alpha(0) - \beta(0)$ for all $t > 0$. Let $\gamma(x, Tx)\gamma(y, Ty) \geq 1$. Then $x, y \in [0, 1]$. Indeed, if $x \notin [0, 1]$ or $y \notin [0, 1]$. So $\gamma(x, Tx) = 0$ or $\gamma(y, Ty) = 0$. That is, $\gamma(x, Tx)\gamma(y, Ty) = 0 < 1$ which is a contradiction. Without any loss of generality we assume that $y \geq x$. We get

$$\begin{aligned}
h(\gamma(x,x),\gamma(y,y),\psi(d(Tx,Ty))) &= (\psi(d(Tx,Ty))+\frac{3}{2})^{\gamma(x,x)\gamma(y,y)}\\
&= 2(Ty-Tx)+1+\frac{3}{2}\\
&= 2y-2x+\frac{5}{2}\\
&= 2(y-x)+\frac{5}{2}\\
&\le 2(y-x)+2+\frac{3}{2}\\
&= F(1,f(\alpha(d(x,y)),\beta(d(x,y))).
\end{aligned}$$

Then the condition (2.2) of Theorem 1.14 holds and T has a fixed point.

But, taking $l=1$, we have that

$$\begin{aligned}
(\psi(d(Tx,Ty))+l)^{\gamma(x,x)\gamma(y,y)} &= 2(Ty-Tx)+1+l\\
&= 2y-2x+2\\
&= 2(y-x)+2\\
&\ge 2(y-x)+2-[(y-x)/2+1]\\
&= \alpha(d(x,y))-\beta(d(x,y))+l.
\end{aligned}$$

That is, the contractive condition of Theorem 2.1 in ([33]) does not hold for this example.

Remark 2.21.

(1) When $h(x,y,z)=(z+l)^{xy}, l>1$, $F(s,t)=st+l$ and $f(s,t)=s-t$ in Theorem 1.14 and $\gamma(x,Tx)\gamma(y,Ty)\ge 1$, Corollary 2.18 can deduce to [33, Corollary 2.3].

(2) Taking $h(x,y,z)=(xy+l)^z, l>0$, $F(s,t)=(1+l)^{st}$, $f(s,t)=s-t$ and $l=1$ in Theorem 1.14, we obtain [33, Theorem 2.4].

(3) Taking $h(x,y,z)=(xy+l)^z$, $l>0$, $F(s,t)=(1+l)^{st}$, $f(s,t)=s-t$, $l=1$ and $\gamma(x,Tx)\gamma(y,Ty)\ge 1$, we have actually [33, Corollary 2.6].

(4) Taking F, h, f, take $h(x,y,z)=xyz$, $F(s,t)=st$, $f(s,t)=s-t$, we have actually [33, Theorem 2.7].

(5) When $h(x,y,z)=(xy+l)^z, l>0$, $F(s,t)=(1+l)^{st}$, $f(s,t)=s-t$ and $l=1$, Theorem 1.14 can reduce to [33, Corollary 2.9].

The following example shows that Theorem 1.14 can be applied to T, but [33, Theorem 2.4] cannot be applied.

Example 2.22. Let X and d be as in Example 2.20. Define $T: X\to X$ by

$$Tx=\begin{cases}1-x, & \text{if } x\in[0,1],\\ e^x, & \text{if } x\in(1,\infty).\end{cases}$$

Let $h(x,y,z)=(xy+1)^z$, $F(s,t)=2^{st}$ and $f(s,t)=s-\frac{t}{1+t}$.

Define also γ, ψ, α and β as in Example 2.20. We shall show that Theorem 1.14 can be applied to T but [33, Theorem 2.4] cannot be applied. Proceeding as in the proof of Example 2.20, $\gamma(0,0)\ge 1$ and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_n)\ge 1$ for all $n\in\mathbb{N}\bigcup\{0\}$ and $x_n\to x$ as $n\to+\infty$, then $\gamma(x,x)\ge 1$. Let $\gamma(x,fx)\gamma(y,fy)\ge 1$. Then $x,y\in[0,1]$. Assume $y\ge x$. We get

$$h(\gamma(x,x),\gamma(y,y),\psi(d(Tx,Ty)))=(\gamma(x,x)\gamma(y,y)+1)^{\psi(d(Tx,Ty))}$$

$$
\begin{aligned}
&= 2^{\psi(d(Tx,Ty))} \\
&= 2^{2(y-x)+1} \\
&\leq 2^{2(y-x)+2-\frac{\frac{y-x}{2}+1}{\frac{y-x}{2}+1+1}} \\
&= 2^{\alpha(d(x,y))-\frac{\beta(d(x,y))}{1+\beta(d(x,y))}} \\
&= F(1, f(\alpha(d(x,y)), \beta(d(x,y))).
\end{aligned}
$$

Then the condition of Corollary 2.18 holds and so T has a fixed point. But

$$
\begin{aligned}
£^{(}\gamma(x,x)\gamma(y,y)+1)^{\psi(d(Tx,Ty))} &= 2^{2(Tx-Ty)+1} \\
&= 2^{2(y-x)+1} \\
&> 2^{\frac{3}{2}(y-x)+1} \\
&> 2^{2(y-x)+2-(\frac{1}{2}(y-x)+1)} \\
&= 2^{\alpha(d(x,y))-\beta(d(x,y))}.
\end{aligned}
$$

Hence, the condition of Theorem 2.4 in [33] does not hold for this example.

The following example shows that Theorem 1.14 can be applied for T but the condition of Theorem 2.7 in [33] does not hold for this example.

Example 2.23. Let X and d be as in Example 2.20. Define $f : X \to X$ by

$$
Tx = \begin{cases} 2-2x, & \text{if } x \in [0,1], \\ e^x + \sin x, & \text{if } x \in (1,\infty). \end{cases}
$$

Let $h(x,y,z) = xyz,\ F(s,t) = st$ and $f(s,t) = s - \frac{t}{2+t}$.

Define also γ, ψ, α and β as in Example 2.20. We shall show that Theorem 1.14 can be applied for T, but [33, Theorem 2.7] cannot be applied. Reviewing the proof of Example 2.20, T is a γ-admissible mapping, $\alpha(0,0) \geq 1$ and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_n) \geq 1$ for all $n \in \mathbb{N}\bigcup\{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\gamma(x,x) \geq 1$. Let $\gamma(x,Tx)\gamma(y,Ty) \geq 1$. Then $x, y \in [0,1]$. Assume $y \geq x$. Then

$$
\begin{aligned}
h(\gamma(x,x),\gamma(y,y),\psi(d(Tx,Ty))) &= \gamma(x,x)\gamma(y,y)\psi(d(Tx,Ty)) \\
&= \psi(d(Tx,Ty)) \\
&= 2(y-x)+1 \\
&\leq \alpha(d(x,y)) - \frac{\beta(d(x,y))}{1+\beta(d(x,y))} \\
&= F(1, f(\alpha(d(x,y)), \beta(d(x,y)))).
\end{aligned}
$$

Then the condition of Theorem 1.14 holds and T has a fixed point. But

$$
\begin{aligned}
\gamma(x,x)\gamma(y,y)\psi(d(Tx,Ty)) &= Ty - Tx + 1 \\
&= 2(y-x)+1 \\
&\geq 2(y-x)+2-[\frac{y-x}{2}+1] \\
&= \alpha(d(x,y)) - \beta(d(x,y)).
\end{aligned}
$$

Clearly, the condition of [33, Theorem 2.7] does not hold for this example.

Theorem 2.24. *Let $(X, d, \preceq)$ be a partially ordered metric space on X such that (X, d) is complete and let $T : X \to X$ be a self mapping. Let $\mu : X \times X \longrightarrow [0, \infty)$ be a function such that T is a non-decreasing and traingular μ-subadmissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$ and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - f(\alpha(s), \beta(s)) > 0 \text{ for all } t > 0 \text{ and } s = t \text{ or } s = 0, \tag{2.7}$$

where f is an element of $\mathcal{C}$ and for all x, $y \in X$ we have

$$\mu(x, Tx) \le 1 \Longrightarrow h(1, \psi(d(Tx, Ty))) \le F(\mu(x, y), f(\alpha(d(x, y)), \beta(d(x, y))), \tag{2.8}$$

where the pair (F, h) is a special upclass of type I. Suppose that either

(i) T is continuous or

(ii) if a non-decreasing sequence $\{x_n\} \subset X$ is such that $x_n \to x$ as $n \to \infty$ and $\mu(x_n, Tx_n) \le 1$ for all n, then $\mu(x, x_n) \le 1$ and $x_n \preceq x$ for $n \in \mathbb{N}$.

If there exists $x_0 \in X$ such that $\mu(x_0, x_0) \le 1$, $\mu(x_0, Tx_0) \le 1$ and $x_0 \preceq Tx_0$, then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\mu(x_0, Tx_0) \le 1$. Define a sequence $\{x_n\} \subset X$ by $x_n = T^n(x_0) = Tx_{n-1}$ for $n \in \mathbb{N}$. Suppose that $x_{n_0} = x_{n_0 - 1}$ for some $n_0 \in N$. Then it is clear that x_{n_0} is a fixed point of T and hence the proof is complete. From now on, we suppose that $x_n \ne x_{n-1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Since T is an α-admissible mapping and $\mu(x_0, Tx_0) \le 1$, we deduce that $\mu(x_1, Tx_1) = \mu(Tx_0, T^2 x_0) \le 1$. By continuing this process, we get that $\mu(x_n, Tx_n) = \mu(x_n, x_{n+1}) \le 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Also from $\mu(x_0, x_0) \le 1$ we obtain $\mu(x_1, x_1) = \mu(Tx_0, Tx_0) \le 1$. By continuing this process, we get $\mu(x_n, x_n) \le 1$ for $n \in \mathbb{N} \cup \{0\}$.

Since T is non-decreasing and $x_0 \preceq Tx_0$, we have

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_{n-1} \preceq x_n \preceq \cdots. \tag{2.9}$$

We will show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $\mu(x_n, x_n) \le 1$, $\mu(x_n, Tx_n) \le 1$ for each $n \in \mathbb{N}$, by (2.8) we have

$$\begin{aligned} h(1, \psi(d(x_n, x_{n+1}))) &= h(1, \psi(d(Tx_{n-1}, Tx_n))) \\ &\le F(\mu(x_{n-1}, x_n), f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n)))) \\ &\le F(1, f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n)))). \end{aligned}$$

Since the pair (F, h) is a special upclass of type I, we have that

$$\psi(d(x_n, x_{n+1})) \le f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n))). \tag{2.10}$$

If $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$, since ψ is non-decreasing, then

$$\psi(d(x_{n-1}, x_n)) \le \psi(d(x_n, x_{n+1})).$$

Combining with (2.10), we obtain that

$$\begin{aligned} \psi(d(x_{n-1}, x_n)) &\le \psi(d(x_n, x_{n+1})) \\ &\le f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n))). \end{aligned}$$

It yields that

$$\psi(d(x_{n-1}, x_n)) \le f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n))).$$

Since $x_{n-1} \ne x_n$, $d(x_{n-1}, x_n) > 0$. Then the above inequality contradicts (2.7). Therefore, our assumption $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$ is wrong. Then we have that

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n).$$

Therefore, the sequence $\{d_n := d(x_n, x_{n+1})\}$ is non-increasing. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d_n = r$. Suppose that $r > 0$. By (2.10), we have

$$\begin{aligned}
\psi(r) &\leq \lim_{n\to\infty} \inf \psi(d_n) \\
&\leq \lim_{n\to\infty} \sup \psi(d_n) \\
&\leq \lim_{n\to\infty} \sup f(\alpha(d(x_{n-1}, x_n)), \beta(d(x_{n-1}, x_n))) \\
&\leq f(\alpha(r), \beta(r)),
\end{aligned}$$

which is a contradiction to (2.7). Thus $r = 0$. Hence

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.11}$$

We shall show that the sequence $\{x_n\}$ is a Cauchy sequence.

Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 1.16 there exist an $\varepsilon > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{n\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{2.12}$$

By [30, Lemma 1.13] and (2.8), taking $x = x_{m(k)}$ and $y = x_{n(k)}$, we have $\mu(x_{m(k)}, x_{n(k)}) \leq 1$ and

$$\begin{aligned}
h(1, \psi(d(x_{m(k)+1}, x_{n(k)+1}))) &= h(1, \psi(d(Tx_{m(k)}, Tx_{n(k)}))) \\
&\leq F(\mu(x_{m(k)}, x_{n(k)}), f(\alpha(d(x_{m(k)}, x_{n(k)})), \beta(d(x_{m(k)}, x_{n(k)})))).
\end{aligned}$$

Therefore,

$$\psi(d(x_{m(k)+1}, x_{n(k)+1})) \leq f(\alpha(d(x_{m(k)}, x_{n(k)})), \beta(d(x_{m(k)}, x_{n(k)}))).$$

Taking the $\liminf$ as $k \to +\infty$ in the above inequality, we have

$$\begin{aligned}
\psi(\varepsilon) &\leq \lim_{n\to\infty} \inf \psi(d(x_{m(k)+1}, x_{n(k)+1})) \\
&\leq \lim_{n\to\infty} \sup \psi(d(x_{m(k)+1}, x_{n(k)+1})) \\
&\leq \lim_{n\to\infty} \sup f(\alpha(d(x_{m(k)}, x_{n(k)})), \beta(d(x_{m(k)}, x_{n(k)}))) \\
&\leq f(\lim_{n\to\infty} \sup \alpha(d(x_{m(k)}, x_{n(k)})), \lim_{n\to\infty} \inf \beta(d(x_{m(k)}, x_{n(k)}))) \\
&\leq f(\alpha(\varepsilon), \beta(\varepsilon)).
\end{aligned}$$

So we have

$$\psi(\varepsilon) \leq f(\alpha(\varepsilon), \beta(\varepsilon)),$$

which contradicts the fact that $\psi(t) - f(\alpha(t), \beta(t)) > 0$ for all $t > 0$. Hence

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = 0,$$

that is, the sequence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x_0 \in X$ such that $x_n \to x_0 \in X$.

Suppose that (i) holds. Then

$$x_0 = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = 0.$$

Suppose that (ii) holds, that is, $\mu(x_0, x_n) \leq 1$ and $x_n \preceq x_0$. Now, from (2.8)

$$h(1, \psi(d(Tx_0, x_{n+1})) = h(1, \psi(d(Tx_0, Tx_n)))$$

$$\begin{aligned}&\leq F(\mu(x_0,x_n), f(\alpha(d(x_0,x_n)),\beta(d(x_0,x_n)))\\&\leq F(1, f(\alpha(d(x_0,x_n)),\beta(d(x_0,x_n))),\end{aligned}$$

which implies that

$$\psi(d(Tx_0,x_{n+1})) \leq f(\alpha(d(x_0,x_n)),\beta(d(x_0,x_n))).$$

Taking the $\liminf$ as $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned}\psi(d(Tx_0,x_0)) &\leq \lim_{n\to\infty}\inf \psi(d(Tx_0,x_{n+1}))\\&= \lim_{n\to\infty}\inf \psi(d(Tx_0,Tx_n))\\&\leq \lim_{n\to\infty}\sup f(\alpha(d(x_0,x_n)),\beta(d(x_0,x_n)))\\&\leq f(\alpha(0),\beta(0)).\end{aligned}$$

Hence $\lim_{n\to\infty} d(Tx_0,x_{n+1}) = \lim_{n\to\infty} d(Tx_0,x_0) = 0$ and so $Tx_0 = x_0$. □

Taking $h(x,y) = (y^k + l)^x$, $l > 1$, $F(s,t) = s^m t^k + l$ and $f(s,t) = s - t$ in Theorem 2.24, we obtain the following corollary.

Corollary 2.25. *Let $(X,d,\preceq)$ be a partially ordered metric space on X such that (X,d) is complete and let $T : X \to X$ be a self mapping. Let $\mu : X \times X \longrightarrow [0,\infty)$ be a function such that T is a non-decreasing and traingular μ-subadmissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$ and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - f(\alpha(s),\beta(s)) > 0 \textit{ for all } t > 0 \textit{ and } s = t \textit{ or } s = 0,$$

where f is an elements of $\mathcal{C}$ and for all x, $y \in X$ we have

$$\mu(x,Tx) \leq 1 \Longrightarrow (\psi(d(Tx,Ty)))^k \leq (\mu(x,y))^m(\alpha(d(x,y)) - \beta(d(x,y)))^k$$

for all comparable $x, y \in X$ where $l \geq 1$. Suppose that either

(i) T is continuous or

(ii) if a non-decreasing sequence $\{x_n\} \subset X$ is such that $x_n \to x$ as $n \to \infty$ and $\mu(x_n,Tx_n) \leq 1$ for all n, then $\mu(x,x_n) \leq 1$ and $x_n \preceq x$ for $n \in \mathbb{N}$.

If there exists $x_0 \in X$ such that $\mu(x_0,x_0) \leq 1$, $\mu(x_0,Tx_0) \leq 1$ and $x_0 \preceq Tx_0$, then T has a fixed point.

When $h(x,y) = (x^n + l)^{y^k}$, $l > 1$, $F(s,t) = (1+l)^{s^m t^k}$ and $f(s,t) = s - t$, we have the following corollary.

Corollary 2.26. *Let $(X,d,\preceq)$ be a partially ordered metric space on X such that (X,d) is complete and let $T : X \to X$ be a self mapping. Let $\mu : X \times X \longrightarrow [0,\infty)$ be a function such that T is a non-decreasing and traingular μ-subadmissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$ and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - f(\alpha(s),\beta(s)) > 0 \textit{ for all } t > 0 \textit{ and } s = t \textit{ or } s = 0,$$

where f is an element of $\mathcal{C}$ and for all x, $y \in X$ we have

$$\mu(x,Tx) \leq 1 \Longrightarrow \psi(d(Tx,Ty))^k \leq (\mu(x,y))^m(\alpha(d(x,y)) - \beta(d(x,y)))^k$$

for all comparable $x, y \in X$ where $l \geq 1$. Suppose that either

(i) T is continuous or

(ii) if a non-decreasing sequence $\{x_n\} \subset X$ is such that $x_n \to x$ as $n \to \infty$ and $\mu(x_n,Tx_n) \leq 1$ for all n, then $\mu(x,x_n) \leq 1$ and $x_n \preceq x$ for $n \in \mathbb{N}$.

If there exists $x_0 \in X$ such that $\mu(x_0, x_0) \leq 1$, $\mu(x_0, Tx_0) \leq 1$ and $x_0 \preceq Tx_0$, then T has a fixed point.

When $h(x, y) = x^n y^k$, $F(s, t) = s^m t^k$, $f(s, t) = \theta s$, $0 < \theta < 1$, we have the following corollary.

Corollary 2.27. *Let $(X, d, \preceq)$ be a partially ordered metric space on X such that (X, d) is complete and let $T : X \to X$ be a self mapping. Let $\mu : X \times X \longrightarrow [0, \infty)$ be a function such that T is a non-decreasing and traingular μ-subadmissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$ and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - f(\alpha(s), \beta(s)) > 0 \text{ for all } t > 0 \text{ and } s = t \text{ or } s = 0,$$

where f is an element of $\mathcal{C}$ and for all x, $y \in X$ we have

$$\mu(x, Tx) \leq 1 \Longrightarrow \psi(d(Tx, Ty))^k \leq (\mu(x, y))^m (\theta\alpha(d(x, y)))^k$$

for all comparable $x, y \in X$ where $l \geq 1$. Suppose that either

(i) T is continuous or

(ii) if a non-decreasing sequence $\{x_n\} \subset X$ is such that $x_n \to x$ as $n \to \infty$ and $\mu(x_n, Tx_n) \leq 1$ for all n, then $\mu(x, x_n) \leq 1$ and $x_n \preceq x$ for $n \in \mathbb{N}$.

If there exists $x_0 \in X$ such that $\mu(x_0, x_0) \leq 1$, $\mu(x_0, Tx_0) \leq 1$ and $x_0 \preceq Tx_0$, then T has a fixed point.

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