Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 1581–1589
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed point and common fixed point theorems on ordered cone metric spaces over Banach algebras

Qi Yan, Jiandong Yin*, Tao Wang

Department of Mathematics, Nanchang University, Nanchang 330031, PR China.

Communicated by R. Saadati

Abstract

The purpose of this paper is to obtain some fixed point and common fixed point results of comparable maps satisfying certain contractive conditions on partially ordered cone metric spaces over Banach algebras. Moreover, an example is given, which shows that our main results are more useful than the presented results in some recent literatures.

Keywords: Fixed points, cone metric spaces over Banach algebras, ordered sets.
2010 MSC: 47H10, 54H25.

1. Introduction

Cone metric spaces were introduced by Huang and Zhang in [5], where they investigated the convergence of a sequence in cone metric spaces in order to introduce the notion of completeness and proved some fixed point theorems for contractive maps on these spaces. Recently, based on the work of Huang and Zhang [5], a few fixed point and common fixed point results of some mappings with certain contractive property on cone metric spaces have been proved (see [1, 2, 6, 7, 8, 9, 11, 12, 13] and the references contained therein).

In the past several years, some existence results of fixed points for some contractive type maps in partially ordered cone metric spaces were investigated (see [3, 4]). In 2013, Liu and Xu [10] introduced the concept of cone metric spaces over Banach algebras by replacing a Banach space E with a Banach algebra $\mathcal{A}$. In this way, they proved some fixed point theorems of generalized Lipschitz maps with weaker and natural conditions on generalized Lipschitz constant k by means of spectral radius.

*Corresponding author
Email addresses: qiyanmath@163.com (Qi Yan), yjdaxf@163.com (Jiandong Yin), taowangmath@163.com (Tao Wang)

Received 2015-09-16

The purpose of this paper is to obtain some fixed point theorems of maps satisfying the contractive conditions given in [3, 4] in the setting of ordered cone metric spaces over Banach algebras. Moreover, we give an example to show that our main results concerning the fixed point theorems in the setting of ordered cone metric spaces over Banach algebras are more useful than the standard results in cone metric spaces presented in the literatures.

2. Preliminaries

In the following, we will review some basic concepts and definitions from [5] and [10].

Let $\mathcal{A}$ always be a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in \mathcal{A}, \alpha \in \mathbb{R}$):

(i) $(xy)z = x(yz)$;
(ii) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(iv) $\|xy\| \leq \|x\|\|y\|$.

The following assumption that a Banach algebra has a unit (*i.e.*, a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$ will be needed throughout the paper. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [14].

For the convenience, we repeat the following proposition from [14].

Proposition 2.1 ([14]). *Let $\mathcal{A}$ be a Banach algebra with a unit e and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Remark 2.2. Here and subsequently, $r(x)$ denotes the spectral radius of $x \in \mathcal{A}$. If $r(x) < 1$, then $\|x^n\| \to 0 (n \to \infty)$.

Now let us recall the concepts of cone and partial ordering for a Banach algebra $\mathcal{A}$. A subset P of $\mathcal{A}$ is called a cone of $\mathcal{A}$ if,

(i) P is non-empty closed and $\{\theta, e\} \subset P$;
(ii) $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
(iii) $P^2 = PP \subset P$;
(iv) $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$, where int P denotes the interior of P. If int $P \neq \emptyset$ then P is called a solid cone.

In the following we always assume that P is a solid cone of $\mathcal{A}$ and $\preceq$ is the partial ordering with respect to P.

Definition 2.3 ([10]). Let X be a non-empty set. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies the followings:

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over a Banach algebra $\mathcal{A}$.

Example 2.4 ([10])**.** Let $\mathcal{A} = R^2, P = \{(x, y) \in \mathcal{A} \mid x, y \geq 0\} \subset R^2, X = R$ and $d : X \times X \to \mathcal{A}$ such that $d(x, y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

See [10] for more examples of cone metric spaces over Banach algebras.

Definition 2.5 ([10])**.** Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and let $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$;
(ii) $\{x_n\}$ is a Cauchy sequence whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$;
(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and $f : X \to X$ be a map. We say that f is continuous if for any $\{x_n\} \subset X, x_n \to x$ implies $f(x_n) \to f(x)(n \to \infty)$.

Definition 2.6. Let $(X, \sqsubseteq)$ be a partially ordered set. We say that $x, y \in X$ are comparable if $x \sqsubseteq y$ or $y \sqsubseteq x$ holds. Similarly, $f : X \to X$ is said to be comparable if for any comparable pair $x, y \in X$, $f(x), f(y)$ are comparable.

Remark 2.7. A map f is said to be nondecreasing with $\sqsubseteq$, if for any $x, y \in X$, $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$. Obviously, a comparable map may not be nondecreasing with $\sqsubseteq$.

Definition 2.8. Let $(X, \sqsubseteq)$ be a partially ordered set. Two maps $f, g : X \to X$ are said to be weakly comparable if both $f(x), gf(x)$ and $g(x), fg(x)$ are comparable for all $x \in X$.

3. Main results

The following lemmas are crucial to the proofs of our main results. We shall appeal to the following lemmas in the sequel [4, 12, 15]. For simplicity, we always assume that $\mathcal{A}$ is a real Banach algebra and P is a solid cone of $\mathcal{A}$ which gives the partial ordering " $\preceq$ " in P.

Lemma 3.1 ([15])**.** *Let $\mathcal{A}$ be a Banach algebra and let x, y be vectors in $\mathcal{A}$. If x and y commute, then the following hold:*

(i) $r(xy) \leq r(x)r(y)$;
(ii) $r(x+y) \leq r(x) + r(y)$;
(iii) $|r(x) - r(y)| \leq r(x-y)$.

Lemma 3.2 ([15])**.** *Let $\mathcal{A}$ be a Banach algebra and let k be a vector in $\mathcal{A}$. If $0 \leq r(k) < 1$, then we have*

$$r((e-k)^{-1}) \leq (1 - r(k))^{-1}.$$

Lemma 3.3 ([12])**.** *If $\mathcal{A}$ is a real Banach space with a solid cone P and if $\|x_n\| \to 0(n \to \infty)$, then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.*

Lemma 3.4 ([4])**.** *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $f : X \to X$ be a continuous and nondecreasing mapping with $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) there exists $k \in (0, 1)$ such that $d(f(x), f(y)) \preceq kd(x, y)$ for each $x, y \in X$ with $y \sqsubseteq x$;
(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq f(x_0)$.

Then f has a fixed point $x^* \in X$.

We can now formulate our main results.

Theorem 3.5. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$. Suppose that $f : X \to X$ is continuous and comparable and the following two assertions hold:*

(i) there exists $k \in P$ with $r(k) \in (0, 1)$ such that $d(f(x), f(y)) \preceq kd(x, y)$ for any comparable pair $x, y \in X$;
(ii) there exists $x_0 \in X$ such that $x_0, f(x_0)$ are comparable.

Then f has a fixed point $x^* \in X$.

Proof. If $f(x_0) = x_0$, then the proof is finished. Assume that $f(x_0) \neq x_0$. From condition (ii) and f is comparable, we deduce that $f^i(x_0)$ and $f^{i+1}(x_0)$ are comparable for any $i \geq 0$. Replacing $x_n = f^n(x_0)$, we recover x_i, x_{i+1} are comparable. By condition (i), it follows that

$$\begin{aligned} d(x_{n+1}, x_n) &\preceq kd(x_n, x_{n-1}) \preceq k^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\preceq k^n d(x_1, x_0). \end{aligned}$$

Let $m > n$, then

$$\begin{aligned} d(x_m, x_n) &\preceq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\preceq (k^{m-1} + \cdots + k^n) d(x_1, x_0) \\ &= (e + k + \cdots k^{m-n-1}) k^n d(x_1, x_0) \\ &\preceq (\sum_{i=0}^{\infty} k^i) k^n d(x_1, x_0) \\ &= (e - k)^{-1} k^n d(x_1, x_0). \end{aligned}$$

We see at once that $\|(e-k)^{-1} k^n d(x_1, x_0)\| \leq \|(e-k)^{-1}\| \|k^n\| \|d(x_1, x_0)\| \to 0 (n \to \infty)$, which is clear by Remark 2.2 that $\|k^n\| \to 0 (n \to \infty)$. By Lemma 3.3, for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $m > n > N$,

$$d(x_m, x_n) \preceq (e-k)^{-1} k^n d(x_1, x_0) \ll c,$$

which implies that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^* (n \to \infty)$. Consequently, the continuity of f implies that x^* is a fixed point of f. □

The following example shows that the contractive condition of Theorem 3.5 is more general than the contractive condition of Lemma 3.4.

Example 3.6. Let $\mathcal{A} = \mathbb{R}^2$, the Euclidean plane. For each $(x_1, x_2) \in \mathcal{A}$, let $\|(x_1, x_2)\| = | x_1 | + | x_2 |$. The multiplication is defined by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1 y_1, x_1 y_2 + x_2 y_1).$$

Then $\mathcal{A}$ is a Banach algebra with unit $e = (1, 0)$.

Let $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ a cone in $\mathcal{A}$. Let $X = \{(x, 0) \in \mathbb{R}^2 : x \geq 0\} \cup \{(0, x) \in \mathbb{R}^2 : x \geq 0\}$ and consider the relation on X as follows: for $(x, y), (z, w) \in X$,

$$(x, y) \sqsubseteq (z, w) \Leftrightarrow \{x \preceq z \ \ and \ \ y \preceq w\},$$

where $\preceq$ is a partial ordering on $\mathbb{R}$ as follows: for $m, n \in \mathbb{R}$,

$$m \preceq n \Leftrightarrow \{(m = n) \ or \ (m, n \in [0,1] \ \ with \ \ m \leq n)\}.$$

Obviously, $\sqsubseteq$ is a partial ordering on X. Let $d : X \times X \to \mathcal{A}$ defined by

$$\begin{aligned} &d((x,0),(y,0)) = \left(\frac{3}{2} \mid x-y \mid, \mid x-y \mid\right), \\ &d((0,x),(0,y)) = (\mid x-y \mid, 2 \mid x-y \mid), \\ &d((x,0),(0,y)) = d((0,y),(x,0)) = \left(\frac{3}{2}x+y, x+2y\right). \end{aligned}$$

It is easy to check that (X, d) is a complete cone metric space.

Define $f : X \to X$ by

$$f(x,0) = (0,x) \ and \ f(0,x) = \begin{cases} \left(2x - \frac{3}{2}, 0\right), & if \ \ x > 1, \\ \left(\frac{x}{2}, 0\right), & if \ \ 0 \leq x \leq 1. \end{cases}$$

It follows immediately that f is continuous and comparable. Also f satisfies the condition (i) of Theorem 3.5 if take $k = (\frac{4}{5}, \alpha)$, where α can be any positive real number larger than $\frac{4}{5}$. Of course

$$r(k) = \lim_{n\to\infty} \left\| \left(\frac{4}{5}, \alpha\right)^n \right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\| \left(\frac{4}{5}\right)^n, \alpha \cdot n \left(\frac{4}{5}\right)^{n-1} \right\|^{\frac{1}{n}} = \frac{4}{5} < 1.$$

Clearly $(0,0) \sqsubseteq f(0,0)$ which shows that the condition (ii) of Theorem 3.5 is satisfied. Therefore, we can apply Theorem 3.5 to this example and get that f has a fixed point.

Remark 3.7. In Example 3.6, f does not satisfy the condition (i) of Lemma 3.4. Hence, Example 3.6 shows that Theorem 3.5 is more powerful than the corresponding result in the setting of ordered cone metric spaces.

Some generalizations of the result are given in the following. For example, by removing the continuity of f in Theorem 3.5, we have the following.

Theorem 3.8. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$. Assume that $f : X \to X$ is comparable and the following two assertions hold:*

(i) there exists $k \in P$ with $r(k) \in (0,1)$ such that $d(f(x), f(y)) \preceq kd(x,y)$ for any comparable pair $x, y \in X$;
(ii) there exists $x_0 \in X$ such that $x_0, f(x_0)$ are comparable;
(iii) if a sequence $\{x_n\}$ converges to x in X and x_i, x_{i+1} are comparable for all $i \geq 0$, then x_i, x are comparable.

Then f has a fixed point $x^* \in X$.

Proof. Let $x_n = f^n(x_0)$, we get that x_n, x_{n+1} are comparable for all $n \geq 0$ and $\{x_n\}$ converges to x^* as in the proof of Theorem 3.5. Now the condition (iii) implies x_n, x^* are comparable. Therefore, the condition (i) gives that

$$\begin{aligned} d(f(x^*), x^*) &\preceq d(f(x^*), f(x_n)) + d(f(x_n), x^*) \\ &\preceq kd(x^*, x_n) + d(x_{n+1}, x^*). \end{aligned}$$

Hence, for each $c \gg \theta$ we have $d(f(x^*), x^*) \ll c$, so $d(f(x^*), x^*) = \theta$, which implies that x^* is a fixed point of f. $\square$

Theorem 3.5 and Theorem 3.8 can be generalized and extended by using the following conditions (1) and (2) instead of (i) respectively: (incidentally, we have generalized versions of Theorem 3 and Theorem 4 of [5], respectively).

(1) let $k \in P$ with $r(k) \in (0,1)$ such that $d(f(x), f(y)) \preceq k(d(f(x), x) + d(f(y), y))$ for any comparable pair $x, y \in X$;
(2) let $k \in P$ with $r(k) \in (0,1)$ such that $d(f(x), f(y)) \preceq k(d(f(x), y) + d(f(y), x))$ for any comparable pair $x, y \in X$.

Theorem 3.9. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$. Let $f : X \to X$ be continuous and comparable and the following two assertions hold:*

(i) there exist $\alpha, \beta, \gamma \in P$ with $r(\alpha) + 2r(\beta) + 2r(\gamma) < 1$ such that

$$d(f(x), f(y)) \preceq \alpha d(x, y) + \beta[d(x, f(x)) + d(y, f(y))] + \gamma[d(x, f(y)) + d(y, f(x))]$$

for any comparable pair $x, y \in X$;

(ii) there exists $x_0 \in X$ such that $x_0, f(x_0)$ are comparable.

Then f has a fixed point $x^* \in X$.

Proof. If $f(x_0) = x_0$, then the proof is finished. Suppose that $f(x_0) \neq x_0$. Since $x_0, f(x_0)$ are comparable and f is comparable, we obtain by induction that $f^i(x)$ and $f^{i+1}(x)$ are comparable for any $i \geq 0$. If take $x_n = f^n(x_0)$, then we have x_i, x_{i+1} are comparable. So we have

$$\begin{aligned} d(x_{n+1}, x_n) \preceq& \alpha d(x_n, x_{n-1}) + \beta[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + \gamma d(x_{n-1}, x_{n+1}) \\ \preceq& \alpha d(x_n, x_{n-1}) + \beta[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &+ \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \end{aligned}$$

that is,

$$(e - \beta - \gamma)d(x_{n+1}, x_n) \preceq (\alpha + \beta + \gamma)d(x_n, x_{n-1}).$$

Since $r(\alpha) + 2r(\beta) + 2r(\gamma) < 1$, then $r(\beta + \gamma) \leq r(\beta) + r(\gamma) < 1$, and $e - \beta - \gamma$ is invertible by Proposition 2.1. Then multiplying both sides with $(e - \beta - \gamma)^{-1}$, it follows that

$$d(x_{n+1}, x_n) \preceq (e - \beta - \gamma)^{-1}(\alpha + \beta + \gamma)d(x_n, x_{n-1}),$$

for all $n \geq 1$. Repeating this relation we get

$$d(x_{n+1}, x_n) \preceq k^n d(x_1, x_0),$$

where $k = (e - \beta - \gamma)^{-1}(\alpha + \beta + \gamma)$.

We claim that $r(k) < 1$.

By Lemma 3.1, we get

$$r(\alpha + \beta + \gamma) + r(\beta + \gamma) \leq r(\alpha) + r(\beta) + r(\gamma) + r(\beta) + r(\gamma) = r(\alpha) + 2r(\beta) + 2r(\gamma) < 1,$$

then $r(\alpha + \beta + \gamma) < 1 - r(\beta + \gamma)$, that is, $\frac{r(\alpha+\beta+\gamma)}{1-r(\beta+\gamma)} < 1$.

Hence, it follows from Lemma 3.1 and 3.2 that

$$\begin{aligned} r(k) &= r[(e - \beta - \gamma)^{-1}(\alpha + \beta + \gamma)] \\ &\leq r[(e - \beta - \gamma)^{-1}]r(\alpha + \beta + \gamma) \\ &\leq [1 - r(\beta + \gamma)]^{-1}r(\alpha + \beta + \gamma) \\ &= \frac{r(\alpha + \beta + \gamma)}{1 - r(\beta + \gamma)} < 1. \end{aligned}$$

Let $m > n$, then

$$\begin{aligned} d(x_m, x_n) &\preceq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\preceq (k^{m-1} + \cdots + k^n) d(x_1, x_0) \\ &= (e + k + \cdots k^{m-n-1}) k^n d(x_1, x_0) \\ &\preceq \left(\sum_{i=0}^{\infty} k^i \right) k^n d(x_1, x_0) \\ &= (e - k)^{-1} k^n d(x_1, x_0). \end{aligned}$$

We see at once that $\|(e-k)^{-1} k^n d(x_1, x_0)\| \leq \|(e-k)^{-1}\| \|k^n\| \|d(x_1, x_0)\| \to 0 (n \to \infty)$, which is clear by the claim and Remark 2.2, $\|k^n\| \to 0 (n \to \infty)$. By Lemma 3.3, it follows that, for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $m > n > N$,

$$d(x_m, x_n) \preceq (e-k)^{-1} k^n d(x_1, x_0) \ll c,$$

which implies that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $x^* \in X$ such that $x_n \to x^* (n \to \infty)$. Finally, the continuity of f implies that x^* is a fixed point of f. □

If we use the condition (iii) instead of continuity of f in Theorem 3.9, we have the following.

Theorem 3.10. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$. Let $f : X \to X$ be comparable and the following two assertions hold:*

(i) there exist $\alpha, \beta, \gamma \in P$ with $r(\alpha) + 2r(\beta) + 2r(\gamma) < 1$ such that

$$d(f(x), f(y)) \preceq \alpha d(x, y) + \beta[d(x, f(x)) + d(y, f(y))] + \gamma[d(x, f(y)) + d(y, f(x))]$$

for any comparable pair $x, y \in X$;

(ii) there exists $x_0 \in X$ such that $x_0, f(x_0)$ are comparable;

(iii) if a sequence $\{x_n\}$ converges to x in X and x_i, x_{i+1} are comparable for all $i \geq 0$, then x_i, x are comparable.

Then f has a fixed point $x^* \in X$.

Proof. Let $x_n = f^n(x_0)$, then x_n, x_{n+1} are comparable for all $n \geq 0$ and $\{x_n\}$ converges to x^* by the proof similar to Theorem 3.9. Now the condition (iii) implies that x_n, x^* are comparable for all n. Therefore, by the condition (i), we have

$$d(x_n, f(x^*)) \preceq \alpha d(x_n, x^*) + \beta[d(x_n, x_{n+1}) + d(x^*, f(x^*))] + \gamma[d(x_n, f(x^*)) + d(x^*, x_n)].$$

Taking $n \to \infty$, we have $d(x^*, f(x^*)) \preceq (\beta + \gamma) d(x^*, f(x^*))$, that is, $(e - \beta - \gamma) d(x^*, f(x^*)) \preceq \theta$. Then multiplying both sides with $(e - \beta - \gamma)^{-1}$, it follows that $d(x^*, f(x^*)) = \theta$. Hence $x^* = f(x^*)$. □

Now we give two common fixed point theorems on ordered cone metric spaces over Banach algebras. The result is still true if we delete the assumption that "there exists $x_0 \in X$ such that $x_0, f(x_0)$ are comparable" of Theorem 3.9. We can rephrase Theorem 3.9 as follows.

Theorem 3.11. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$. Let $f, g : X \to X$ be two weakly comparable maps and the following two assertions hold:*

(i) there exist $\alpha, \beta, \gamma \in P$ with $r(\alpha) + 2r(\beta) + 2r(\gamma) < 1$ such that

$$d(f(x), g(y)) \preceq \alpha d(x, y) + \beta[d(x, f(x)) + d(y, g(y))] + \gamma[d(x, g(y)) + d(y, f(x))]$$

for any comparable pair $x, y \in X$;

(ii) f or g is continuous.

Then f and g have a common fixed point $x^* \in X$.

Proof. Let x_0 be an arbitrary point of X and define a sequence $\{x_n\}$ in X as follows: $x_{2n+1} = f(x_{2n})$ and $x_{2n+2} = g(x_{2n+1})$ for all $n \geq 0$. Note that f and g are weakly comparable, we have $x_1 = f(x_0)$ and $x_2 = g(x_1) = gf(x_0)$ are comparable, by a similar argument, $x_2 = g(x_1)$, $x_3 = f(x_2) = fg(x_1)$ are comparable, and continuing this process, we have that x_n, x_{n+1} are comparable for all $n \geq 1$.

It follows from condition (i) that

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) =& d(f(x_{2n}), g(x_{2n+1})) \\
\preceq& \alpha d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma d(x_{2n}, x_{2n+2}) \\
\preceq& \alpha d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\
&+ \gamma[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})],
\end{aligned}$$

that is,

$$(e - \beta - \gamma)d(x_{2n+1}, x_{2n+2}) \preceq (\alpha + \beta + \gamma)d(x_{2n}, x_{2n+1}).$$

Since $r(\alpha) + 2r(\beta) + 2r(\gamma) < 1$, then $r(\beta + \gamma) \leq r(\beta) + r(\gamma) < 1$, by Proposition 2.1, $e - \beta - \gamma$ is invertible. Then multiplying both sides with $(e - \beta - \gamma)^{-1}$, we have

$$d(x_{2n+1}, x_{2n+2}) \preceq (e - \beta - \gamma)^{-1}(\alpha + \beta + \gamma)d(x_{2n}, x_{2n+1})$$

for all $n \geq 1$. Repeating this relation, we get

$$d(x_{n+1}, x_n) \preceq k^n d(x_1, x_0),$$

where $k = (e - \beta - \gamma)^{-1}(\alpha + \beta + \gamma)$.

Let $m > n$, then

$$\begin{aligned}
d(x_m, x_n) &\preceq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\
&\preceq (k^{m-1} + \cdots + k^n)d(x_1, x_0) \\
&= (e + k + \cdots k^{m-n-1})k^n d(x_1, x_0) \\
&\preceq \left(\sum_{i=0}^{\infty} k^i\right) k^n d(x_1, x_0) \\
&= (e - k)^{-1} k^n d(x_1, x_0),
\end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence. As X is complete, there exists $x^* \in X$ such that $x_n \to x^*(n \to \infty)$.

Without loss of generality, we can assume that f is continuous. Then it is clear that x^* is a fixed point of f. We show now that x^* is also a fixed point of g. Since x^*, x^* are comparable, by using the condition (i) for $x = y = x^*$, we have

$$d(f(x^*), g(x^*)) \preceq \alpha d(x^*, x^*) + \beta[d(x^*, f(x^*)) + d(x^*, g(x^*))] + \gamma[d(x^*, f(x^*)) + d(x^*, g(x^*))],$$

and $d(x^*, g(x^*)) \preceq (\beta + \gamma)d(x^*, g(x^*))$, that is, $(e - \beta - \gamma)d(x^*, f(x^*)) \preceq \theta$. Then multiplying both sides with $(e - \beta - \gamma)^{-1}$, we get that $d(x^*, g(x^*)) = \theta$. Hence $x^* = g(x^*)$. Therefore, we have proved that f and g have a common fixed point. □

Theorem 3.12. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$. Let $f, g : X \to X$ be two weakly comparable maps and suppose that the following two assertions hold:*

(i) *there exist* $\alpha, \beta, \gamma \in P$ *with* $r(\alpha) + 2r(\beta) + 2r(\gamma) < 1$ *such that*

$$d(f(x), g(y)) \preceq \alpha d(x, y) + \beta[d(x, f(x)) + d(y, g(y))] + \gamma[d(x, g(y)) + d(y, f(x))]$$

for any comparable pair $x, y \in X$*;*

(ii) *if a sequence* $\{x_n\}$ *converges to* x *in* X *and* x_i, x_{i+1} *are comparable for all* $i \geq 0$*, then* x_i, x *are comparable.*

Then f *and* g *have a common fixed point* $x^* \in X$*.*

Proof. The above theorem can be proved in same way as Theorem 3.10 and Theorem 3.11. So we omit it. □

Acknowledgements

The authors thank the editor and the referees for their valuable comments and suggestions. This research was supported by the National Natural Science Foundation of China(11261039) and the Provincial Natural Science Foundation of Jiangxi(20132BAB201009).

References

[1] M. Abbas, G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341** (2008), 416–420. 1

[2] M. Abbas, B. Rhoades, T. Nazir, *Common fixed points for four maps in cone metric spaces*, Appl. Math. Comput., **216** (2010), 80–86. 1

[3] I. Altun, B. Damjanović, D. Djorić, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett., **23** (2010), 310–316. 1

[4] I. Altun, G. Durmaz, *Some fixed point theorems on ordered cone metric spaces*, Rend. Circ. Mate. Palermo, **58** (2009), 319–325. 1, 3, 3.4

[5] L. G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468–1476. 1, 2, 3

[6] D. Ilić, V. Rakočević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl., **341** (2008), 876–882. 1

[7] D. Ilić, V. Rakočević, *Quasi-contraction on a cone metric space*, Appl. Math. Lett., **22** (2009), 728–731. 1

[8] S. Janković, Z. Kaselburg, S. Radenović, *On the cone metric space: a survey*, Nonliner Anal., **74** (2011), 2591–2601. 1

[9] Z. Kadelburg, S. Radenović, V. Rakočević, *A note on the equivalence of some metric and cone metric fixed point results*, Appl. Math. Lett., **24** (2011), 370–374. 1

[10] H. Liu, S. Xu, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz maps*, Fixed Point Theory Appl., **2013** (2013), 10 pages. 1, 2, 2.3, 2.4, 2, 2.5

[11] S. Radenović, *Common fixed points under contractive conditions in cone metric spaces*, Comput. Math. Appl., **58** (2009) 1273–1278. 1

[12] S. Radenović, B. E. Rhoades, *Fixed point theorem for two non-self maps in cone metric spaces*, Comput. Math. Appl., **57** (2009), 1701–1707. 1, 3, 3.3

[13] S. Rezapour, R. Hamlbarani, *Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mapping'*, J. Math. Anal. Appl., **345** (2008), 719–724. 1

[14] W. Rudin, *Functional Analysis, 2nd edn*, McGraw-Hill, New York, (1991). 2, 2.1

[15] S. Xu, S. Radenović, *Fixed point theorems of generalized Lipschitz maps on cone metric spaces over Banach algebras without assumption of normality*, Fixed Point Theory Appl., **2014** (2014), 12 pages. 3, 3.1, 3.2