Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 1619–1635
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Almost strongly θ-e-continuous functions

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Communicated by C. Park

Abstract

We introduce and investigate a new class of functions called almost strongly θ-e-continuous functions, containing the classes of almost strongly θ-precontinuous [J. H. Park, S. W. Bae, Y. B. Park, Chaos Solitons Fractals, **28** (2006), 32–41], almost strongly θ-semicontinuous [Y. Beceren, S. Yüksel, E. Hatir, Bull. Calcutta Math. Soc., **87** (1995), 329–334] and strongly θ-e-continuous functions [M. Özkoç, G. Aslım, Bull. Korean Math. Soc., **47** (2010), 1025–1036]. Several characterizations concerning almost strongly θ-e-continuous functions are obtained. Also we investigate the relationships between almost strongly θ-e-continuous functions and separation axioms and almost strongly e-closedness of graphs of functions.

Keywords: Almost strong θ-e-continuity, e-open, e-θ-open, almost e-regular, almost strongly e-closed.
2010 MSC: 54C08, 54C10.

1. Introduction

The concept of continuity is the most important subject in topology. In 2008, the notion of e-continuous functions was introduced and studied by Ekici [8] and in 2010, the notion of strongly θ-e-continuous functions was introduced by Özkoç and Aslım [19]. In 1984, Noiri and Kang introduced the notion of almost strong θ-continuity. Recently, three generalizations of almost strong θ-continuity are obtained by Beceren et al. [4], Park et al. [21] and Noiri and Zorlutuna [18]. The aim of this paper is to introduce and investigate a new class of functions, called almost strongly θ-e-continuous functions, which contains the classes of almost strongly θ-semicontinuous functions, almost strongly θ-precontinuous functions and strongly θ-e-continuous functions.

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Received 2015-11-01

We introduce and investigate some fundamental properties of almost strongly θ-e-continuous functions defined via e-open sets introduced by Ekici [8] in a topological space. It turns out that almost strong θ-e-continuity is stronger than θ-e-continuity [11] and weaker than strong θ-e-continuity [19], almost strong θ-semicontinuity [4] and almost strong θ-precontinuity [21]. Moreover, we obtain some results related to separation axioms and graphs properties.

2. Preliminaries

Throughout the paper, X and Y always mean topological spaces on which no separation axioms are assumed, unless explicitly stated. Let X be a topological space and A a subset of X. The closure and interior of A are denoted by $cl(A)$ and $int(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A point $x \in X$ is said to be δ-cluster point of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighborhood U of x. The set of all δ-cluster points of A is called the δ-closure [25] of A and is denoted by $\delta\text{-}cl(A)$. If $A = \delta\text{-}cl(A)$, then A is called δ-closed, and the complement of a δ-closed set is called δ-open. A subset A is called semiopen [12] (resp. b-open [3], e-open [8], preopen [13], α-open [15], a-open [7], β-open [1]) if $A \subset cl(int(A))$ (resp. $A \subset cl(int(A)) \cup int(cl(A))$, $A \subset cl(int_\delta(A)) \cup int(cl_\delta(A))$, $A \subset int(cl(A))$, $A \subset int(cl(int(A)))$, $A \subset int(cl(int_\delta(A)))$, $A \subset cl(int(cl(A)))$). The complement of a semiopen (resp. b-open, e-open, preopen, α-open, a-open, β-open) set is called semiclosed (resp. b-closed, e-closed, preclosed, α-closed, a-closed, β-closed). The intersection of all e-closed sets of X containing A is called the e-closure [8] of A and is denoted by $e\text{-}cl(A)$. The semiclosure, preclosure, b-closure and α-closure are similarly defined and are denoted by $scl(A)$, $pcl(A)$, $bcl(A)$ and $\alpha\text{-}cl(A)$, respectively. The union of all e-open sets of X contained in A is called the e-interior [8] of A and is denoted by $e\text{-}int(A)$. A subset A is said to be e-regular [19] if it is e-open and e-closed.

A point x of X is called an e-θ-cluster point of A if $e\text{-}cl(U) \cap A \neq \emptyset$ for every e-open set U containing x. The set of all e-θ-cluster points of A is called the e-θ-closure [19] of A and is denoted by $e\text{-}cl_\theta(A)$. A subset A is said to be e-θ-closed if $A = e\text{-}cl_\theta(A)$. The complement of an e-θ-closed set is called an e-θ-open set. Also it is noted in [19] that

$$e\text{-regular} \Rightarrow e\text{-}\theta\text{-open} \Rightarrow e\text{-open}.$$

The family of all e-open (resp. e-closed, e-regular, e-θ-open, e-θ-closed) subsets of X is denoted by $eO(X)$ (resp. $eC(X)$, $eR(X)$, $e\theta O(X)$, $e\theta C(X)$). The family of all e-open (e-closed, e-regular, e-θ-open, e-θ-closed) sets of X containing a point x of X is denoted by $eO(X,x)$ (resp. $eC(X,x)$, $eR(X,x)$, $e\theta O(X,x)$, $e\theta C(X,x)$).

Lemma 2.1 ([2])**.** *Let X be a topological space. If A is a preopen set in X, then $scl(A) = int(cl(A))$.*

Lemma 2.2 ([19])**.** *Let X be a topological space and $A \subset X$ and $\{A_\alpha | \alpha \in \Lambda\} \subset \mathcal{P}(X)$. Then the following statements hold:*

(1) $A \in eO(X)$ if and only if $e\text{-}cl(A) \in eR(X)$.

(2) A is e-θ-open in X if and only if for each $x \in A$, there exists $W \in eR(X,x)$ such that $W \subset A$.

(3) If A_α is e-θ-open in X for each $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is e-θ-open in X.

(4) $A \in eR(X)$ if and only if A is e-θ-open and e-θ-closed.

Lemma 2.3 ([17])**.** *Let X be a topological space. Then the following statements hold:*

(1) $\alpha\text{-}cl(V) = cl(V)$ for each β-open set V of X.

(2) $pcl(V) = cl(V)$ for each semi-open set V of X.

Lemma 2.4. *Let A be a subset of a space X. The set A is e-θ-open in X if and only if for each $x \in A$, there exists a $U \in eO(X)$ containing x such that $x \in e\text{-}cl(U) \subset A$.*

Proof. It can be proved directly using Lemma 2.2. □

Lemma 2.5 ([11])**.** *Let X be a topological space and $A \subset X$. Then:*

(1) $e\text{-}cl_\theta(X \setminus A) = X \setminus e\text{-}int_\theta(A)$.

(2) $e\text{-}int_\theta(X \setminus A) = X \setminus e\text{-}cl_\theta(A)$.

Lemma 2.6. *Let X be a topological space. Then the following statements hold:*
(1) $V \in \beta O(X) \Rightarrow \alpha\text{-}cl(V) \in SO(X)$.
(2) $V \in SO(X) \Rightarrow \alpha\text{-}cl(V) = pcl(V)$.

Proof. *(1)* Let $V \in \beta O(X)$. We have

$$\begin{aligned} V \in \beta O(X) &\Rightarrow V \subset cl(int(cl(V))) \\ &\Rightarrow \alpha\text{-}cl(V) \subset \alpha\text{-}cl(cl(int(cl(V)))) \\ &\overset{\text{Lemma2.3}}{\Longrightarrow} \alpha\text{-}cl(V) \subset cl(int(cl(V))) = cl(int(\alpha\text{-}cl(V))). \end{aligned}$$

(2) Let $V \in SO(X)$. We have

$$\left.\begin{array}{r} \left.\begin{array}{r} \alpha\text{-}cl(V) = V \cup cl(int(cl(V))) \\ V \in SO(X) \Rightarrow V \subset cl(int(V)) \end{array}\right\} \Rightarrow \alpha\text{-}cl(V) \subset V \cup cl(int(V)) = pcl(V) \\ V \subset X \Rightarrow pcl(V) \subset \alpha\text{-}cl(V) \end{array}\right\} \Rightarrow \alpha\text{-}cl(V) = pcl(V).$$ □

Lemma 2.7 ([20])**.** *In a space X, the intersection of an a-open set and an e-open set is an e-open set.*

3. Almost Strongly θ-e-continuous Functions

Definition 3.1. A function $f : X \to Y$ is said to be almost strongly θ-e-continuous (briefly, a.st.θ.e.c.) if for each $x \in X$ and each open set V containing $f(x)$, there exists an e-open set U in X containing x such that $f[e\text{-}cl(U)] \subset int(cl(V))$.

Theorem 3.2. *For a function $f : X \to Y$, the followings are equivalent:*

(1) f is a.st.θ.e.c.,

(2) for each $x \in X$ and each regular open set V containing $f(x)$, there exists an e-open set U in X containing x such that $f[e\text{-}cl\,(U)] \subset V$,

(3) for each $x \in X$ and each regular open set V containing $f(x)$, there exists an e-regular set U in X containing x such that $f[U] \subset V$,

(4) for each $x \in X$ and each regular open set V containing $f(x)$, there exists an e-θ-open set U in X containing x such that $f[U] \subset V$,

(5) $f^{-1}[G] \in e\theta O(X)$ for every regular open set G of Y,

(6) $f^{-1}[F] \in e\theta C(X)$ for every regular closed set F of Y,

(7) $f^{-1}[G] \in e\theta O(X)$ for every δ-open set G of Y,

(8) $f^{-1}[F] \in e\theta C(X)$ for every δ-closed set F of Y,

(9) $f[e\text{-}cl_\theta(A)] \subset cl_\delta(f[A])$ for every subset A of X,

(10) $e\text{-}cl_\theta(f^{-1}[B]) \subset f^{-1}[cl_\delta(B)]$ for every subset B of Y,

(11) $e\text{-}cl_\theta(f^{-1}[cl(int(cl(B)))]) \subset f^{-1}[cl(B)]$ for every subset B of Y,

(12) $e\text{-}cl_\theta(f^{-1}[V]) \subset f^{-1}[cl(V)]$ for every β-open set V of Y,

(13) $e\text{-}cl_\theta(f^{-1}[V]) \subset f^{-1}[cl(V)]$ for every semi-open set V of Y,

(14) $e\text{-}cl_\theta(f^{-1}[V]) \subset f^{-1}[\alpha\text{-}cl(V)]$ for every β-open set V of Y,

(15) $e\text{-}cl_\theta(f^{-1}[V]) \subset f^{-1}[pcl(V)]$ *for every semi-open set* V *of* Y*,*

(16) $e\text{-}cl_\theta(f^{-1}[cl(int(V))]) \subset f^{-1}[F]$ *for every closed set* F *of* Y*,*

(17) $e\text{-}cl_\theta(f^{-1}[cl(int(V))]) \subset f^{-1}[cl(V)]$ *for every closed set* V *of* Y*,*

(18) $f^{-1}[V] \subset e\text{-}int_\theta(f^{-1}[scl(V)])$ *for every open set* V *of* Y*,*

(19) $f^{-1}[V] \subset e\text{-}int_\theta(f^{-1}[int(cl(V))])$ *for every preopen set* V *of* Y*,*

(20) $f^{-1}[V] \subset e\text{-}int_\theta(f^{-1}[scl(V)])$ *for every preopen set* V *of* Y*,*

(21) $f^{-1}[V] \subset e\text{-}int_\theta(f^{-1}[int(cl(V))])$ *for every open set* V *of* Y*,*

(22) $f : X \to Y_s$ *is st.θ.e.c., where* Y_s *denotes the semi regularization of* Y*.*

Proof. *(1)* ⇒*(2)*: Let $x \in X$ and $V \in RO(Y, f(x))$. We have

$$\left.\begin{array}{r}(x \in X)(V \in RO(Y, f(x)))\\ RO(Y, f(x)) \subset \mathcal{U}(Y, f(x))\end{array}\right\} \Rightarrow \left.\begin{array}{r}(x \in X)(V \in \mathcal{U}(Y, f(x))\\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in eO(X, x))(f[e - cl\,(U)] \subset int(cl(V)) = V).$$

(2)⇒*(3)*: Let $x \in X$ and $V \in RO(Y, f(x))$. We have

$$\left.\begin{array}{r}(x \in X)\,(V \in RO(Y, f(x)))\\ \text{Hypothesis}\end{array}\right\} \Rightarrow (\exists U' \in eO(X, x))(f[e - cl\,(U)] \subset V), \tag{3.1}$$

$$U' \in eO(X, x) \Rightarrow U = e - cl(U) \in eR(X, x) \tag{3.2}$$

(3.1),(3.2)⇒ $(\exists U \in eR(X, x))(f[U] \subset V)$.

(3)⇒*(4)*: Let $x \in X$ and $V \in RO(Y, f(x))$. We have

$$\left.\begin{array}{r}(x \in X)(V \in RO(Y, f(x)))\\ \text{Hypothesis}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists U \in eR(X, x))(f[U] \subset V)\\ eR(X, x) \subset e\theta O(X, x)\end{array}\right\} \Rightarrow (\exists U \in e\theta O(X, x))(f[U] \subset V).$$

(4)⇒*(5)*: Let $G \in RO(Y, f(x))$ and $x \notin f^{-1}[G]$. We have

$$\left.\begin{array}{r}(G \in RO(Y, f(x)))\,(x \notin f^{-1}[G])\\ \text{Hypothesis}\end{array}\right\} \Rightarrow (\exists U \in e\theta O(X, x))(f[U] \subset G)$$
$$\left.\begin{array}{r}\Rightarrow (\exists U \in e\theta O(X, x))(x \in U \subset f^{-1}[G])\\ \text{Lemma2.2}\end{array}\right\} \Rightarrow$$
$$\Rightarrow \left(\bigcup_{x \in f^{-1}[G]} U \in e\theta O(X)\right)\left(\bigcup_{x \in f^{-1}[G]} U = f^{-1}[G]\right) \Rightarrow f^{-1}[G] \in e\theta O(X).$$

(5)⇒*(6)*: Let $F \in RC(Y)$. We have

$$\begin{array}{rcl}F \in RC(Y) & \Leftrightarrow & X \setminus F \in RO(Y)\\ & \Leftrightarrow & f^{-1}[X \setminus F] \in e\theta O(X)\\ & \Leftrightarrow & X \setminus f^{-1}[F] \in e\theta O(X)\\ & \Leftrightarrow & f^{-1}[F] \in e\theta C(X).\end{array}$$

(6)⇒(7): Let $V \in \delta O(Y)$. We have

$$\begin{array}{lll} V \in \delta O(Y) & \Rightarrow & X \setminus V \in \delta C(Y) \\ & \Rightarrow & X \setminus V = cl_\delta(X \setminus V) \\ & \Rightarrow & \left.\begin{array}{r} X \setminus V = \bigcap\{F|(W \subset F)(F \in RC(Y))\} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \end{array}$$

$$\Rightarrow (X \setminus V \subset F \in RC(Y) \Rightarrow f^{-1}[F] \in e\theta C(X)) \left(f^{-1}[X \setminus V] = \bigcap_{X \setminus V \subset F \in RC(Y)} f^{-1}[F] \right)$$

$$\Rightarrow f^{-1}[X \setminus V] \in e\theta C(X)$$
$$\Rightarrow X \setminus f^{-1}[V] \in e\theta C(X)$$
$$\Rightarrow f^{-1}[V] \in e\theta O(X).$$

(7)⇒(8): Let $F \in \delta C(Y)$. We have

$$\begin{array}{lll} F \in \delta C(Y) & \Rightarrow & X \setminus F \in \delta O(Y) \\ & \Rightarrow & f^{-1}[X \setminus F] \in e\theta O(X) \\ & \Rightarrow & X \setminus f^{-1}[F] \in e\theta O(X) \\ & \Rightarrow & f^{-1}[F] \in e\theta C(X). \end{array}$$

(8)⇒(9): Let $A \subset X$. We have

$$\left.\begin{array}{r} A \subset X \Rightarrow cl_\delta(f[A]) \in \delta C(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \left.\begin{array}{r} f^{-1}[cl_\delta(f[A])] \in e\theta C(X) \\ x \notin f^{-1}[cl_\delta(f[A])] \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in eO(X,x))(e - cl\,(U) \cap f^{-1}[cl_\delta(f[A])] = \emptyset).$$
$$\Rightarrow (\exists U \in eO(X,x))(e\text{-}cl\,(U) \cap A = \emptyset).$$
$$\Rightarrow x \notin e\text{-}cl_\theta(A).$$

Then $e\text{-}cl_\theta(A) \subset f^{-1}[cl_\delta(f[A])] \Rightarrow f^{-1}[e\text{-}cl_\theta(A)] \subset cl_\delta(f[A])$.

(9)⇒(10): Let$B \subset Y$. We have

$$\left.\begin{array}{r} B \subset Y \Rightarrow f^{-1}[B] \subset X \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f[e - cl_\theta(f^{-1}[B])] \subset cl_\delta(f[f^{-1}[B]]) \subset cl_\delta(B) \Rightarrow e\text{-}cl_\theta(f^{-1}[B]) \subset f^{-1}[cl_\delta(B)]$$

.

(10)⇒(11): Let $B \subset Y$. We have

$$\left.\begin{array}{r} B \subset Y \Rightarrow cl(int(cl(B))) \in RC(Y) \Rightarrow cl(int(cl(B))) \in \delta C(Y) \\ cl(int(cl(B))) \subset cl(B) \end{array}\right\} \Rightarrow$$

$$\Rightarrow e\text{-}cl_\theta(f^{-1}[cl(int(cl(B)))]) \subset f^{-1}[cl_\delta(cl(int(cl(B))))] \subset f^{-1}[cl_\delta(cl_\delta$$
$$\Rightarrow e\text{-}cl_\theta(f^{-1}[cl(int(cl(B)))]) \subset f^{-1}[cl_\delta(int(cl(B)))] = f^{-1}[cl(int(cl(B)))]$$
$$\Rightarrow e\text{-}cl_\theta(f^{-1}[cl(int(cl(B)))]) \subset f^{-1}[cl(B)].$$

(11)⇒(12): Let $V \in \beta O(Y)$. We have

$$\left.\begin{array}{r} V \in \beta O(Y) \overset{[2]}{\Rightarrow} cl(V) \in RC(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\Rightarrow e\text{-}cl_\theta(f^{-1}[V]) \subset e\text{-}cl_\theta(f^{-1}[cl(V)]) = e\text{-}cl_\theta(f^{-1}[cl(int(cl(V)))]) \subset f^{-1}[cl(V)].$$

(12)⇒(13): This is obvious since every semiopen set is β-open.

(13)⇒(14): Let $V \in \beta O(Y)$. We have

$$\left.\begin{array}{r} V \in \beta O(Y) \overset{\text{Lemma 2.6}}{\Rightarrow} \alpha\text{-}cl(V) \in SO(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\begin{array}{l} \Rightarrow e\text{-}cl_\theta(f^{-1}[V]) \subset e\text{-}cl_\theta(f^{-1}[\alpha\text{-}cl(V)]) \subset e\text{-}cl_\theta(f^{-1}[cl(\alpha\text{-}cl(V))]) \subset f^{-1}[cl(V)] \\ \Rightarrow e\text{-}cl_\theta(f^{-1}[V]) \subset f^{-1}[cl(V)] \overset{\text{Lemma2.3}}{=} f^{-1}[\alpha\text{-}cl(V)]. \end{array}$$

(14)⇒(15): Let $V \in SO(Y)$. We have

$$\left.\begin{array}{r} V \in SO(Y) \Rightarrow V \in \beta O(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow e\text{-}cl_\theta(f^{-1}[V]) \subset f^{-1}[\alpha\text{-}cl(V)] \\ V \in SO(Y) \overset{\text{Lemma 2.6}}{\Longrightarrow} \alpha\text{-}cl(V) = pcl(V) \end{array}\right\} \Rightarrow e\text{-}cl_\theta(f^{-1}[V]) \subset f^{-1}[pcl(V)].$$

(15)⇒(16): Let $V \in C(Y)$. We have

$$\left.\begin{array}{r} V \in C(Y) \Rightarrow cl(int(V)) \in SO(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow e\text{-}cl_\theta(f^{-1}[cl(int(V))]) \subset f^{-1}[pcl(int(cl(V)))] \subset f^{-1}[V].$$

(16)⇒(17): Let $V \in \sigma$. We have

$$\left.\begin{array}{r} V \in \sigma \Rightarrow cl(V) \in C(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow e\text{-}cl_\theta(f^{-1}[cl(int(cl(V)))]) \subset f^{-1}[cl(V)]\, e\text{-}cl_\theta(f^{-1}[cl(V)]) \subset f^{-1}[cl(V)].$$

(17)⇒(18): Let $V \in \sigma$. We have

$$V \in \sigma \Rightarrow Y \setminus cl(V) \in \sigma \overset{\text{Lemmas 2.1,2.5}}{\Rightarrow}$$

$$\left.\begin{array}{r} \Rightarrow X \setminus e\text{-}int_\theta(f^{-1}[scl(V)]) = e\text{-}cl_\theta(f^{-1}[Y \setminus int(cl(V))]) = e\text{-}cl_\theta(f^{-1}[cl(Y \setminus cl(V))]) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\begin{array}{l} \Rightarrow X \setminus e\text{-}int_\theta(f^{-1}[scl(V)]) \subset f^{-1}[Y \setminus cl(V)] \subset X \setminus f^{-1}[V] \\ \Rightarrow f^{-1}[V] \subset e\text{-}int_\theta(f^{-1}[scl(V)]). \end{array}$$

(18)⇒(19): Let $V \in PO(Y)$. We have

$$\left.\begin{array}{r} V \in PO(Y) \Rightarrow scl(V) = int(cl(V)) \\ \text{Hypothesis} \end{array}\right\} \overset{\text{Lemma 2.1}}{\Rightarrow}$$

$$\Rightarrow f^{-1}[V] \subset f^{-1}[scl(V)] \subset e\text{-}int_\theta(f^{-1}[scl(V)]) \subset e\text{-}int_\theta(f^{-1}[int(cl(V))]).$$

(19)⇒(20) and *(20)⇒(21)* are clear.

(21)⇒(22): Let $x \in X$ and $V \in O(Y_S, f(x))$. We have

$$\left.\begin{array}{r} (x \in X)\,(V \in O(Y_S, f(x))) \Rightarrow (\exists G \in RO(Y))(f(x) \in G \subset V) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow x \in f^{-1}[G] \subset e\text{-}int_\theta(f^{-1}[G])$$

$$\Rightarrow f^{-1}[G] \in e\theta O(X)$$

$$\overset{\text{Lemma 2.2}}{\Rightarrow} (\exists U \in eO(X,x))(e\text{-}cl(U) \subset f^{-1}[G])$$
$$\Rightarrow (\exists U \in eO(X,x))(f[e\text{-}cl(U)] \subset G \subset V).$$

(22)⇒(1): Let $V \in O(Y)$ and $x \in f^{-1}[V]$. We have

$$\left.\begin{array}{r}(V \in O(Y))\,(x \in f^{-1}[V]) \Rightarrow f(x) \in V \subset int(cl(V)) \in \sigma \\ \text{Hypothesis}\end{array}\right\} \Rightarrow (\exists U \in eO(X,x))(e\text{-}cl(U) \subset f^{-1}[int(cl(V))])$$
$$\Rightarrow (\exists U \in eO(X,x))(f[e\text{-}cl(U)] \subset int(cl(V))).$$

□

Definition 3.3. Let A be a subset of a topological space X. The e-θ-frontier of A is defined by $e\text{-}Fr_\theta(A) = e\text{-}cl_\theta(A) \setminus e\text{-}int_\theta(A)$.

Theorem 3.4. *The set of all points $x \in X$ at which a function $f : X \to Y$ is not a.st.θ.e.c. coincides with the union of the e-θ-frontiers of the inverse images of regular open sets of Y containing $f(x)$.*

Proof. Let $A := \{x \mid f$ is not a.st.θ.e.c. at a point x of $X\}$. Then

$$\begin{aligned} x \in A &\Rightarrow (\exists V \in RO(Y, f(x)))(\forall U \in eO(X,x))(f[e\text{-}cl(U)] \not\subset V) \\ &\Rightarrow (\exists V \in RO(Y, f(x)))(\forall U \in eO(X,x))(e\text{-}cl(U) \not\subset f^{-1}[V]) \\ &\Rightarrow (\exists V \in RO(Y, f(x)))(\forall U \in eO(X,x))(e\text{-}cl(U) \cap \left(X \setminus f^{-1}[V]\right) \neq \emptyset) \\ &\Rightarrow x \in e\text{-}cl_\theta(X \setminus f^{-1}[V]) \\ &\Rightarrow x \in X \setminus e\text{-}int_\theta(f^{-1}[V]) \\ &\Rightarrow x \notin e\text{-}int_\theta(f^{-1}[V]), \end{aligned} \tag{3.3}$$

$$f(x) \in V \Rightarrow x \in f^{-1}[V] \subset e - cl_\theta(f^{-1}[V]) \Rightarrow x \in e - cl_\theta(f^{-1}[V]) \tag{3.4}$$

(3.3), (3.4)⇒ $x \in e\text{-}Fr_\theta(f^{-1}[V])$.
Then we have $A \subset \bigcup \left\{e\text{-}Fr_\theta(f^{-1}[V]) | f(x) \in V \in RO(Y)\right\}$.

$$\left.\begin{array}{r} x \notin A \Rightarrow f \text{ is a.st.}\theta\text{.e.c. at} x \\ f(x) \in V \in RO(Y)\end{array}\right\} \Rightarrow x \in f^{-1}[V] \in e\theta O(X)$$

$$\begin{aligned} &\Rightarrow x \in e\text{-}int_\theta(f^{-1}[V]) \\ &\Rightarrow x \notin e\text{-}Fr_\theta(f^{-1}[V]) \\ &\Rightarrow x \notin \bigcup \left\{e\text{-}Fr_\theta(f^{-1}[V]) | f(x) \in V \in RO(Y)\right\}. \end{aligned}$$

Then we have $\bigcup \left\{e\text{-}Fr_\theta(f^{-1}[V]) | f(x) \in V \in RO(Y)\right\} \subset A$. □

4. Comparisons and Some Properties

Definition 4.1. A function $f : X \to Y$ is called almost strongly θ-continuous [17] (resp. almost strongly θ-semicontinuous [4], almost strongly θ-precontinuous [21], almost strongly θ-b-continuous [18]), if for each $x \in X$ and each open set V containing $f(x)$, there is an open (resp. semi-open, preopen, b-open) set U containing x such that $f[cl(U)] \subset int(cl(V))$ (resp. $f[scl(U)] \subset int(cl(V))$, $f[pcl(U)] \subset int(cl(V))$, $f[bcl(U)] \subset int(cl(V))$).

Definition 4.2. A function $f : X \to Y$ is called strongly θ-e-continuous [19] (resp. e-continuous [8]) if for each $x \in X$ and each open set V containing $f(x)$, there is an e-open set U containing x such that $f[e\text{-}cl(U)] \subset V$ (resp. $f[U] \subset V$).

Definition 4.3. A function $f : X \to Y$ is called θ-e-continuous [11] if for each $x \in X$ and each open set V containing $f(x)$, there is an e-open set U containing x such that $f[e\text{-}cl(U)] \subset cl(V)$.

Remark 4.4. From Definitions 4.1, 4.2 and 4.3, we have the following diagram.

$$
\begin{array}{ccccccc}
 & & \text{a.st.}\theta.p.\text{c.} & & \text{st.}\theta.e.\text{c.} & & \\
 & \nearrow & \downarrow & \searrow & \downarrow & & \\
\text{a.st.}\theta.\text{c.} & & \text{a.st.}\theta.b.\text{c.} & & \text{a.st.}\theta.e.\text{c.} & & e.\text{c.} \\
 & \searrow & \uparrow & \nearrow & \downarrow & \nearrow & \\
 & & \text{a.st.}\theta.s.\text{c.} & & \theta.e.\text{c.} & &
\end{array}
$$

However, none of these implications is reversible as shown by the following examples.

Example 4.5. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$.

(a) Define the function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = f(b) = a$, $f(c) = f(d) = c$. Then f is a.st.$\theta.e$.c. on X, but it is not a.st.$\theta.p$.c. at the point d of X.

(b) Define the function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = a$, $f(b) = c$, $f(c) = f(d) = d$. Then f is a.st.$\theta.e$.c. on X, but it is not a.st.$\theta.s$.c. at the point a of X.

Example 4.6. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$ and $\sigma = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$.

(a) Define a function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = f(c) = f(d) = a$, $f(b) = c$. Then f is $\theta.e$.c. on X, but it is not a.st.$\theta.e$.c. at the point b of X.

(b) Define a function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = f(b) = f(d) = d$, $f(c) = a$. Then f is a.st.$\theta.e$.c. on X, but it is not a.st.$\theta.b$.c. at the point d of X.

Example 4.7. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = f(b) = b$, $f(c) = d$, $f(d) = c$. Then f is a.st.$\theta.e$.c. on X, but it is not st.$\theta.e$.c. at the point d of X.

Example 4.8. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$ and $\sigma = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = f(d) = a$, $f(b) = f(c) = c$. Then f is a.st.$\theta.b$.c. on X, but it is not a.st.$\theta.e$.c. at the point c of X.

The family of regular open sets of a space (X, τ) forms a base for a smaller topology τ_s on X, called semi-regularization of τ. The space (X, τ) is said to be semi-regular if $\tau_s = \tau$ [14].

A space (X, τ) is called almost regular [23] if for any regular open set $U \subset X$ and each point $x \in U$, there is a regular open set V of X such that $x \in V \subset cl(V) \subset U$.

Theorem 4.9. *Let $f : X \to Y$ be a function. Then the following statements hold:*
(a) If $f : X \to Y$ e.c. and Y is almost regular, then f is a.st.θ.e.c.
(b) If $f : X \to Y$ is a.st.θ.e.c. and Y is semi-regular, then f is st.θ.e.c.

Proof. (a) Let f be e.c. and Y almost regular.We have

$$
\left.\begin{array}{r}(x \in X)\,(V \in RO(Y, f(x)))\\ Y \text{ is almost regular}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists W \in RO(Y, f(x)))(W \subset cl(W) \subset V)\\ f \text{ is } e.\text{c.}\end{array}\right\} \Rightarrow
$$

$$
\left.\begin{array}{r}\Rightarrow (\exists U \in eO(X, x))(f[U] \subset W \Rightarrow U \subset f^{-1}[W])\\ y \notin cl(W) \Rightarrow (\exists G \in \mathcal{U}(y))(G \cap W = \emptyset) \Rightarrow f^{-1}[G] \cap f^{-1}[W] = \emptyset\end{array}\right\} \Rightarrow f^{-1}[G] \cap U = \emptyset ...(1)
$$

$$\left.\begin{array}{r} G \in \mathcal{U}(y) \\ f \text{ is e.c.} \end{array}\right\} \Rightarrow f^{-1}[G] \in eO(X) ...(2)$$

$(1),(2) \Rightarrow f^{-1}[G] \cap e\text{-}cl(U) = \emptyset \Rightarrow G \cap f[e\text{-}cl(U)] = \emptyset \Rightarrow y \notin f[e\text{-}cl(U)]$.

(b) Let f be a.st.θ.e.c. and Y semi-regular. We have

$$\left.\begin{array}{r} (x \in X)(V \in \mathcal{U}(Y, f(x)) \\ Y \text{ is semi-regular} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists W \in RO(X,x))(W \subset V) \\ f \text{ is a.st.}\theta\text{.e.c.} \end{array}\right\} \Rightarrow (\exists W \in eO(X,x))(f[e\text{-}cl(U)] \subset W \subset V).$$

□

Theorem 4.10. *Let Y be a semi-regular space. Then $f : X \to Y$ is a.st.θ.e.c. if and only if $f : X \to Y$ is st.θ.e.c.*

Proof. It follows clearly from Theorem 4.9. □

Corollary 4.11 ([19])**.** *Let Y be a regular space. Then the following statements are equivalent for a function $f : X \to Y$:*

(1) f is st.θ.e.c.,
(2) f is a.st.θ.e.c.,
(3) f is θ.e.c.,
(4) f is e.c.

Recall that a space X is called submaximal if each dense subset of X is open in X. A space X is called extremally disconnected if the closure of each open subset of X is open in X. In an extremally disconnected submaximal regular space, open, preopen, semiopen, b-open and e-open sets are equivalent. Then we have the following corollary:

Corollary 4.12 ([19])**.** *Let X be an extremally disconnected submaximal regular space and let Y be a regular space. Then the following statements are equivalent for a function $f : X \to Y$:*

(1) f is almost strongly θ-continuous,
(2) f is almost strongly θ-precontinuous,
(3) f is almost strongly θ-semicontinuous,
(4) f is almost strongly θ-b-continuous,
(5) f is almost strongly θ-e-continuous,
(6) f is strongly θ-e-continuous,
(7) f is strongly θ-continuous,
(8) f is b-continuous,
(9) f is e-continuous.

5. Fundamental Properties

Lemma 5.1. *Let X be a topological space and X_0 an a-open set in X. Then:*

(a) $X_0 \cap eO(X) := \{X_0 \cap E | E \in eO(X)\} = eO(X_0)$.
(b) If $A \subset X_0$ and $A \in eO(X_0)$, then $A \in eO(X)$.
(c) If $F \subset X_0$ and $F \in eC(X_0)$, then $F \in eC(X)$.

Proof. (a) [20]
(b) Let $A \in eO(X_0)$. Then

$$\begin{array}{rcl} A \in eO(X_0) & \overset{(a)}{\Rightarrow} & A \in X_0 \cap eO(X) \\ & \Rightarrow & (\exists E \in eO(X))(A = X_0 \cap E) \\ & \Rightarrow & A \in eO(X). \end{array}$$

(c) Let $F \in eC(X_0)$. Then

$$F \in eC(X_0) \Rightarrow X \setminus F \in eO(X_0) \overset{(b)}{\Rightarrow} X \setminus F \in eO(X) \Rightarrow F \in eC(X).$$

□

Lemma 5.2. *If $A \subset X_0 \subset X$ and X_0 is an a-open set in X, then $e\text{-}cl(A) \cap X_0 = e\text{-}cl_{X_0}(A)$, where $e\text{-}cl_{X_0}(A)$ denotes the e-closure of A in the subspace X_0.*

Proof. Let $x \in e\text{-}cl(A) \cap X_0$ and $U \in eO(X_0, x)$. We have

$$\left.\begin{array}{r}(x \in e\text{-}cl(A) \cap X_0)\,(U \in eO(X_0, x)) \overset{\text{Lemma 5.1}}{\Rightarrow} (\exists V \in eO(X, x))\,(U = V \cap X_0) \\ x \in e\text{-}cl(A)\end{array}\right\} \Rightarrow$$

$\Rightarrow \emptyset \neq V \cap A = U \cap A \Rightarrow x \in e\text{-}cl_{X_0}(A)$. Then we have $e\text{-}cl(A) \cap X_0 \subset e\text{-}cl_{X_0}(A)$.

$$(x \in e\text{-}cl_{X_0}(A))\,(U \in eO(X, x)) \overset{\text{Lemma 5.1}}{\Rightarrow} (U \cap X_0 \in eO(X, x))\,(\emptyset \neq A \cap (U \cap X_0) = A \cap U)$$
$$\Rightarrow x \in e\text{-}cl(A)...(1)$$
$$x \in e\text{-}cl_{X_0}(A) \subset X_0 \Rightarrow x \in X_0...(2)$$

$(1),(2) \Rightarrow x \in e\text{-}cl(A) \cap X$. Then we have $e\text{-}cl_{X_0}(A) \subset e\text{-}cl(A) \cap X_0$.

□

Lemma 5.3. *Let $G \subset X_0 \subset X$ and X_0 be an a-open set in X. If G is an e-θ-open set in X_0, then G is an e-θ-open set in X.*

Proof. Let $G \in e\theta O(X_0, x)$. Then

$$\begin{array}{lcl} G \in e\theta O(X_0, x) & \overset{\text{Lemma 2.2}}{\Rightarrow} & (\exists U \in eO(X_0, x))\,(U \subset e\text{-}cl(U) \subset G) \\ & \overset{\text{Lemma 2.2}}{\Rightarrow} & e\text{-}cl_{X_0}(U) \in eC(X_0) \\ & \overset{\text{Lemma 5.1}}{\Rightarrow} & (U \in eO(X))(e\text{-}cl_{X_0}(U) \in eC(X)) \\ & \Rightarrow & x \in U \subset e\text{-}cl(U) \subset e\text{-}cl(e\text{-}cl_{X_0}(U)) = e\text{-}cl_{X_0}(U) \subset G \\ & \Rightarrow & x \in e\text{-}int_\theta(G). \end{array}$$

□

Lemma 5.4. *If X_0 is an a-open set and U is an e-θ-open set in X, then $U \cap X_0$ is an e-θ-open set in the relative topology of X_0.*

Proof. Let X_0 be an a-open set in X and $U \in e\theta O(X)$. Then

$$\left.\begin{array}{r} x \in U \cap X_0 \Rightarrow (x \in U)\,(x \in X_0) \\ U \in e\theta O(X) \end{array}\right\} \overset{\text{Lemma 2.2}}{\Rightarrow} (\exists T \in eO(X, x))(e - cl(T) \subset U)$$

$$\begin{array}{l} \overset{\text{Lemma 5.1}}{\Rightarrow} (T \cap X_0 \in eO(X_0, x))(T \cap X_0 \subset e - cl(T) \cap X_0 \subset U \cap X_0) \\ \overset{\text{Lemma 5.2}}{\Rightarrow} (T \cap X_0 \in eO(X_0, x))(T \cap X_0 \subset e - cl_{X_0}(T \cap X_0) \\ \quad = e - cl(T \cap X_0) \cap X_0 \subset e - cl(T) \cap X_0 \subset U \cap X_0) \\ \quad \Rightarrow x \in e - int_\theta(U \cap X_0). \end{array}$$

□

Corollary 5.5. *If X_0 is an a-open set and U is an e-θ-open set in X, then $U \cap X_0$ is an e-θ-open set in X.*

Theorem 5.6. *Let $\{U_\alpha \mid \alpha \in \Lambda\}$ be an a-open cover of a topological space X. A function $f : (X, \tau) \to (Y, \sigma)$ is a.st.θ.e.c. if and only if the restriction $f|_{U_\alpha} : (U_\alpha, \tau_{U_\alpha}) \to (Y, \sigma)$ is a.st.θ.e.c. for each $\alpha \in \Lambda$.*

Proof. Necessity. Let f be a.st.θ.e.c. and $\alpha_0 \in \Lambda$ and $x \in U_{\alpha_0}$. Then

$$\left.\begin{array}{r}(f(x) \in V \in \sigma)\,(f \text{ a.st.}\theta\text{.e.c.}) \Rightarrow (\exists G \in eO(X,x))\,(f[e\text{-}cl(G)] \subset int(cl(V))) \\ W := G \cap U_{\alpha_0}\end{array}\right\} \Rightarrow$$

$$\overset{\text{Lemma 5.1,5.2}}{\Rightarrow} (x \in W \in eO(U_{\alpha_0}))\,(e - cl_{U_{\alpha_0}}(W) \subset e - cl(W))$$

$$\Rightarrow (W \in eO(U_{\alpha_0}, x))\left(f|_{U_{\alpha_0}}\left[e\text{-}cl_{U_{\alpha_0}}(W)\right] = f\left[e\text{-}cl_{U_{\alpha_0}}(W)\right] \subset f\left[e\text{-}cl(W))\right] \subset int(cl(V))\right).$$

Sufficiency. Let $f|_{U_\alpha}$ be a.st.θ.e.c. for all $\alpha \in \Lambda$ and $V \in RO(Y)$. Then

$$\left.\begin{array}{r}V \in RO(Y) \\ f|_{U_\alpha} \text{ is a.st.}\theta\text{.e.c.}\end{array}\right\} \overset{\text{Theorem 3.2}}{\Rightarrow} (\forall \alpha \in \Lambda)\left((f|_{U_\alpha})^{-1}[V] \in e\theta O(U_\alpha)\right)$$

$$\begin{aligned}&\overset{\text{Lemma 5.3}}{\Rightarrow} (\forall \alpha \in \Lambda)\left((f|_{U_\alpha})^{-1}[V] \in e\theta O(X)\right) ...(1)\\ &\Rightarrow f^{-1}[V] = f^{-1}[V] \cap X = f^{-1}[V] \cap \left(\bigcup_{\alpha\in\Lambda} U_\alpha\right) = \bigcup\{f^{-1}[V] \cap U_\alpha | \alpha \in \Lambda\}\\ &\Rightarrow f^{-1}[V] = \bigcup\{(f|_{U_\alpha})^{-1}[V] | \alpha \in \Lambda\} ...(2)\end{aligned}$$

$$(1), (2) \Rightarrow f^{-1}[V] \in e\theta O(X).$$

□

Definition 5.7. A function $f : X \to Y$ is called an R-map [6] if the preimage of every regular open subset of Y is regular open in X.

Definition 5.8. A function $f : X \to Y$ is called δ-continuous [16] if for each $x \in X$ and each open set V containing $f(x)$, there is an open set U containing x such that $f[int(cl(U))] \subset int(cl(V))$.

Theorem 5.9. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then:*

(1) If f is a.st.θ.e.c. and g is an R-map, then $g \circ f$ is a.st.θ.e.c.

(2) If f is a.st.θ.e.c. and g is δ-continuous, then $g \circ f$ is a.st.θ.e.c.

Proof. Clear. □

Theorem 5.10. *Let $f : X \to Y$ be a function and $g : Y \to Z$ an injective R-map which preserves regular open sets. Then f is a.st.θ.e.c. if and only if $g \circ f$ is a.st.θ.e.c.*

Proof. Necessity. It follows from Theorem 5.9.

Sufficiency. Let $g \circ f$ be a.st.θ.e.c. and let g be an injective R-map which preserves regular open sets.

$$\left.\begin{array}{r}V \in RO(Y) \overset{\text{Hypothesis}}{\Rightarrow} g[V] \in RO(Z) \\ g \text{ is } R\text{-map and injective}\end{array}\right\} \Rightarrow V = g^{-1}[g[V]] \in RO(Y)$$

$$\left.\begin{array}{r}\Rightarrow f^{-1}[V] = f^{-1}\left[g^{-1}[g[V]]\right] = (g \circ f)^{-1}[g[V]] \\ g \circ f \text{ is a.st.}\theta\text{.e.c.}\end{array}\right\} \Rightarrow f^{-1}[V] \in e\theta O(X).$$

□

Theorem 5.11. *Let $\{Y_\alpha | \alpha \in \Lambda\}$ be a family of spaces. If a function $f : X \to \Pi Y_\alpha$ is a.st.θ.e.c., then $P_\alpha \circ f : X \to Y_\alpha$ is a.st.θ.e.c. for each $\alpha \in \Lambda$, where P_α is the projection of ΠY_α onto Y_α.*

Proof. This is obvious from Theorem 5.9 because every open continuous surjection P_α is an R-map. □

6. Separation Axioms

Definition 6.1. A space X is called almost e-regular [11] if for any regular closed set $F \subset X$ and any point $x \in X \setminus F$, there exist disjoint e-open sets U and V such that $x \in U$ and $F \subset V$.

Theorem 6.2. *The following statements are equivalent for a space X:*

(1) X is almost e-regular,

(2) for each $x \in X$ and for each regular open set U of X containing x, there exists $V \in eO(X)$ such that $x \in V \subset e\text{-}cl(V) \subset U$,

(3) for each regular closed set F of X, $F = \cap\{e\text{-}cl(V) | F \subset V \text{ and } V \in eO(X)\}$,

(4) for each subset $A \subset X$ and each regular closed set F such that $A \cap F = \emptyset$, there exist disjoint $U, V \in eO(X)$ such that $A \cap U \neq \emptyset$ and $F \subset V$,

(5) for each subset $A \subset X$ and each regular open set U such that $A \cap U \neq \emptyset$, there exists $W \in eO(X)$ such that $A \cap W \neq \emptyset$ and $e\text{-}cl(W) \subset U$.

Proof. It can be proved directly. □

Theorem 6.3. *If a continuous function $f : X \to X$ is a.st.θ.e.c., then X is almost e-regular.*

Proof. Let f be the identity function. Then f is continuous and a.st.θ.e.c. so,

$$\left.\begin{array}{r} x \in U \in RO(X) \\ f \text{ is identity and a.st.}\theta.e.\text{c.} \end{array}\right\} \Rightarrow x \in f^{-1}[U] = U \in e\theta O(X)$$

$$\overset{\text{Lemma 2.2}}{\Rightarrow} (\exists V \in eO(X, x))(V \subset e - cl(V) \subset U).$$

□

Theorem 6.4. *An R-map $f : X \to X$ is a.st.θ.e.c. if and only if X is almost e-regular.*

Proof. Necessity. Obvious.
Sufficiency. Let f be an R-map and X be almost e-regular.

$$\left.\begin{array}{r} (x \in X)(V \in RO(Y, f(x))) \\ f \text{ is } R\text{-map} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (x \in f^{-1}[V] \in RO(X)) \\ X \text{ is almost } e\text{-regular} \end{array}\right\} \overset{\text{Theorem 6.2}}{\Rightarrow}$$

$$\begin{array}{l} \Rightarrow (\exists U \in eO(X, x))(e - cl(U) \subset f^{-1}[V]) \\ \Rightarrow (\exists U \in eO(X, x))(f[e\text{-}cl(U)] \subset V). \end{array}$$

□

Definition 6.5. A space is called e-regular [19] if for any closed set $F \subset X$ and any point $x \in X \setminus F$, there exist disjoint e-open sets U and V such that $x \in U$ and $F \subset V$.

Definition 6.6. A function $f : X \to Y$ is called almost continuous [24] if the preimage of every regular open subset of Y is open in X.

Theorem 6.7. *If $f : X \to Y$ is almost continuous and X is e-regular, then f is a.st.θ.e.c.*

Proof. Let $x \in X$ and let $V \in RO(Y, f(x))$. Then

$$\left.\begin{array}{r}(x \in X)(V \in RO(Y, f(x)))\\ f \text{ is almost continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r} x \in f^{-1}[V] \in \tau\\ X \text{ is } e\text{-regular}\end{array}\right\} \overset{[19]}{\Rightarrow}$$

$$\begin{aligned}&\Rightarrow (\exists U \in eO(X,x))(e - cl(U) \subset f^{-1}[V])\\ &\Rightarrow (\exists U \in eO(X,x))(f[e\text{-}cl(U)] \subset V).\end{aligned}$$

□

Theorem 6.8. *Let $f : X \to Y$ be a function and let $g : X \to X \times Y$, given by $g(x) = (x, f(x))$ for each $x \in X$ be graph function. Then g is a.st.θ.e.c. if and only if f is a.st.θ.e.c. and X is almost e-regular.*

Proof. Necessity. Let $x \in X$ and let $V \in RO(Y, f(x))$. Then

$$\left.\begin{array}{r}(x \in X)\,(V \in RO(Y, f(x))) \Rightarrow g(x) = (x, f(x)) \in X \times V\\ X \times V \in RO(X \times Y)\\ g \text{ is a.st.}\theta\text{.e.c.}\end{array}\right\} \Rightarrow (\exists U \in eR(X,x))(g\,[U] \subset X \times V)$$

$$\Rightarrow (\exists U \in eR(X,x))(f\,[U] \subset V). \text{ Then } f \text{ is a.st.}\theta\text{.e.c.}$$

$$\left.\begin{array}{r}U \in RO(X,x) \Rightarrow g(x) \in U \times Y \in RO(X \times Y)\\ g \text{ is a.st.}\theta\text{.e.c.}\end{array}\right\} \Rightarrow (\exists W \in eO(X,x))(g\,[e\text{-}cl(W)] \subset U \times Y)$$

$$\Rightarrow (\exists W \in eO(X,x))(W \subset e\text{-}cl(W) \subset U). \text{ Then } X \text{ is almost } e\text{-regular.}$$

Sufficiency. Let $x \in X$ and let $V \in RO(X \times Y, g(x))$. Then

$$\left.\begin{array}{r}(x \in X)(V \in RO(X \times Y, g(x))) \Rightarrow (\exists V_1 \in RO(X))\,(\exists V_2 \in RO(Y))\,(g(x) = (x, f(x)) \in V_1 \times V_2 \subset V)\\ f \text{ is a.st.}\theta\text{.e.c.}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U_0 \in eR(X,x))(f[U_0] \subset V_2)...(1)$$

$U := U_0 \cap V_1 \overset{\text{Lemma 5.4}}{\Rightarrow} U \in e\theta O(V_1) \overset{\text{Lemma 5.3}}{\Rightarrow} U \in e\theta O(X)...(2)$

$(1), (2) \Rightarrow (\exists U \in e\theta O(X))\,(g[U] \subset U \times f\,[U] \subset U \times f\,[U_0] \subset V_1 \times V_2 \subset V).$ □

Definition 6.9. A space X is said to be:

(1) rT_0 [10] if for each pair of distinct points x and y in X, there exists a regular open set $U \in RO(X)$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

(2) e-T_2 [7] if for each pair of distinct points x and y in X, there exist e-open sets U and V of X containing x and y, respectively, such that $U \cap V = \emptyset$.

Theorem 6.10. *If $f : X \to Y$ is an a.st.θ.e.c. injection and Y is rT_0, then X is e-T_2.*

Proof. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Then

$$\left.\begin{array}{r}(x_1, x_2 \in X)(x_1 \neq x_2)(f \text{ is injective}) \Rightarrow f(x_1) \neq f(x_2)\\ Y \text{ is} rT_0\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists V \in RO(Y, f(x_1)))\,(\exists W \in RO(Y, f(x_2)))\,(f(x_1) \notin W \vee f(x_2) \notin V).$$

Case I. Let $V \in RO(Y, f(x_1))$ and $f(x_2) \notin V$.

$$\left.\begin{array}{r}V \in RO(Y, f(x_1))\\ f \text{ is a.st.}\theta\text{.e.c.}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists U \in eO(X, x_1))(f[e\text{-}cl(U)] \subset V)\\ f(x_2) \notin V\end{array}\right\} \Rightarrow f(x_2) \notin f[e\text{-}cl(U)]$$

$$\Rightarrow x_2 \notin e\text{-}cl(U) \Rightarrow x_2 \in X \setminus e\text{-}cl(U).$$

Case II. It can be proved similarly. □

Corollary 6.11. *If $f : X \to Y$ is an a.st.θ.e.c. injection and Y is Hausdorff, then X is e-T_2.*

Proof. It is obvious since every Hausdorff space is rT_0. □

Theorem 6.12. *Let $f, g : X \to Y$ be functions and Y a Hausdorff space. If f is a.st.θ.e.c. and g is an R-map, then the set $A = \{x \in X \mid f(x) = g(x)\}$ is e-closed in X.*

Proof. Let $x \notin A$. Then

$$\left.\begin{array}{r} x \notin A \Rightarrow f(x) \neq g(x) \\ Y \text{ is Hausdorff} \end{array}\right\} \Rightarrow (\exists V_1 \in \mathcal{U}(f(x)))\,(\exists V_2 \in \mathcal{U}(g(x)))\,(V_1 \cap V_2 = \emptyset)$$

$$\Rightarrow (\exists V_1 \in \mathcal{U}(f(x)))\,(\exists V_2 \in \mathcal{U}(g(x)))\,(int(cl(V_1)) \cap int(cl(V_2)) = \emptyset)...(1)$$

$$\left.\begin{array}{r} int(cl(V_1)) \in RO(Y, f(x_1)) \\ f \text{ is a.st.}\theta\text{.e.c.} \end{array}\right\} \Rightarrow (\exists G \in eO(X, x))(f[e - cl(G)] \subset int(cl(V_1)))...(2)$$

$$\left.\begin{array}{r} int(cl(V_2)) \in RO(Y, f(x_2)) \\ g \text{ is } R\text{-map} \end{array}\right\} \Rightarrow g^{-1}[int(cl(V_2))] \in RO(X, x)...(3)$$

$$U := G \cap g^{-1}[int(cl(V_2))] \overset{\text{Lemma 2.7}}{\Rightarrow} U \in eO(X, x)...(4)$$

$$(1), (2), (3), (4) \Rightarrow (U \in eO(X, x))\,(U \cap A = \emptyset) \Rightarrow x \notin e\text{-}cl(A).$$

□

7. Preservation Properties

Definition 7.1. A space X is called:

(1) nearly compact [22] (resp. nearly countable compact [9]) if every regular open cover (resp. countable regular open cover) of X has a finite subcover.

(2) e-closed [19] (resp. countable e-closed [19]) if every cover (resp. countable cover) of X by e-open sets has a finite subcover whose e-closures cover X.

A subset A of a space X is said to be e-closed [19] (resp. N-closed [5]) relative to X if for every cover $\{V_\alpha | \alpha \in I\}$ of A by e-open (resp. regular open) sets of X, there exists a finite subset I_0 of I such that $A \subset \bigcup\{e\text{-}cl(V_\alpha) | \alpha \in I_0\}$ (resp. $A \subset \bigcup\{V_\alpha | \alpha \in I_0\}$).

Theorem 7.2. *If $f : X \to Y$ is an a.st.θ.e.c. function and A is e-closed relative to X, then $f[A]$ is N-closed relative to Y.*

Proof. It can be proved directly. □

Corollary 7.3. *Let $f : X \to Y$ be an a.st.θ.e.c. surjection. Then the following statements hold:*
(1) If X is e-closed, then Y is nearly compact.
(2) If X is countable e-closed, then Y is nearly countable compact.

Definition 7.4. The graph $G(f)$ of a function $f : X \to Y$ is said to be θ-e-closed [11] if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in eO(X, x)$ and an open set V containing y such that $(e\text{-}cl(U) \times cl(V)) \cap G(f) = \emptyset$.

Definition 7.5. The graph $G(f)$ of a function $f : X \to Y$ is said to be almost strongly e-closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in eO(X, x)$ and a regular open set V containing y such that $(e\text{-}cl(U) \times V) \cap G(f) = \emptyset$.

Corollary 7.6. *If the graph $G(f)$ of a function $f : X \to Y$ is θ-e-closed, then it is almost strongly e-closed.*

Lemma 7.7. *The graph $G(f)$ of a function $f : X \to Y$ is almost strongly e-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in eO(X, x)$ and a regular open set V containing y such that $f[e\text{-}cl(U)] \cap V = \emptyset$.*

Proof. It follows immediately from the definition. □

Definition 7.8. The graph $G(f)$ of a function $f : X \to Y$ is said to be strongly e-closed [19] if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in eO(X, x)$ and an open set V containing y such that $(e\text{-}cl(U) \times V) \cap G(f) = \emptyset$.

It is obvious that if the graph of a function is almost strongly e-closed, then it is strongly e-closed.

Theorem 7.9. *If $f : X \to Y$ is a.st.θ.e.c. and Y is Hausdorff, then the graph $G(f)$ of f is almost strongly e-closed in $X \times Y$.*

Proof. Let $(x, y) \notin G(f)$. Then

$$\left.\begin{array}{r}(x,y) \notin G(f) \Rightarrow y \neq f(x) \\ Y \text{ is Hausdorff}\end{array}\right\} \Rightarrow (\exists V_1 \in \mathcal{U}(Y, f(x)))\,(\exists V_2 \in \mathcal{U}(Y, y))\,(V_1 \cap V_2 = \emptyset)$$

$$\left.\begin{array}{r}\Rightarrow (\exists V_1 \in \mathcal{U}(f(x)))\,(\exists V_2 \in \mathcal{U}(g(x)))\,(int(cl(V_1)) \cap int(cl(V_2)) = \emptyset) \\ f \text{ is a.st.}\theta.e.\text{c.}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in eO(X, x))(f[e\text{-}cl(U)] \cap int(cl(V_2)) = \emptyset).$$

Then $G(f)$ is almost strongly e-closed in $X \times Y$ by Lemma 7.7. □

Theorem 7.10. *If a function $f : X \to Y$ has an almost strongly e-closed graph, then $f[K]$ is δ-closed in Y for each subset K which is e-closed relative to X.*

Proof. Let f be a.st.θ.e.c. and $y \notin f[K]$. Then

$$\left.\begin{array}{r}y \notin f[K] \Rightarrow (\forall x \in K)((x,y) \notin G(f)) \\ G(f) \text{ is almost strongly } e\text{-closed}\end{array}\right\} \overset{\text{Lemma 7.7}}{\Rightarrow} (\exists U_x \in eO(X, x))(\exists V_x \in RO(Y, y))(f[e\text{-}cl(U_x)] \cap V_x = \emptyset)$$

$$\left.\begin{array}{r}\Rightarrow (\{U_x | x \in K\} \subset eO(X))(K \subset \bigcup\{U_x | x \in K\}) \\ K \text{ is } e\text{-closed relative to } X\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists K^* \subset K)(|K^*| < \mathcal{N}_0)(K \subset \bigcup\{e\text{-}cl(U_x) | x \in K^*\}) \\ V := \bigcap_{x \in K^*} V_x \in RO(Y, y)\end{array}\right\} \Rightarrow$$

$$\Rightarrow (V \in RO(Y, y)) \left(f[K] \cap V \subset \left(\bigcup_{x \in K^*} f\,[e\text{-}cl(U_x)] \right) \cap V = \emptyset \right)$$

$$\Rightarrow (V \in RO(Y, y))\,(f[K] \cap V = \emptyset) \Rightarrow x \notin cl_\delta(f[K]).$$

□

Corollary 7.11. *If $f : X \to Y$ is an a.st.θ.e.c. function and Y is Hausdorff, then $f[K]$ is δ-closed in Y for each subset K which is e-closed relative to X.*

Theorem 7.12 ([19])**.** *Let X be a submaximal extremally disconnected regular space and Y be a compact Hausdorff space. Then the following statements are equivalent:*

(1) f is strongly θ-e-continuous,

(2) $G(f)$ is strongly e-closed in $X \times Y$,
(3) f is strongly θ-continuous,
(4) f is continuous,
(5) f is e-continuous.

Corollary 7.13. *Let X be a submaximal extremally disconnected regular space and Y a compact Hausdorff space. Then the following properties are equivalent:*
(1) f is strongly θ-e-continuous,
(2) f is almost strongly θ-e-continuous,
(3) $G(f)$ is almost strongly e-closed in $X \times Y$,
(4) $G(f)$ is strongly e-closed in $X \times Y$,
(5) f is strongly θ-continuous,
(6) f is continuous,
(7) f is e-continuous.

Proof. *(2)⇒(3)*: It follows from Theorem 7.9. Other implications follow from Theorem 7.12. □

Acknowledgements

The authors are very grateful to the referee for his observations and comments which improved the value of the paper.

This study is dedicated to Professor Zekeriya GÜNEY on the occasion of his 67th birthday.

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