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Some fixed point theorems in generalized quasi-partial metric spaces

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Abstract

In this paper, a new concept of generalized quasi-partial metric spaces is presented. Some fixed point results due to Karapinar *et. al.*, [E. Karapinar, I. M. Erhan, A. Öztürk, Math. Comput. Modelling, **57** (2013), 2442–2448] are extended in the setting of the generalized quasi-partial metric spaces.

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1. Introduction

Matthews introduced the concept of a partial metric space by replacing the condition $d(x,x) = 0$ with the condition $d(x,x) \leqslant d(x,y)$ for all x, y [12, 13]. The partial metric space is a generalization of the metric space and has applications in theoretical computer science [3]. A lot more generalized metric spaces were put forward by many researchers of fixed point theory, for example, Hitzler and Seda have focused fixed point theorems on dislocated metric spaces defined by themselves [7], more relevant results based on such spaces followed in recent years [2, 10, 17, 19]. Czerwik presented the notion of b-metric space [5]. Nakano [16] introduced the notion of modular spaces as a generalize of metric spaces in 1950. Corresponding fixed point theorems were studied in the above generalized metric spaces (see, e.g.[1, 4, 8, 11, 14, 15] and the references therein). Especially, as a further generalization for the metric spaces and partial metric spaces, Karapinar et al. introduced the notion of a quasi-partial metric space and discussed the existence of fixed points of self-mappings T on quasi-partial metric spaces [9]: any mapping T of a complete quasi-partial metric space X into

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itself that satisfies, for mappings R from X to a complete quasi-partial metric space Y and $\psi : R(X) \to \mathbb{R}^+$, there exist $x \in X$ and $c > 0$ such that the inequality $\max\{qp^*(y, Ty), cqp^*(Ry, RTy)\} \leqslant \psi(Ry) - \psi(RTy)$ for all $y \in O(x, T)$, has a fixed point if and only if $G(x) = qp^*(x, Tx)$ is T-orbitally lower semi-continuous at x. Very recently, Gupta and Gautam (see [6]) have focused on this subject and have generalized some fixed point theorems from the class of quasi-partial metric spaces to the class of quasi-partial b-metric spaces.

In this paper, inspired by [9], we introduce generalized quasi-partial metric spaces (GQPMS) and generalize some fixed point theorems on quasi-partial metric spaces to generalized quasi-partial metric spaces. In the meantime, some examples are provided to verify the effectiveness of the results.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}^+$ denotes the set of all nonnegative real numbers.
We begin with the following definition as a recall from[7, 18].

Definition 2.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies the following conditions:
(d_1) $d(x, x) = 0$ for all $x \in X$;
(d_2) $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \in X$;
(d_3) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_4) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.
If d satisfies conditions (d_1), (d_2) and (d_4), then d is called a *quasi-metric* on X. If d satisfies conditions (d_2), (d_3) and (d_4), then d is called a *dislocated metric* on X. If it satisfies conditions (d_2) and (d_4), it is called a *dislocated quasi-metric*. If d satisfies conditions (d_1)-(d_4), then d is called a (standard) *metric* on X.

The concept of a quasi-partial metric space was introduced by Karapinar et al.

Definition 2.2 ([9])**.** A *quasi-partial metric* on a nonempty set X is a function $qp : X \times X \to \mathbb{R}^+$, satisfying the following conditions:
(QP_1) If $qp(x, x) = qp(x, y) = qp(x, y)$, then $x = y$;
(QP_2) $qp(x, x) \leqslant qp(x, y)$;
(QP_3) $qp(x, x) \leqslant qp(y, x)$;
(QP_4) $qp(x, y) + qp(z, z) \leqslant qp(x, z) + qp(z, y)$ for all $x, y, z \in X$.

A *quasi-partial metric space* is a pair (X, qp) such that X is a nonempty set and qp is a quasi-partial metric on X.

For each quasi-partial metric $qp : X \times X \to \mathbb{R}^+$, the function $d_q : X \times X \to \mathbb{R}^+$ defined by

$$d_q(x, y) = qp(x, y) + qp(y, x) - qp(x, x) - qp(y, y)$$

is a (standard) metric on X.
The next Lemma has shown the relationship between quasi-partial metric and standard metric.

Lemma 2.3 ([9])**.** *Let (X, qp) be a quasi-partial metric space and (X, d_q) be the corresponding metric space. Then (X, qp) is complete if and only if (X, d_q) is complete.*

For each quasi-partial metric $qp : X \times X \to \mathbb{R}^+$, the function $d_{qp} : X \times X \to \mathbb{R}^+$ defined by

$$d_{qp}(x, y) = qp(x, y) - qp(x, x)$$

is a dislocated quasi-metric.
We introduce the concept of *generalized dislocated quasi-metric*, a generalization for dislocated quasi-metric, which is shown as follows:

Definition 2.4. Let X be a nonempty set. Suppose that the mapping $gdq : X \times X \to [0, \infty)$ satisfies the following conditions:

(gdq_1) $gd_q(x, y) = gd_q(y, x) = 0$ implies $x = y$ for all $x, y \in X$;

(gdq_2) If $(x, y) \in X \times X$, $\{x_n\}_{n=0}^{\infty} \in C(gd_q, X, x)$, then

$$gd_q(x, y) \leqslant \limsup_{n\to\infty} gd_q(x_n, y),$$

where

$$C(gd_q, X, x) = \Big\{\{x_n\}_{n=0}^{\infty} \subset X : \lim_{n\to\infty} gd_q(x, x_n) = \lim_{n\to\infty} gd_q(x_n, x) = 0\Big\}.$$

Then gd_q is called a *generalized dislocated quasi-metric* (or simply gd_q-*metric*) on X.

The pair (X, gd_q) is then called a *generalized dislocated quasi-metric space.*

Remark 2.5. Obviously, if the set $C(gd_q, X, x)$ is empty for every $x \in X$, then (X, gd_q) is a generalized dislocated quasi-metric space if and only if (gdq_1) is satisfied.

Proposition 2.6. *Any dislocated quasi-metric on X is a generalized dislocated quasi-metric on X.*

Proof. Let d be a dislocated quasi-metric on X. We have just to proof that d satisfies the property (gdq_2). Let $x \in X$ and $\{x_n\}_{n=0}^{\infty} \in C(gd_q, X, x)$. For every $y \in X$, by the property(d_2), we have

$$d(x, y) \leqslant d(x, x_n) + d(x_n, y)$$

for every natural number n. Thus we have $d(x, y) \leqslant \limsup_{n\to\infty} d(x_n, y)$. The property ($\mathrm{gdq}_2$) is then satisfied. □

Definition 2.7. Let (X, gd_q) be a generalized dislocated quasi-metric. Then

(i) A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges to $x \in X$ if and only if $\lim_{n\to\infty} gd_q(x, x_n) = \lim_{n\to\infty} gd_q(x_n, x) = 0$.

(ii) A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is called a *Cauchy sequence* if and only if $\lim_{n,m\to\infty} gd_q(x_m, x_n)$ and $\lim_{n,m\to\infty} gd_q(x_n, x_m)$ exist (and are finite).

(iii) The generalized dislocated quasi-metric space (X, gd_q) is said to be *complete* if every Cauchy sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges with respect to $\mathscr{T}_{gd_q}$ to a point $x \in X$ such that $\lim_{n\to\infty} gd_q(x, x_n) = \lim_{n\to\infty} gd_q(x_n, x) = 0$.

We denote simply gd_q-converges to x by $x_n \xrightarrow{gd_q} x$.

3. Generalized quasi-partial metric spaces

We introduce the concept of generalized quasi-partial metric space and give some properties on such spaces in this section.

Let X be a nonempty set and $qp^* : X \times X \to \mathbb{R}^+$ be a given mapping. For every $x \in X$, let us define the set

$$\mathscr{C}(qp^*, X, x) = \Big\{\{x_n\}_{n=0}^{\infty} \subset X : \lim_{n\to\infty} qp^*(x, x_n) = \lim_{n\to\infty} qp^*(x_n, x) = qp^*(x, x) \text{ and } \lim_{n\to\infty} qp^*(x_n, x_n) \text{ exists}\Big\}.$$

Definition 3.1. A *generalized quasi-partial metric* on a nonempty set X is a function $qp^* : X \times X \to \mathbb{R}^+$, satisfying the following conditions:

(GQP_1) If $qp^*(x, x) = qp^*(x, y) = qp^*(y, y)$, then $x = y$.

(GQP_2) $qp^*(x, x) \leqslant qp^*(x, y)$.

(GQP_3) $qp^*(x, x) \leqslant qp^*(y, x)$.

(GQP_4) If $(x, y) \in X \times X$, $\{x_n\}_{n=0}^{\infty} \in \mathscr{C}(qp^*, X, x)$, then

$$qp^*(x, y) + \lim_{n\to\infty} qp^*(x_n, x_n) \leqslant qp^*(x, x) + \limsup_{n\to\infty} qp^*(x_n, y). \tag{3.1}$$

A *generalized quasi-partial metric space* (GQPMS) is a pair (X, qp^*) such that X is a nonempty set and qp^* is a generalized quasi-partial metric on X.

Remark 3.2. At least, there exists a constant sequence $\{x_n = x\}_{n=0}^{\infty} \in \mathscr{C}(qp^*, X, x)$ such that $\mathscr{C}(qp^*, X, x)$ is nonempty for every $x \in X$. In this case, the Inequality (3.1) reduces to $qp^*(x,y) \leqslant \limsup_{n\to\infty} qp^*(x_n, y)$ due to $\lim_{n\to\infty} qp^*(x_n, x_n) = qp^*(x,x)$.

Proposition 3.3. *Any quasi-partial metric on X is a generalized quasi-partial metric on X.*

Proof. Let qp be a quasi-partial metric on X. We should just proof that QP_4 satisfies the property (GQP_4). Let $x \in X$ and $\{x_n\}_{n=0}^{\infty} \in \mathscr{C}(qp, X, x)$. For every $y \in X$, by the property(QP_4), we have

$$qp(x,y) + qp(x_n, x_n) \leqslant qp(x, x_n) + qp(x_n, y)$$

for every natural number n. Thus we have

$$\begin{aligned} qp(x,y) + \lim_{n\to\infty} qp(x_n, x_n) &\leqslant \limsup_{n\to\infty} qp(x,x) + \limsup_{n\to\infty} qp(x_n, y) \\ &= qp(x,x) + \limsup_{n\to\infty} qp(x_n, y). \end{aligned}$$

The property (GQP_4) is then satisfied. □

Next we provide an example of generalized quasi-partial metric space as follow:

Example 3.4. Let $X = \{n - \frac{1}{n} : n \in \mathbb{N}\}$ and define

$$qp^*(x,y) = (x-y)^2 + x$$

for any $(x,y) \in X \times X$.

If $qp^*(x,x) = qp^*(x,y) = qp^*(y,y)$, that is, $x = (x-y)^2 + x = y$, then it is obvious that GQP_1 holds for any $(x,y) \in X \times X$. In addition, it is easy to calculate

$$qp^*(x,x) = x \leqslant (x-y)^2 + x = qp^*(x,y).$$

Let $x = n - \frac{1}{n}, y = m - \frac{1}{m}$ for any $m, n \in \mathbb{N}$, then

$$qp^*(x,x) = n - \frac{1}{n}, \quad qp^*(y,x) = (m-n)^2(1 + \frac{1}{mn})^2 + 1 - \frac{1}{m}.$$

Calculating

$$qp^*(y,x) - qp^*(x,x) = [(m-n)(1 + \frac{1}{mn}) + \frac{1}{2}]^2 - \frac{1}{4} \geqslant 0,$$

thus, $qp^*(x,x) \leqslant qp^*(y,x)$ is true, hence GQP_2 and GQP_3 hold for any $(x,y) \in X \times X$. Moreover, for any $x = n - \frac{1}{n} (n \in \mathbb{N})$, we do not find any sequence belonging to $\mathscr{C}(qp^*, X, x)$ except for a constant sequence $\{\eta_i = x\}_{i=0}^{\infty}$. Additionally,

$$\begin{aligned} qp^*(x,y) &= (x-y)^2 + x \\ &\leqslant x + \limsup_{i\to\infty} (\eta_i - y)^2 \\ &= x + \limsup_{i\to\infty} [(\eta_i - y)^2 + \eta_i] - \limsup_{i\to\infty} \eta_i \\ &= qp^*(x,x) + \limsup_{i\to\infty} qp^*(\eta_i, y) - \limsup_{i\to\infty} qp^*(\eta_i, \eta_i) \end{aligned}$$

is true for any $(x,y) \in X \times X$, that is, (GQP_4) holds, hence (X, qp^*) is a generalized quasi-partial metric space, but since

$$qp^*(5 - \frac{1}{5}, 1 - \frac{1}{1}) + qp^*(2 - \frac{1}{2}, 2 - \frac{1}{2}) = \frac{1467}{50} > \frac{972}{50} = qp^*(5 - \frac{1}{5}, 2 - \frac{1}{2}) + qp^*(2 - \frac{1}{2}, 1 - \frac{1}{1}),$$

(QP_4)(triangle inequality) is not true, thus (X, qp^*) is not a quasi-partial metric space.

Remark 3.5. Proposition 3.3 and Example 3.4 indicate that quasi-partial metric spaces are generalized quasi-partial metric spaces, but conversely this is not true.

Denote $\mathscr{T}_{qp^*}$ as the topology induced by the generalized quasi-partial metric qp^*. Next we define *convergent sequence*, *Cauchy sequence*, *completeness of space* and *continuous mapping* in generalized quasi-partial metric spaces.

Definition 3.6. Let (X, qp^*) be a generalized quasi-partial metric. Then

(i) A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges to $x \in X$ if and only if $qp^*(x,x) = \lim_{n\to\infty} qp^*(x, x_n) = \lim_{n\to\infty} qp^*(x_n, x)$.

(ii) A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is called a *Cauchy sequence* if and only if $\lim_{n,m\to\infty} qp^*(x_m, x_n)$ and $\lim_{n,m\to\infty} qp^*(x_n, x_m)$ exist (and are finite).

(iii) The generalized quasi-partial metric space (X, qp^*) is said to be *complete* if every Cauchy sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges with respect to $\mathscr{T}_{qp^*}$ to a point $x \in X$ such that

$$qp^*(x,x) = \lim_{m,n\to\infty} qp^*(x_m, x_n) = \lim_{m,n\to\infty} qp^*(x_n, x_m).$$

(iv) A mapping $f : X \to X$ is said to be *continuous* at $x \in X$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon)$.

The relationship between generalized quasi-partial metric and generalized dislocated quasi-metric will be shown in next proposition.

Proposition 3.7. *For each generalized quasi-partial metric $qp^* : X \times X \to \mathbb{R}^+$, the function $gd_q : X \times X \to \mathbb{R}^+$ defined by*

$$gd_q(x,y) = qp^*(x,y) - qp^*(x,x) \tag{3.2}$$

is a generalized dislocated quasi-metric.

Proof. If $gd_q(x,y) = gd_q(y,x) = 0$, then $qp^*(x,y) = qp^*(x,x) = qp^*(y,y)$, it follows that $x = y$.

If $\{x_n\}_{n=0}^{\infty} \in \mathscr{C}(qp^*, X, x)$, then

$$qp^*(x,y) + \lim_{n\to\infty} qp^*(x_n, x_n) \leqslant \limsup_{n\to\infty} qp^*(x_n, y) \tag{3.3}$$

for all $(x,y) \in X \times X$, on the other hand, following Eq. (3.2)

$$gd_q(x_n, y) = qp^*(x_n, y) - qp^*(x_n, x_n)$$

is true for every $n \in \mathbb{N}$. Thus

$$\begin{aligned} \limsup_{n\to\infty} gd_q(x_n, y) &= \limsup_{n\to\infty} qp^*(x_n, y) - \limsup_{n\to\infty} qp^*(x_n, x_n) \\ &= \limsup_{n\to\infty} qp^*(x_n, y) - \lim_{n\to\infty} qp^*(x_n, x_n), \end{aligned} \tag{3.4}$$

subsequently, by Eq. (3.2), Inequality (3.3) and Eq. (3.4)

$$\begin{aligned} gd_q(x,y) &= qp^*(x,y) - qp^*(x,x) \\ &\leqslant \limsup_{n\to\infty} qp^*(x_n, y) - \lim_{n\to\infty} qp^*(x_n, x_n) \\ &= \limsup_{n\to\infty} gd_q(x_n, y). \end{aligned}$$

□

We denote simply qp^*-converges to x by $x_n \xrightarrow{qp^*} x$. We state the uniqueness of the limit of a sequence in a generalized quasi-partial metric space.

Proposition 3.8. *Let (X, qp^*) be a complete generalized quasi-partial metric space, $\{x_n\}_{n=0}^{\infty}$ be a sequence in X. If $\{x_n\}_{n=0}^{\infty} \in \mathscr{C}(qp^*, X, x) \cap \mathscr{C}(qp^*, X, y)$ and $qp^*(x, y) = qp^*(y, x)$ for $x, y \in X$, then $x = y$. In other words, if $x_n \xrightarrow{qp^*} x$, $x_n \xrightarrow{qp^*} y$ then $x = y$.*

Proof. Assume that $x_n \xrightarrow{qp^*} x$ and $x_n \xrightarrow{qp^*} y$ in (X, qp^*), then

$$qp^*(x, x) = \lim_{n\to\infty} qp^*(x_n, x) = \lim_{n\to\infty} qp^*(x, x_n)$$

and

$$qp^*(y, y) = \lim_{n\to\infty} qp^*(x_n, y) = \lim_{n\to\infty} qp^*(y, x_n).$$

Since qp^* is complete, it is obvious that

$$\begin{aligned} qp^*(x, x) &= \lim_{n\to\infty} qp^*(x_n, x) = \lim_{n\to\infty} qp^*(x, x_n), \\ &= \lim_{m,n\to\infty} qp^*(x_m, x_n) = \lim_{m,n\to\infty} qp^*(x_n, x_m) \end{aligned}$$

and

$$\begin{aligned} qp^*(y, y) &= \lim_{n\to\infty} qp^*(x_n, y) = \lim_{n\to\infty} qp^*(y, x_n), \\ &= \lim_{m,n\to\infty} qp^*(x_m, x_n) = \lim_{m,n\to\infty} qp^*(x_n, x_m). \end{aligned}$$

On the other hand, using GQP_4, we have

$$\begin{aligned} qp^*(x, y) &\leqslant \limsup_{n\to\infty} qp^*(x_n, y) + qp^*(x, x) - \lim_{n\to\infty} qp^*(x_n, x_n), \\ &= \lim_{n\to\infty} qp^*(x_n, y) \\ &= qp^*(y, y) \end{aligned}$$

and

$$\begin{aligned} qp^*(y, x) &\leqslant \limsup_{n\to\infty} qp^*(x_n, x) + qp^*(y, y) - \lim_{n\to\infty} qp^*(x_n, x_n), \\ &= \lim_{n\to\infty} qp^*(x_n, x) \\ &= qp^*(x, x). \end{aligned}$$

In combination with GQP_2 and GQP_3,

$$qp^*(y, y) = qp^*(x, y), qp^*(x, x) = qp^*(y, x).$$

Subsequently, from the condition $qp^*(x, y) = qp^*(y, x)$, we get $qp^*(x, x) = qp^*(x, y) = qp^*(y, y)$ which implies from the property (GQP_1) that $x = y$. □

Definition 3.9. We called the generalized quasi-partial metric space satisfying the condition $qp^*(x, y) = qp^*(y, x)$ a *generalized partial metric space.*

Lemma 3.10. *Let (X, qp^*) be a generalized quasi-partial metric space and (X, gd_q) be the corresponding generalized dislocated quasi-metric space. Then (X, gd_q) is complete if (X, qp^*) is complete.*

Proof. Since (X, qp^*) is complete, every Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ in X converges with respect to $\mathscr{T}_{qp^*}$ to a point $x \in X$ such that

$$qp^*(x, x) = \lim_{m,n\to\infty} qp^*(x_n, x_m) = \lim_{m,n\to\infty} qp^*(x_m, x_n). \tag{3.5}$$

Consider a Cauchy sequence $\{x_n\}_{i=0}^{\infty}$ in (X, gd_q). We will show that $\{x_n\}_{n=0}^{\infty}$ is Cauchy in (X, qp^*). Since $\{x_n\}_{n=0}^{\infty}$ is Cauchy in (X, gd_q), $\lim_{n\to\infty} gd_q(x_n, x_m)$ exists and is finite. On the other hand,

$$gd_q(x_n, x_m) = qp^*(x_n, x_m)qp^*(x_n, x_n)$$

and

$$gd_q(x_m, x_n) = qp^*(x_m, x_n)qp^*(x_m, x_m)$$

hold for any $m, n \in \mathbb{N}$, hence $\lim_{m,n\to\infty} qp^*(x_n, x_m)$ and $\lim_{m,n\to\infty} qp^*(x_m, x_n)$ exist and are finite. Therefore, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, qp^*). Because (X, qp^*) is complete, therefore the sequence $\{x_n\}_{i=0}^{\infty}$ converges with respect to $\mathscr{T}_{qp^*}$ to a point $x \in X$ such that (3.5) holds. In addition,

$$\lim_{n\to\infty} gd_q(x, x_n) = \lim_{n\to\infty} qp^*(x, x_n)qp^*(x, x) = 0.$$

Similarly,

$$\lim_{n\to\infty} gd_q(x_n, x) = \lim_{n\to\infty} qp^*(x_n, x) \lim_{n\to\infty} qp^*(x_n, x_n) = 0.$$

Hence

$$\lim_{n\to\infty} gd_q(x, x_n) = \lim_{n\to\infty} gd_q(x_n, x) = 0.$$

□

Lemma 3.11. *Let (X, qp^*) be a generalized quasi-partial metric space. Then the following hold.*
(i) *If $qp^*(x, y) = 0$, then $x = y$.*
(ii) *If $x \neq y$, then $qp^*(x, y) > 0$ and $qp^*(y, x) > 0$.*

The proof is similar to the case of quasi-partial metric space [9], thus we omit it.

4. Main results

In this paper, some fixed point results (see [9]) on quasi-partial metric spaces are extended to generalized quasi-partial metric spaces.

Definition 4.1 ([9])**.** Let $T : X \to X$ be a self-mapping on X, $O(x, T) = \{x, Tx, T^2x, \ldots\}$ is called a *orbit of* x. A mapping $G : X \to \mathbb{R}^+$ is T-orbitally lower semi-continuous at x if $\{x_n\}_{n=0}^{\infty}$ is a sequence in $O(x, T)$ and $\lim_{n\to\infty} x_n = z$ implies $G(z) \leqslant \liminf_{n\to\infty} G(x_n)$.

The following two lemmas are very useful in the proof of the main theorems.

Lemma 4.2. *Let (X, qp^*) be a generalized quasi-partial metric space. Assume that there exist $x \in X$ and $x_n \in O(x, T)$ such that $\limsup_{i\to\infty} qp^*(\eta_i, y) < \infty$ holds for all $y \in O(x, T)$ and $\{\eta_i\}_{i=0}^{\infty} \in \mathscr{C}(qp^*, X, x_k)$, then*

$$qp^*(x_n, y) < \infty$$

for all $y \in O(x, T)$.

Proof. If $\{\eta_i\}_{i=0}^{\infty} \in \mathscr{C}(qp^*, X, x_n)$, then using GQP_4,

$$\begin{aligned} qp^*(x_n, y) &\leqslant qp^*(x_n, x_n) + \limsup_{i\to\infty} qp^*(\eta_i, y) - \limsup_{i\to\infty} qp^*(\eta_i, \eta_i) \\ &= \limsup_{i\to\infty} qp^*(\eta_i, x_n) + \limsup_{i\to\infty} qp^*(\eta_i, y) - \limsup_{i\to\infty} qp^*(\eta_i, \eta_i) \\ &\leqslant \limsup_{i\to\infty} qp^*(\eta_i, x_n) + \limsup_{i\to\infty} qp^*(\eta_i, y), \end{aligned}$$

subsequently, $qp^*(x_n, y) < \infty$ since $\limsup_{i\to\infty} qp^*(\eta_i, x_n) < \infty$ and $\limsup_{i\to\infty} qp^*(\eta_i, y) < \infty$. □

Lemma 4.3. *Let (X, qp^*) be a generalized quasi-partial metric spaces. Assume that there exist $x \in X$ and $x_n \in O(x,T)$ such that $\limsup_{i\to\infty} qp^*(\eta_i^k, y) < \infty$ for all $y \in O(x,T)$, $\{\eta_i^k\}_{i=0}^\infty \in \mathscr{C}(qp^*, X, x_k)$ and $n \leqslant k < m$, then there exists $0 < C_n < \infty$ such that*

$$qp^*(x_n, x_m) \leqslant C_n \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1})$$

holds for all $m > n (m, n \in \mathbb{N})$.

Proof. Set $x_n, x_m \in O(x,T)$, $m > n$ $(m, n \in \mathbb{N})$. Considering the two cases:

Case (i) $qp^*(x_n, x_m) = 0$. In this case, obviously,

$$0 = qp^*(x_n, x_m) \leqslant C_n \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1})$$

for arbitrary $C_n > 0$.

Case (ii) $qp^*(x_n, x_m) > 0$. In this case, let us prove that

$$0 < qp^*(x_n, x_m) < \infty \text{and} 0 < \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}) < \infty.$$

Note that when $x_n, x_m \in O(x,T)$ and $\{\eta_i^n\}_{i=0}^\infty \in \mathscr{C}(qp^*, X, x_n)$, $\limsup_{i\to\infty} qp^*(\eta_i^n, x_m) < \infty$ holds, therefore, we get from Lemma 4.2,

$$qp^*(x_n, x_m) < \infty$$

for all $m > n$.

Assume that $\sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}) = 0$, then

$$qp^*(x_k, x_{k+1}) = 0.$$

From Lemma 3.11, we derive $x_k = x_{k+1}$ for $n \leqslant k < m$, that is,

$$x_n = x_{n+1} = \ldots = x_m,$$

hence

$$qp^*(x_n, x_m) = 0,$$

which contradicts $qp^*(x_n, x_m) > 0$.

Thus

$$\sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}) > 0.$$

On the other hand, because $\limsup_{i\to\infty} qp^*(\eta_i^k, y) < \infty$ for all $y \in O(x,T)$, $\{\eta_i^k\}_{i=0}^\infty \in \mathscr{C}(qp^*, X, x_k)$ and $k \in \{n, n+1, \ldots, m-1\}$, in combination with Lemma 4.2, it can be deduced

$$qp^*(x_k, x_{k+1}) < \infty$$

for every $k \in \{n, n+1, \ldots, m-1\}$.

If $m < \infty$, then

$$0 < \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}) < \infty.$$

Therefore there exists $0 < qp^*(x_n, x_m) \Big/ \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}) \leqslant C_n < \infty$ such that

$$qp^*(x_n, x_m) \leqslant C_n \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1})$$

holds.

If $m = \infty$, then

$$\sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}) = \sum_{k=n}^{\infty} qp^*(x_k, x_{k+1}) < \infty$$

or

$$\sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}) = \sum_{k=n}^{\infty} qp^*(x_k, x_{k+1}) = \infty.$$

Considering the first case, we take $qp^*(x_n, x_m) \Big/ \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}) \leqslant C_n < \infty$. As for the second case, $qp^*(x_n, x_m) \leqslant C_n \sum_{k=n}^{\infty} qp^*(x_k, x_{k+1})$ holds for arbitrary C_n $(0 < C_n < \infty)$. □

Theorem 4.4. *Let (X, qp^*) be generalized quasi-partial metric spaces and $T : X \to X$ be a self-mapping, then the following hold*

(i) There exists a mapping $\psi : X \to \mathbb{R}^+$ such that

$$qp^*(x, Tx) \leqslant \psi(x) - \psi(Tx) \tag{4.1}$$

holds for all $x \in X$ if and only if

$$\sum_{n=0}^{\infty} qp^*(T^n x, T^{n+1} x)$$

converges for all $x \in X$.

(ii) There exists a mapping $\psi : X \to \mathbb{R}^+$ such that

$$qp^*(y, Ty) \leqslant \psi(y) - \psi(Ty)$$

holds for all $y \in O(x, T)$ if and only if

$$\sum_{n=0}^{\infty} qp^*(T^n y, T^{n+1} y)$$

converges for all $y \in O(x, T)$.

Proof. Proof of (i). First, let us prove the necessity of (i). Take $x \in X$ and let

$$qp^*(x, Tx) \leqslant \psi(x) - \psi(Tx).$$

Denote the sequence $\{x_n\}_{n=0}^{\infty}$ in the following way:

$$x_0 = x, x_{n+1} = Tx_n = T^{n+1} x \quad \text{for all} \quad n \in \{0, 1, 2, \ldots, \},$$

thus

$$\sum_{k=0}^{n} qp^*(x_k, x_{k+1}) = \sum_{k=0}^{n} qp^*(T^k x_0, T^{k+1} x_0).$$

Set

$$S_n = \sum_{k=0}^{n} qp^*(x_k, x_{k+1}).$$

By (4.1), we obtain

$$\begin{aligned} S_n &\leqslant \sum_{k=0}^{n}[\psi(T^k x_0) - \psi(T^{k+1}x_0)] \\ &= \psi(x_0) - \psi(T^{k+1}x_0) \\ &\leqslant \psi(x_0) = \psi(x), \end{aligned}$$

which implies $\{S_n\}$ is bounded. On the other hand, $\{S_n\}$ is non-decreasing by definition and hence it is convergent.

Next we prove the sufficiency of (i). Define

$$\psi(x) = \sum_{k=0}^{\infty}[\psi(T^k x) - \psi(T^{k+1}x)], \quad S_n(x) = \sum_{k=0}^{n}[\psi(T^k x) - \psi(T^{k+1}x)].$$

Calculating

$$\begin{aligned} S_n(x) - S_n(Tx) &= \sum_{k=0}^{n} qp^*(T^k x, T^{k+1}x) - \sum_{k=0}^{n} qp^*(T^{k+1}x, T^{k+2}x) \\ &= qp^*(x, Tx) - qp^*(x_{k+1}, x_{k+2}). \end{aligned} \tag{4.2}$$

Moreover, since $\sum\limits_{n=0}^{\infty} qp^*(T^n x, T^{n+1}x)$ converges for all $x \in X$, then

$$\lim_{n\to\infty} qp^*(T^n x, T^{n+1}x) = 0 \quad \text{and} \quad \lim_{n\to\infty} S_n(x) = \psi(x).$$

Letting $n \to \infty$ in Eq. (4.2), we get

$$\psi(x) - \psi(Tx) = qp^*(x, Tx).$$

Proof of (ii). It can easily be proved using part (i). □

We present an example of a generalized quasi-partial metric instead of quasi-partial metric to illustrate Theorem 4.4.

Example 4.5. Let $X = \left[0, \frac{\pi}{6}\right]$ and define

$$qp_b^*(x, y) = \tan|x - y| + x$$

for any $(x, y) \in X \times X$.
If $qp^*(x, x) = qp^*(x, y) = qp^*(y, y)$, that is, $x = \tan|x - y| + x = y$, then it is obvious that (GQP_1) holds for any $(x, y) \in X \times X$. In addition, since

$$qp^*(x, x) = x \leqslant \tan|x - y| + x = qp^*(x, y)$$

and

$$\begin{aligned} qp^*(x, x) = x &= |x - y + y| \\ &\leqslant |y - x| + y \\ &\leqslant \tan|y - x| + y \\ &= qp^*(y, x) \end{aligned}$$

are true, then (GQP_2) and (GQP_3) hold for any $(x, y) \in X \times X$. Moreover, we observe that for every

$x, y \in X$, if sequence $\{x_n\}_{n=0}^{\infty} \in \mathscr{C}(qp^*, X, x)$, then

$$\begin{aligned} qp^*(x,y) &= \tan|x-y| + x \\ &\leqslant x + \limsup_{n\to\infty} \tan|x_n - y| \\ &= x + \limsup_{n\to\infty}(\tan|x_n-y| + x_n) - \limsup_{n\to\infty} x_n \\ &= qp^*(x,x) + \limsup_{n\to\infty} qp^*(x_n,y) - \limsup_{n\to\infty} qp^*(x_n,x_n), \end{aligned}$$

that is, (GQP_4) holds, hence (X, qp^*) is a generalized quasi-partial metric space, but since

$$qp^*(\frac{\pi}{6},0) + qp^*(\frac{\pi}{18},\frac{\pi}{18}) = \tan\frac{\pi}{6} + \frac{2\pi}{9} > \tan\frac{\pi}{9} + \tan\frac{\pi}{18} + \frac{2\pi}{9} = qp^*(\frac{\pi}{6},\frac{\pi}{18}) + qp^*(\frac{\pi}{18},0),$$

(QP_4) (triangle inequality) does not hold, thus (X, qp^*) is not a quasi-partial metric space.

Define $T : X \to X$ as $Tx = \dfrac{x}{2}$ for all $x \in X$, we can verify that the series $\sum_{n=0}^{\infty} qp^*(T^n x, T^{n+1}x)$ is convergent. In fact

$$\begin{aligned} \sum_{n=0}^{\infty} qp^*(T^n x, T^{n+1}x) &= \sum_{n=0}^{\infty} qp^*(\frac{x}{2^n}, \frac{x}{2^{n+1}}) \\ &= \sum_{n=0}^{\infty} \tan\left|\frac{x}{2^n} - \frac{x}{2^{n+1}}\right| + \left|\frac{x}{2^n}\right| \\ &= \sum_{n=0}^{\infty} \tan\frac{x}{2^{n+1}} + \frac{x}{2^n}. \end{aligned}$$

Because $0 \leqslant \dfrac{x}{2^{n+1}} < \dfrac{\pi}{6}(n \in \mathbb{N} \cup \{0\})$ and it is not difficult to verify that $\tan x \leqslant \dfrac{4}{3}x$ when $x \in \left[0, \dfrac{\pi}{6}\right]$, therefore

$$\begin{aligned} \sum_{n=0}^{\infty} qp^*(T^n x, T^{n+1}x) &= \sum_{n=0}^{\infty} \tan\frac{x}{2^{n+1}} + \frac{x}{2^n} \\ &\leqslant \sum_{n=0}^{\infty} \frac{4}{3}\left(\frac{x}{2^{n+1}} + \frac{x}{2^n}\right) \\ &= \sum_{n=0}^{\infty} \frac{x}{2^{n-1}} \\ &= 4x. \end{aligned}$$

In addition,

$$\begin{aligned} qp^*(x, Tx) &= \tan\frac{x}{2} + x \\ &\leqslant \frac{2x}{2} + x \\ &= 2x. \end{aligned}$$

Define $\psi(x) = 4x$, then $\psi(x) - \psi(Tx) = 2x$, the conditions of Theorem 4.4 are satisfied.

The statement on the conditions for the existence of fixed points of operators in the setting generalized quasi-partial metric spaces will be given in subsequent theorem.

Theorem 4.6. *Let (X, qp^*) and (Y, qp^*) be complete generalized quasi-partial metric spaces. Given mappings $R : X \to Y$, $T : X \to X$ and $\psi : R(X) \to \mathbb{R}^+$. If there exist $x \in X$ and $c > 0$ such that*

$$\max\{qp^*(y, Ty), cqp^*(Ry, RTy)\} \leqslant \psi(Ry) - \psi(RTy) \tag{4.3}$$

holds for all $y \in O(x, T)$ and moreover, assume that $\limsup_{i\to\infty} qp^*(\eta_i^k, z) < \infty$ for all $z \in O(x, T)$ and every $k \in \mathbb{N}$ when $\{\eta_i^k\}_{i=0}^{\infty} \in \mathscr{C}(qp^*, X, x_k)$, then *(i)* $\lim\limits_{n\to\infty} T^n x = \zeta$ *exists.*

(ii) $T(\zeta) = \zeta$ *iff* $G(x) = qp^*(x, Tx)$ *is* T*-orbitally lower semi-continuous at* x*.*

(iii) There exists $C > 0$ *such that* $qp^*(x, T^n x) \leqslant C\psi(Rx)$*.*

(iv) If $y \to qp^*(\zeta, y)$ *is continuous for* $\zeta \in O(x, T)$*, then there exists* $C > 0$ *such that* $qp^*(T^n x, \zeta) \leqslant C\psi(R^n x)$ *and* $q(x, \zeta) \leqslant C\psi(Rx)$*.*

Proof. Proof of (i).
Denote $x_{n+1} = Tx_n = T^{n+1}x, x_0 = x$. For every fixed n and all $m > n$ ($m, n \in \{0, 1, \ldots\}$), according to Lemma 4.3, there exists $0 < C_n < \infty$ such that

$$qp^*(x_n, x_m) \leqslant C_n \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}). \tag{4.4}$$

Taking $C = \max\{C_0, C_1, C_2, \ldots\}$, then

$$qp^*(x_n, x_m) \leqslant C \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}). \tag{4.5}$$

Next, we will prove that the sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.
Following [9], set $S_n(x) = C \sum\limits_{k=0}^{n} qp^*(x_k, x_{k+1})$. Using Inequality (4.3),

$$\begin{aligned} qp^*(T^k x, T^{k+1}x) &\leqslant \max\{qp^*(T^k x, T^{k+1}x), cqp^*(RT^k x, RT^{k+1}x)\} \\ &\leqslant \psi(RT^k x) - \psi(RT^{k+1}x), \end{aligned}$$

thus

$$\begin{aligned} S_n(x) &\leqslant C \sum_{k=0}^{n} [\psi(RT^k x) - \psi(RT^{k+1}x)] \\ &= C(\psi(Rx) - \psi(RT^{k+1}x)) \\ &\leqslant C\psi(Rx), \end{aligned}$$

consequently, $\sum\limits_{k=0}^{\infty} qp^*(x_k, x_{k+1})$ is convergent. Taking the limit as $n, m \to \infty$ on the two sides of Inequality (4.5), we obtain

$$\lim_{m,n\to\infty} qp^*(x_n, x_m) = \lim_{m,n\to\infty} (S_{m-1} - S_n) = 0.$$

Using similar arguments, one can show that

$$\lim_{m,n\to\infty} qp^*(x_m, x_n) = 0,$$

that is, the sequence $\{x_n\}_{n=0}^{\infty}$ is Cauchy in (X, qp^*). Since (X, qp^*) is complete, (X, gd_q) is also complete by Lemma 3.10 and hence $\lim\limits_{n\to\infty} gd_q(T^n x, \zeta) = \lim\limits_{n\to\infty} gd_q(\zeta, T^n x) = 0$, that is, $\lim\limits_{n\to\infty} T^n x = \zeta$. Moreover, we get

$$\lim_{n\to\infty} qp^*(T^n x, T^{n+1}x) = \lim_{n\to\infty} qp^*(\zeta, \zeta) = 0.$$

Proof of necessity of (ii).
Suppose that $T\zeta = \zeta$ and $\{x_n\}_{n\to\infty}^{\infty} \in O(x, T)$ with $x_n \xrightarrow{qp^*} x$. Using Lemma 3.10,

$$qp^*(x, x) = \lim_{m,n\to\infty} qp^*(x_n, x_m) = \lim_{m,n\to\infty} qp^*(x_m, x_n)$$

$$\Rightarrow \lim_{n\to\infty} gd_q(T^n x,\zeta) = \lim_{n\to\infty} gd_q(\zeta, T^n x) = 0.$$

Then $G(\zeta) = qp^*(\zeta, T\zeta) = qp^*(\zeta,\zeta) \leqslant \liminf_{n\to\infty} qp^*(x_n, Tx_n) = \liminf_{n\to\infty} G(x_n)$, that is, $G(x) = qp^*(x, Tx)$ is T-orbitally lower semi-continuous at x.

Proof of sufficiency of (ii).

Assume that $x_n \xrightarrow{qp^*} x$ and G is T-orbitally lower semi-continuous at x. It can be derived that

$$\begin{aligned} 0 \leqslant qp^*(\zeta, T\zeta) &= G(\zeta) \leqslant \liminf_{n\to\infty} G(x_n) \\ &= \liminf_{n\to\infty} qp^*(T^n x, T^{n+1}x) \\ &= \liminf_{n\to\infty} qp^*(x_n, x_{n+1}) \\ &= qp^*(\zeta,\zeta) = 0. \end{aligned}$$

It follows from Lemma 3.11 that $T\zeta = \zeta$.

Proof of (iii).

The same as the proof of (i), there exists $C > 0$ such that

$$qp^*(x, x_n) \leqslant C \sum_{k=0}^{n-1} qp^*(x_k, x_{k+1}). \tag{4.6}$$

Using Eq. (4.3),

$$\begin{aligned} qp^*(T^k x, T^{k+1}x) &\leqslant \max\{qp^*(T^k x, T^{k+1}x), cqp^*(RT^k x, RT^{k+1}x)\} \\ &\leqslant \psi(RT^k x) - \psi(RT^{k+1}x), \end{aligned}$$

thus

$$\begin{aligned} qp^*(x, x_n) &\leqslant C \sum_{k=0}^{n-1} qp^*(x_k, x_{k+1}), \\ &\leqslant C \sum_{k=0}^{n-1} [\psi(RT^k x) - \psi(RT^n x)] \\ &= C(\psi(Rx) - \psi(RT^{k+1}x)) \\ &\leqslant C\psi(Rx). \end{aligned}$$

Proof of (iv).

Because $y \to qp^*(\zeta, y)$ is continuous for every fixed $\zeta \in O(x,T)$, therefore letting $n \to \infty$, $q(x,\zeta) \leqslant C\psi(Rx)$ holds. We have shown in (4.5)

$$qp^*(x_n, x_m) \leqslant C \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}). \tag{4.7}$$

Similar to the proof of (iii), we derive

$$\begin{aligned} qp^*(x_n, x_m) &\leqslant C \sum_{k=n}^{m-1} qp^*(x_k, x_{k+1}), \\ &\leqslant C \sum_{k=n}^{m-1} [\psi(RT^k x) - \psi(RT^{k+1}x)] \\ &= C(\psi(R^n x) - \psi(RT^m x)) \\ &\leqslant C\psi(R^n x) \end{aligned}$$

for $m > n$. Letting $m \to \infty$, then the inequality $q(T^n x, \zeta) \leqslant C\psi(R^n x)$ follows. □

We give an illustrative example for above fixed point theorem in the setting of generalized quasi-partial metric spaces.

Example 4.7. Let $X = \left[0, \frac{\pi}{6}\right]$ and define $qp_b^*(x,y) = \tan|x-y| + x$ for any $(x,y) \in X \times X$, then (X, qp^*) is a generalized quasi-partial metric space. Define $T : X \to X$ as $Tx = \frac{x}{2}$ for all $x \in X$; $R : X \to Y$ as $Rx = 2x$ and $\psi : R(X) \to \mathbb{R}^+$ as $\psi(x) = 2x$. Then for $c = \frac{1}{2}$ and $y \in \left[0, \frac{\pi}{6}\right]$,

$$\begin{aligned} \max\left\{qp^*(y,Ty), cqp^*(Ry,RTy)\right\} &= \max\left\{qp^*(y,\frac{y}{2}), \frac{1}{2}qp^*(2y,y)\right\} \\ &= \max\left\{\tan\frac{y}{2} + y, \frac{1}{2}\tan y + y\right\}. \end{aligned}$$

Since $\tan\theta \leqslant 2\theta$ for $\theta \in \left[0, \frac{\pi}{6}\right]$, then

$$\begin{aligned} \max\left\{qp^*(y,Ty), cqp^*(Ry,RTy)\right\} &\leqslant 2y \\ &= \psi(2y) - \psi(y) \\ &= \psi(Ry) - \psi(RTy). \end{aligned}$$

For every $x \in X$ and $k \in \mathbb{N}$, when $\{\eta_i^k\}_{i=0}^{\infty} \in \mathscr{C}(qp^*, X, x_k)$, let us prove $\limsup_{i\to\infty} qp^*(\eta_i^k, z) < \infty$ for all $z \in O(x,T)$. Indeed

$$\begin{aligned} \limsup_{i\to\infty} qp^*(\eta_i^k, z) &= \lim_{i\to\infty}(\tan|\eta_i^k - z| + \eta_i^k) \\ &\leqslant \lim_{i\to\infty}(2|\eta_i^k - z| + \eta_i^k) \\ &\leqslant \frac{2\pi}{6} + \frac{\pi}{6} \\ &= \frac{\pi}{2} < \infty. \end{aligned}$$

Let $C = \frac{\pi}{2}$, we now prove that (i)-(iv) of the above theorem hold:

(i). $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \frac{x}{2^n} = 0 = \zeta$ exists.

(ii). By (i), we get $\zeta = 0$. Therefore $T(\zeta) = T(0) = 0 = \zeta$ holds trivially. Hence whenever $G(x) = qp^*(x,Tx)$ is T-orbitally lower semi-continuous at x implies $T\zeta = \zeta$.
Conversely, let $T\zeta = \zeta$ and we show that G is T-orbitally lower semi-continuous at x. Let $T\zeta = \zeta$ and $\{x_n\}_{n\to\infty}^{\infty} \in O(x,T)$ with $x_n \xrightarrow{qp^*} \zeta = 0$, we have

$$G(\zeta) = qp^*(\zeta, T\zeta) = qp^*(\zeta,\zeta) = \zeta = 0.$$

On the other hand,

$$\begin{aligned} 0 &= \liminf_{n\to\infty} \frac{x_n}{2} + x_n \\ &\leqslant \liminf_{n\to\infty} \tan\frac{x_n}{2} + x_n \\ &= \liminf_{n\to\infty} qp^*(x_n, \frac{x_n}{2}) = qp^*(x_n, Tx_n) \\ &= \liminf_{n\to\infty} G(x_n). \end{aligned}$$

Hence $G(\zeta) = \liminf_{n\to\infty} G(x_n)$.
(iii).

$$\begin{aligned} qp^*(x, T^n x) &= qp^*(x, \frac{x}{2^n}) \\ &= \tan\left|x - \frac{x}{2^n}\right| + x \\ &\leqslant 2x - \frac{x}{2^{n-1}} + x \\ &\leqslant 3x - \frac{x}{2^{n-1}} \\ &< \frac{\pi}{2} \times 4x = C\psi(Rx). \end{aligned}$$

(iv). Let $m > n$, then

$$\begin{aligned} qp^*(T^n x, T^m x) &= qp^*(\frac{x}{2^n}, \frac{x}{2^m}) \\ &= \tan\left|\frac{x}{2^n} - \frac{x}{2^m}\right| + \left|\frac{x}{2^n}\right| \\ &= \tan\frac{(2^{m-n}-1)x}{2^m} + \frac{x}{2^n} \\ &\leqslant \frac{(2^{m-n}-1)x}{2^{m-1}} + \frac{x}{2^n} \\ &= \frac{(2^{m-1}-2^n)x}{2^n \times 2^{m-1}}. \end{aligned}$$

Moreover, since $0 < \frac{(2^{m-1}-2^n)x}{2^{m-1}} < 1 < \frac{\pi}{2}$, subsequently, $qp^*(T^n x, T^m x) < \frac{\pi}{2} \times \frac{x}{2^n} = C\psi(RT^n x)$.

When taking $X = Y$, $g = id_X$ and $c = 1$ in Theorem 4.6, we can obtain the following corollary immediately.

Corollary 4.8. *Let (X, qp^*) be a complete generalized quasi-partial metric spaces. Given mappings $T : X \to X$ and $\psi : X \to \mathbb{R}^+$. If there exists $x \in X$ such that*

$$qp^*(y, Ty) \leqslant \psi(y) - \psi(Ty) \tag{4.8}$$

holds for all $y \in O(x, T)$ and moreover, assume that $\limsup_{i\to\infty} qp^*(\eta_i^k, z) < \infty$ for all $z \in O(x, T)$ and every $k \in \mathbb{N}$ when $\{\eta_i^k\}_{i=0}^{\infty} \in \mathscr{C}(qp^*, X, x_k)$, then

(i) $\lim_{n\to\infty} T^n x = \zeta$ exists.

(ii) $T(\zeta) = \zeta$ iff $G(x) = qp^*(x, Tx)$ is T-orbitally lower semi-continuous at x.

(iii) There exists $C > 0$ such that $qp^*(x, T^n x) \leqslant C\psi(x)$.

(iv) If $y \to qp^*(\zeta, y)$ is continuous for $\zeta \in O(x, T)$, then there exists $C > 0$ such that $qp^*(T^n x, \zeta) \leqslant C\psi(T^n x)$ and $q(x, \zeta) \leqslant C\psi(x)$.

As a corollary of Theorem 4.6, we can state the subsequent facts.

Corollary 4.9. *Let (X, qp^*) be a complete generalized quasi-partial metric spaces. Given mappings $T : X \to X$ and $\psi : X \to \mathbb{R}^+$. If there exist $x \in X$ and $0 < \alpha < 1$ such that*

$$qp^*(Ty, T^2 y) \leqslant qp^*(y, Ty) \tag{4.9}$$

holds for all $y \in O(x, T)$ and moreover, assume that $\limsup_{i\to\infty} qp^*(\eta_i^k, z) < \infty$ for all $z \in O(x, T)$ and every $k \in \mathbb{N}$ when $\{\eta_i^k\}_{i=0}^{\infty} \in \mathscr{C}(qp^*, X, x_k)$, then
(i) $\lim_{n\to\infty} T^n x = \zeta$ exists.
(ii) $T(\zeta) = \zeta$ iff $G(x) = qp^*(x, Tx)$ is T-orbitally lower semi-continuous at x.
(iii) There exists $C > 0$ such that $qp^*(x, T^n x) \leqslant \frac{C}{1-\alpha} qp^*(x, Tx)$.

Proof. Following [9], taking $y = T^n x$ in (4.9), then

$$qp^*(T^{n+1}x, T^{n+2}x) \leqslant \alpha qp^*(T^{n+1}x, T^{n+2}x)$$

and

$$qp^*(T^n x, T^{n+1}x) - \alpha qp^*(T^n x, T^{n+1}x) \leqslant qp^*(T^n x, T^{n+1}x) - qp^*(T^{n+1}x, T^{n+2}x),$$

thus

$$qp^*(T^nx, T^{n+1}x) \leqslant \frac{1}{1-\alpha}\Big(qp^*(T^nx, T^{n+1}x) - qp^*(T^{n+1}x, T^{n+2}x)\Big).$$

Set $\psi(y) = \dfrac{1}{1-\alpha}qp^*(y, Ty)$ for all $y \in O(x,T)$, then

$$qp^*(y, Ty) \leqslant \psi(y) - \psi(Ty).$$

The assertions (i)-(iii) follow immediately from Corollary 4.8. □

Because quasi-partial metric spaces are special generalized quasi-partial metric spaces, therefore if we apply Theorem 4.6, Corollary 4.8 and Corollary 4.9 to the setting of quasi-partial metric spaces respectively, then the following several corollaries can be stated.

Corollary 4.10. *Let (X, qp) and (Y, qp) be complete quasi-partial metric spaces. Given mappings $R : X \to Y$, $T : X \to X$ and $\psi : R(X) \to \mathbb{R}^+$. If there exist $x \in X$ and $c > 0$ such that*

$$\max\{qp(y, Ty), cqp(Ry, RTy)\} \leqslant \psi(Ry) - \psi(RTy) \tag{4.10}$$

holds for all $y \in O(x,T)$ and moreover, assume that $\limsup\limits_{i\to\infty} qp(\eta_i^k, z) < \infty$ for all $z \in O(x,T)$ and every $k \in \mathbb{N}$ when $\{\eta_i^k\}_{i=0}^{\infty} \in \mathscr{C}(qp, X, x_k)$, then

(i) $\lim\limits_{n\to\infty} T^nx = \zeta$ *exists.*

(ii) $T(\zeta) = \zeta$ *iff* $G(x) = qp(x, Tx)$ *is T-orbitally lower semi-continuous at x.*

(iii) There exists $C > 0$ such that $qp(x, T^nx) \leqslant C\psi(Rx)$.

(iv) If $y \to qp(\zeta, y)$ is continuous for $\zeta \in O(x,T)$, then there exists $C > 0$ such that $qp(T^nx, \zeta) \leqslant C\psi(R^nx)$ and $q(x, \zeta) \leqslant C\psi(Rx)$.

Corollary 4.11. *Let (X, qp) be a complete quasi-partial metric spaces. Given mappings $T : X \to X$ and $\psi : X \to \mathbb{R}^+$. If there exists $x \in X$ such that*

$$qp(y, Ty) \leqslant \psi(y) - \psi(Ty) \tag{4.11}$$

holds for all $y \in O(x,T)$ and moreover, assume that $\limsup\limits_{i\to\infty} qp(\eta_i^k, z) < \infty$ for all $z \in O(x,T)$ and every $k \in \mathbb{N}$ when $\{\eta_i^k\}_{i=0}^{\infty} \in \mathscr{C}(qp, X, x_k)$, then

(i) $\lim\limits_{n\to\infty} T^nx = \zeta$ *exists.*

(ii) $T(\zeta) = \zeta$ *iff* $G(x) = qp(x, Tx)$ *is T-orbitally lower semi-continuous at x.*

(iii) $qp(x, T^nx) \leqslant \psi(x)$.

(iv) If $y \to qp(\zeta, y)$ is continuous for $\zeta \in O(x,T)$, then there exists $C > 0$ $qp(T^nx, \zeta) \leqslant C\psi(T^nx)$ and $q(x, \zeta) \leqslant C\psi(x)$.

Corollary 4.12. *Let (X, qp) be a complete generalized quasi-partial metric spaces. Given mappings $T : X \to X$ and $\psi : X \to \mathbb{R}^+$. If there exist $x \in X$ and $0 < \alpha < 1$ such that*

$$qp(Ty, T^2y) \leqslant \alpha qp(y, Ty) \tag{4.12}$$

holds for all $y \in O(x,T)$ and moreover, assume that $\limsup\limits_{i\to\infty} qp(\eta_i^k, z) < \infty$ for all $z \in O(x,T)$ and every $k \in \mathbb{N}$ when $\{\eta_i^k\}_{i=0}^{\infty} \in \mathscr{C}(qp, X, x_k)$, then *(i)* $\lim_{n\to\infty} T^n x = \zeta$ *exists.*

(ii) $T(\zeta) = \zeta$ *iff* $G(x) = qp(x, Tx)$ *is* T*-orbitally lower semi-continuous at* x*.*

(iii) There exists $C > 0$ *such that* $qp(x, T^n x) \leqslant \dfrac{C}{1-\alpha} qp(x, Tx)$*.*

Remark 4.13. Because quasi-partial metric spaces are special generalized quasi-partial metric spaces and triangle inequality are satisfied on such spaces, as a consequence, Corollary 4.10, Corollary 4.11, and Corollary 4.12 are stated in simpler and better formations in the setting of quasi-partial metric spaces in [9].

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