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Calculation of generalized period constants via time-angle difference for complex analytic systems with resonant ratio

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Abstract

In the case of a critical point being a center, the isochronicity problem (or linearizability problem) is far to be solved in general. A progressive way to find necessary conditions for isochronicity is to compute period constants. In this paper, we establish a new recursive algorithm of calculation of the so-called generalized period constants. Furthermore, we verify the new algorithm by the existing results for the Lotka-Volterra system with $3 : -2$ resonance. Finally, the algorithm is applied to solve the linearizability problem for the Lotka-Volterra system in the ratio $4 : -5$.

Keywords: Generalized period constant, recursive algorithm, linearizability, Lotka-Volterra system.
2010 MSC: 34C05, 34C07.

1. Introduction

Consider a planar autonomous analytic differential system in the form of linear center perturbed by higher degree terms, that is

$$\frac{dx}{dt} = -y + \sum_{\alpha+\beta=2}^{\infty} A_{\alpha\beta}x^{\alpha}y^{\beta} = -y + X(x,y), \qquad \frac{dy}{dt} = x + \sum_{\alpha+\beta=2}^{\infty} B_{\alpha\beta}x^{\alpha}y^{\beta} = x + Y(x,y), \tag{1.1}$$

where X and Y are real polynomial functions whose series expansions in a neighborhood of the origin start with terms at least second degree. This system can be regarded as a perturbation of the canonical linear center

$$\frac{dx}{dt} = -y, \qquad \frac{dy}{dt} = x. \tag{1.2}$$

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Conversion to polar coordinates shows that near the origin either all non-stationary trajectories of the system (1.1) are ovals (in which case the origin is called a *center*) or they are all spirals (in which case the origin is called a *focus*). If all solutions near $x = 0, y = 0$ are periodic (that is, the origin is a center), the problem then arises to determine whether the period of oscillations is constant for all solutions near the origin. A center with such property is called an *isochronous* center. It follows form Poincaré and Lyapunov that the center of system (1.1) is isochronous if and only if it is *linearizable*, that is, there exists an analytic coordinates $U = x + \sum_{k+j=2}^{\infty} c_{kj}x^k y^j, V = y + \sum_{k+j=2}^{\infty} d_{kj}x^k y^j$, such that it reduces system (1.1) to the linear system $\frac{dU}{dt} = -V, \frac{dV}{dt} = U$. Many mathematicians have attached their attention to the isochronicity problem and made a systematic research. We do not mention any contribution here, for more details, see the survey [4].

The method to characterize isochronous center is not unique. Usually, there are two active methods: one is the calculation of isochronous constants (See [1, 2, 3]) while another is to compute period constants, which can be obtained recurrently. The authors of [5] pointed out that it was obviously a much more difficult problem to compute period constants. The vanishing of all of the period constants is a necessary and a sufficient condition for isochronicity. Although theoretically the isochronous center problem can be solved by letting all period constants be zero, it is not the fact in practice which is due to the difficulty of computing the period constants. Questions relating to calculation of period constants have been studied by a number of authors (See [7, 8, 10, 11]). However, only the first few ones can be given. The main trouble lies in the large amount of computations that involved which break down the capacity of computers.

In [12], Liu and Huang gave a new algorithm to compute period constants of complex polynomial systems, and the period constants of real polynomial systems are the special case of them. Wang and Liu generalized and developed the algorithm mentioned above in [16], they established a new algorithm to compute generalized period constants for general complex polynomial differential system with a resonant critical point.

In this paper, on the basis of the work of [13], we develop a new method of computation of generalized period constants for the following system

$$\frac{dz}{dT} = pz + \sum_{k=2}^{\infty} Z_k(z,w) = Z(z,w), \qquad \frac{dw}{dT} = -qw - \sum_{k=2}^{\infty} W_k(z,w) = -W(z,w), \tag{1.3}$$

with two complex straight line solutions $z = 0, w = 0$, i.e., $a_{0k} = b_{0k} = 0, k = 2, 3, \cdots$, where $p, q \in \mathbb{Z}^+, (p,q) = 1$, z, w, T are complex parameters, and

$$Z_k(z,w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^\alpha w^\beta, \qquad W_k(z,w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^\alpha z^\beta. \tag{1.4}$$

We now describe more precisely the organization of this paper. In Section 2, we come back to some preliminary knowledge which is necessary to demonstrate the results in Sections 3 and 4. Among them, we introduce a new detection criterion-node point value to decern linearizability. In Section 3, we derive a new recursive formula to compute generalized period constants of the origin of system (1.3). In the last section, to illustrate the effectiveness of our new method of computing generalized period constants, we use it to study the necessary conditions for linearizability of the Lotka-Volterra system with $3 : -2$ resonance. Moreover, as an application, we investigate the linearizability problem for the Lotka-Volterra system with $4 : -5$ resonance.

The technique used in this paper is different from more than the usual ones. Our recursive algorithm is new and origin. With the algorithm, in order to obtain the generalized period constants of a system, one only needs to force addition, subtraction, multiplication and division to the coefficients of the system which avoids complex integrating operation and solving equations. It is symbolic and easy to realize with computer algebra system such as *Mathematica* or *Maple*. What's also worth mentioning is that, for the first time, node point value is introduced to characterize the linearizability of the Lotka-Volterra system.

2. Some preliminary results

First of all, we briefly restate some main notions and results for system (1.3).

Lemma 2.1 ([6, 9]). *System* (1.3) *is normalizable at the origin if and only if there exists an analytic change of variables*

$$\xi = z + \Phi(z, w) = z + o(|z, w|), \qquad \eta = w + \Psi(z, w) = w + o(|z, w|) \tag{2.1}$$

bringing the system to its normal form

$$\frac{d\xi}{dT} = p\xi \left(1 + \sum_{i=1}^{\infty} p_i U^i\right), \qquad \frac{d\eta}{dT} = -q\eta \left(1 + \sum_{i=1}^{\infty} q_i U^i\right), \tag{2.2}$$

where $U = \xi^m \eta^n$.

We write $\mu_0 = \tau_0 = 0, \mu_k = p_k - q_k, \tau_k = p_k + q_k, k = 1, 2, \cdots$.

Definition 2.2 ([16]). For any positive integer k, μ_k is called k-th singular point quantity of the origin of system (1.3). If system (1.3) is real planar differential system, μ_k is the k-th saddle quantity. If system (1.3) is concomitant system of (1.1), μ_k is the k-th focus quantity. Moreover, the origin of system (1.3) is called generalized center if $\mu_k = 0, k = 1, 2, \cdots$.

Definition 2.3 ([16]). For any positive integer k, τ_k is called k-th generalized period constant of the origin of system (1.3). And the origin of system (1.3) is called generalized isochronous center if $\mu_k = \tau_k = 0, k = 1, 2, \cdots$.

The next lemmas are some results of a general nature concerning with linearizability of quadratic systems with resonance saddles.

Lemma 2.4 ([6]). *The system*

$$\dot{x} = x + c_{20}x^2 + c_{11}xy + c_{02}y^2, \qquad \dot{y} = -\lambda y + d_{02}y^2 \tag{2.3}$$

for $\lambda > 0$ *is always linearizable if* $1/\lambda \in \mathbb{N}$.

For the Lotka-Volterra system

$$\dot{x} = x(1 + ax + by), \qquad \dot{y} = y(-\lambda + cx + dy), \tag{2.4}$$

we have,

Lemma 2.5 ([9]). *System* (2.4) *is linearizable if* $\lambda \in \mathbb{Q}$ *and* $\frac{c}{a} + \lambda = k \in \mathbb{N}, 2 \le k < \lambda + 1$.

There exists a symmetric condition for $0 < \lambda < 1$ *by means of change*

$$(x, y, \lambda, t, a, b, c, d) \to \left(y, x, \frac{1}{\lambda}, -\lambda t, d, c, b, a\right). \tag{2.5}$$

Lemma 2.6 ([9]). *System* (2.4) *is linearizable if* $\lambda > 1, -\frac{c}{a} = n \in \mathbb{N}^*$ *and one of the following conditions is satisfied:*

E1. $\lambda \in \mathbb{R}\backslash\mathbb{Q}$ *and* $1 \le n < \lambda$*;*

E2. $\lambda \in \mathbb{Q}$ *and* $1 \le n < \lambda - 1$*;*

E3. $\lambda \in \mathbb{Q}$, $\lambda - 1 < n < \lambda$ *and* $\lambda \ne n + \frac{1}{q}$*;*

E4. $\lambda = \frac{p}{q} = n + \frac{1}{q}$ *and* $\frac{d}{b} = \frac{-p}{q-1}$.

Change (2.5) *gives the corresponding conditions for* $0 < \lambda < 1$.

The following ideas come from [14].

We consider the real planar system

$$\frac{dz}{dT} = \gamma z + \sum_{k+j=2}^{\infty} A_{kj} z^k w^j, \qquad \frac{dw}{dT} = n\gamma w + \sum_{k+j=2}^{\infty} B_{kj} z^k w^j, \tag{2.6}$$

where the functions of the right hand side are analytic in the neighborhood of the origin, $k \geq 0, j \geq 0, \gamma \neq 0$ and $n(> 1)$ is a positive integer.

In [15], the authors proved that,

Lemma 2.7. *For system* (2.6), *one can derive successively the terms of a convergent power series*

$$u(z, w) = \sum_{\alpha+\beta=1}^{\infty} p_{\alpha\beta} z^\alpha w^\beta = z + h.o.t., \qquad v(z, w) = \sum_{\alpha+\beta=1}^{\infty} q_{\alpha\beta} z^\alpha w^\beta = w + h.o.t., \tag{2.7}$$

in a neighborhood of the origin, where $\alpha \geq 0, \beta \geq 0, p_{10} = q_{01} = 1, p_{01} = q_{10} = 0, q_{n0} = 0$, such that by using the transformation

$$u = u(z, w), \qquad v = v(z, w), \tag{2.8}$$

system (2.6) *reduces to the norm form*

$$\frac{du}{dT} = \gamma u, \qquad \frac{dv}{dT} = n\gamma v + \sigma u^n. \tag{2.9}$$

System (2.9) has the first integral

$$\frac{v}{u^n} - \frac{\sigma}{\gamma} \ln u = h, \tag{2.10}$$

where h is a constant.

It is easy to see that when $\sigma = 0$, system (2.6) can be linearized. So that, we would like to introduce the following.

Definition 2.8 ([14])**.** We say that σ is a node point value of the origin of system (2.6).

To compute $u(z, w), v(z, w)$ and σ, we have the recursive formulae as follows:

Theorem 2.9 ([14])**.** *When $\alpha + \beta > 1, p_{\alpha\beta}$ of* (2.7) *are given uniquely by*

$$p_{\alpha\beta} = \frac{1}{\gamma(1 - \alpha - n\beta)} \sum_{k+j=2}^{\alpha+\beta} [(\alpha - k + 1) A_{kj} p_{\alpha-k+1,\beta-j} + (\beta - j + 1) B_{kj} p_{\alpha-k,\beta-j+1}]. \tag{2.11}$$

When $2 \leq \alpha + \beta \leq n - 1, q_{\alpha\beta}$ of (2.7) *are given uniquely by*

$$q_{\alpha\beta} = \frac{1}{\gamma(n - \alpha - n\beta)} \sum_{k+j=2}^{\alpha+\beta} [(\alpha - k + 1) A_{kj} q_{\alpha-k+1,\beta-j} + (\beta - j + 1) B_{kj} q_{\alpha-k,\beta-j+1}]. \tag{2.12}$$

Furthermore, σ is given uniquely by

$$\sigma = \sum_{k=2}^{n} [(n - k + 1) A_{k0} q_{n-k+1,0} + B_{k0} q_{n-k,1}]. \tag{2.13}$$

Theorem 2.10 ([14])**.** *If $\sigma = 0$, then q_{n0} can take arbitrary constant. When q_{n0} is given, for all pairs (α, β), if $\alpha + \beta \geq 2$ and $(\alpha, \beta) \neq (n, 0)$, then, $q_{\alpha\beta}$ are given uniquely by* (2.12).

Corollary 2.11 ([14])**.** *If $B_{20} = B_{30} = \cdots = B_{n-1,0} = 0$, then $q_{20} = q_{30} = \cdots = q_{n-1,0} = 0$ and $\sigma = B_{n0}$.*

An immediate consequence of Corollary 2.11 is the following.

Corollary 2.12 ([14])**.** *If $B_{20} = B_{30} = \cdots = B_{n-1,0} = B_{n0} = 0$, then $\sigma = 0$.*

3. Calculation of period constants

This section is devoted to giving a new recursive algorithm which allows to compute all the generalized period constants of the origin of system (1.3).

Theorem 3.1. *Suppose that system* (1.3) *is a complex system having two complex straight line solutions* $z = 0, w = 0$. *Then one can derive successively the terms of the following formal series*

$$G(z,w) = \sum_{k=1}^{\infty} g_k(z,w) = \sum_{k=1}^{\infty} \sum_{\alpha+\beta=k} c_{\alpha\beta} z^\alpha w^\beta, \tag{3.1}$$

where $c_{kk}, k = 1, 2, \cdots$ *can take arbitrary constants, such that*

$$\frac{dG}{dT} + \frac{p+q}{2} i + \frac{d\theta}{dT} = \frac{1}{2i} \sum_{m=1}^{\infty} \tau'_m (zw)^m. \tag{3.2}$$

For all $p\alpha \neq q\beta$, $c_{\alpha\beta}$ *is given by the following recursive formula*

$$c_{\alpha\beta} = \frac{1}{q\beta - p\alpha} \left\{ \sum_{k+j=1}^{\alpha+\beta-1} [(\alpha - k) a_{k+1,j} - (\beta - j) b_{j+1,k}] c_{\alpha-k,\beta-j} - \frac{i}{2} (a_{\alpha+1,\beta} + b_{\beta+1,\alpha}) \right\}, \tag{3.3}$$

and for any positive integer m, τ'_m *is given by the following recursive formula*

$$\tau'_m = a_{qm+1,pm} + b_{pm+1,qm} + 2i \sum_{k+j=1}^{(p+q)m-1} [(qm - k) a_{k+1,j} - (pm - j) b_{j+1,k}] c_{qm-k,pm-j}. \tag{3.4}$$

In addition, for $\alpha < 0$ *or* $\beta < 0$, *one defines* $a_{\alpha\beta} = b_{\alpha\beta} = c_{\alpha\beta} = 0$.

Proof. From expression (1.4), for any integer l, we have

$$Z_{l+1}(z,w) = \sum_{\alpha+\beta=l} a_{\alpha+1,\beta} z^{\alpha+1} w^\beta, \qquad W_{l+1}(z,w) = \sum_{\alpha+\beta=l} b_{\beta+1,\alpha} w^{\beta+1} z^\alpha. \tag{3.5}$$

Observe that the transformation of polar coordinate

$$z = re^{i\theta}, \qquad w = re^{-i\theta}, \qquad T = it, \qquad i = \sqrt{-1}, \tag{3.6}$$

hence

$$\frac{d\theta}{dT} = \frac{1}{2i} \left(\frac{1}{z} \frac{dz}{dT} - \frac{1}{w} \frac{dw}{dT} \right). \tag{3.7}$$

By straight computation, we have for system (1.3)

$$\begin{aligned}
&\frac{dG}{dT} + \frac{p+q}{2} i + \frac{d\theta}{dT} \\
&= \sum_{m=1}^{\infty} \left[\left(\frac{\partial g_m}{\partial z} pz - \frac{\partial g_m}{\partial w} qw \right) + \sum_{l=1}^{m-1} \left(\frac{\partial g_{m-l}}{\partial z} Z_{l+1} - \frac{\partial g_{m-l}}{\partial w} W_{l+1} \right) - \frac{i}{2} \frac{wZ_{m+1} + zW_{m+1}}{zw} \right] \\
&= \sum_{m=1}^{\infty} \left\{ \sum_{\alpha+\beta=m} (p\alpha - q\beta) c_{\alpha\beta} z^\alpha w^\beta + \sum_{l=1}^{m-1} \left[\sum_{\alpha+\beta=m-l} \sum_{k+j=l} (\alpha a_{k+1,j} - \beta b_{j+1,k}) c_{\alpha\beta} z^{\alpha+k} w^{\beta+j} \right] \right. \\
&\quad \left. - \frac{i}{2} \sum_{\alpha+\beta=m} (a_{\alpha+1,\beta} + b_{\beta+1,\alpha}) z^\alpha w^\beta \right\}
\end{aligned}$$

$$= \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m} \{(p\alpha - q\beta)c_{\alpha\beta} + \sum_{k+j=1}^{\alpha+\beta-1} [(\alpha-k)a_{k+1,j} - (\beta-j)b_{j+1,k}]c_{\alpha-k,\beta-j} - \frac{i}{2}(a_{\alpha+1,\beta} + b_{\beta+1,\alpha})\} z^\alpha w^\beta.$$

Denote that

$$s_{\alpha\beta} = \sum_{k+j=1}^{\alpha+\beta-1} [(\alpha-k)a_{k+1,j} - (\beta-j)b_{j+1,k}]c_{\alpha-k,\beta-j} - \frac{i}{2}(a_{\alpha+1,\beta} + b_{\beta+1,\alpha}). \tag{3.8}$$

For all integers $p\alpha \neq q\beta$ and m, we take $c_{\alpha\beta} = \frac{s_{\alpha\beta}}{q\beta - p\alpha}$ and $\tau'_m = 2is_{qm,pm}$. □

The relations between τ_m and τ'_m $(m = 1, 2, \cdots)$ are as follows:

Theorem 3.2. *Denote that* $\tau'_0 = 0$. *For all positive integers* m, $\tau'_0 = \tau'_1 = \cdots = \tau'_{m-1} = 0$ *if and only if* $\tau_0 = \tau_1 = \cdots = \tau_{m-1} = 0$, $\tau_m = \tau'_m$.

Proof. We see from expressions (3.2) and (3.6) that

$$\frac{p+q}{2} = \frac{d\theta}{dt} - \frac{1}{2}\sum_{k=1}^{\infty} \tau'_k r^{2k} + \frac{dG}{dt}. \tag{3.9}$$

Formally integrating the two sides of (3.9) from 0 to $\tau(2\pi, h)$, it follows

$$\frac{p+q}{2}\tau(2\pi, h) = 2\pi - \frac{1}{2}\int_0^{\tau(2\pi,h)} \sum_{k=1}^{\infty} \tau'_k r^{2k} dt = \pi[2 - \tau'_m h^{2m} + o(h^{2m})]. \tag{3.10}$$

Expression (3.10) implies the result of this theorem. □

4. Verification and application

As a verification of the new algorithm obtained in Section 3, let's consider linearizability of the Lotka-Volterra system with $3 : -2$ resonance.

Case $3 : -2$

The corresponding system is

$$\frac{dz}{dT} = 3z + a_1 zw + a_2 z^2, \quad \frac{dw}{dT} = -2w - b_1 wz - b_2 w^2. \tag{4.1}$$

Theorem 4.1. *System* (4.1) *is linearizable if and only if one of the following conditions is satisfied:*

(1) $b_1 = 0$;
(2) $a_1 = 0$;
(3) $2a_1 + b_2 = 0$;
(4) $a_2 + b_1 = 0, \quad a_1 + b_2 = 0$;
(5) $a_2 - 3b_1 = 0, \quad a_1 - b_2 = 0$.

Proof. We prove the necessity by computing the generalized period constants τ_k of system (4.1). The necessary conditions for system (4.1) to be linearizable are $\tau_k = 0$ for any $k \geq 1$. Usually, one needs only a finite number of τ_k to reach necessary conditions.

Applying recursive formulae (3.3) and (3.4) to system (4.1) and executing calculations by Mathematica, we obtain the first three generalized period constants as follows:

$$\begin{aligned}
\tau_1 =& \frac{a_1 b_1 (2a_1 + b_2)}{24}(a_1 a_2 + 5a_1 b_1 - 3a_2 b_2 + b_1 b_2),\\
\tau_2 =& \frac{a_1 b_1 (2a_1 + b_2)}{13226976}(-83430 a_1^4 a_2^3 - 30666 a_1^4 a_2^2 b_1 + 1247156 a_1^4 a_2 b_1^2 + 1763674 a_1^4 b_1^3 - 765724 a_1^3 b_1^3 b_2\\
&- 1977003 a_1^2 b_1^3 b_2^2 - 437904 a_1 b_1^3 b_2^3 + 204093 b_1^3 b_2^4),
\end{aligned}$$

$$\begin{aligned}\tau_3 =&\frac{a_1b_1(2a_1+b_2)}{6177349100482560}(-88413693207840a_1^7a_2^5+1424204793214812a_1^7a_2^4b_1\\&-4347304652288379a_1^7a_2^3b_1^2\\&-16461059659171969a_1^7a_2^2b_1^3+74035050499453839a_1^7a_2b_1^4\\&+109431808333112977a_1^7b_1^5-111384091208069408a_1^6b_1^5b_2\\&-145356575873116382a_1^5b_1^5b_2^2+58479246464930400a_1^4b_1^5b_2^3\\&+59887200036254436a_1^3b_1^5b_2^4-16906150380762648a_1^2b_1^5b_2^5\\&-6871563863754750a_1b_1^5b_2^6+2265676736483160b_1^5b_2^7).\end{aligned}\tag{4.2}$$

The greatest common factor of τ_1, τ_2, τ_3 is

$$G_1 = PolynomialGCD[\tau_1, \tau_2, \tau_3] = \frac{a_1b_1(2a_1+b_2)}{6177349100482560}.\tag{4.3}$$

Let $F_1 = \tau_1/G_1, F_2 = \tau_2/G_1, F_3 = \tau_3/G_1$.

We denote the resultant of the polynomials $poly_1$ and $poly_2$ with respect to the variable x by $Resultant[poly_1, poly_2, x]$. From the algebraic theory, $Resultant[poly_1, poly_2, x] = 0$ is a necessary condition for $poly_1 = poly_2 = 0$.

Factually, by computing the resultants of the polynomials F_1, F_2, F_3 with respect to b_2, we get

$$\begin{aligned}R_{1,2} &= Resultant[F_1, F_2, b_2] = -C_{69}a_1^4(a_2-3b_1)(a_2+b_1)^2 f_{12}(a_2, b_1),\\ R_{1,3} &= Resultant[F_1, F_3, b_2] = C_{106}a_1^7(a_2-3b_1)(a_2+b_1)^2 f_{13}(a_2, b_1),\\ R_{2,3} &= Resultant[F_2, F_3, b_2] = -C_{71}a_1^{28}b_1^{20}(a_2-3b_1)(a_2+b_1)^2 f_{23}(a_2, b_1),\end{aligned}\tag{4.4}$$

where C_{69}, C_{106}, C_{71} are all positive integers, whose digits are 69, 106, 71, respectively. $f_{12}(a_2, b_1)$, $f_{13}(a_2, b_1)$, $f_{23}(a_2, b_1)$ are polynomials with respect to a_2, b_1.

$$PolynomialGCD[R_{1,2}, R_{1,3}, R_{2,3}] = C_{64}a_1^4(a_2-3b_1)(a_2+b_1)^2.\tag{4.5}$$

Denote $G_2 = C_{64}(a_2-3b_1)(a_2+b_1)^2$. Let $F_{12} = R_{1,2}/G_2/a_1^4, F_{13} = R_{1,3}/G_2/a_1^7, F_{23} = R_{2,3}/G_2/a_1^{28}/b_1^{20}$. The resultants of the polynomials F_{12}, F_{13}, F_{23} with respect to a_2 are respectively

$$\begin{aligned}R_{12,13} &= Resultant[F_{12}, F_{13}, a_2] = C_{321}b_1^{36},\\ R_{12,23} &= Resultant[F_{12}, F_{23}, a_2] = -C_{588}b_1^{72},\\ R_{13,23} &= Resultant[F_{13}, F_{23}, a_2] = -C_{1925}b_1^{162}.\end{aligned}$$

It is clear that $R_{12,13} = R_{12,23} = R_{13,23} = 0$ if and only if $b_1 = 0$, thus $\{G_1 = 0, G_2 = 0\}$ is a complete set of necessary conditions for $\tau_1 = \tau_2 = \tau_3 = 0$. From $G_1 = 0$ or $G_2 = 0$, we have $b_1 = 0$, or $a_1 = 0$, or $2a_1 + b_2 = 0$, or $a_2 - 3b_1 = 0$, or $a_2 + b_1 = 0$. Substituting each condition into expression (4.2) and solving the equation set $\tau_1 = \tau_2 = \tau_3 = 0$, then we can accomplish the necessity of the theorem.

We can see that conditions $(1) - (5)$ are identical with the corresponding results of Theorem B in [9]. The proof of the sufficiency of these conditions will not be given here. □

Additionally, as an application of our new method, we are going to discuss the linearizability problem of the Lotka-Volterra system with $4 : -5$ resonance in the following.

Case $4 : -5$

The corresponding system is

$$\frac{dz}{dT} = 4z + a_1zw + a_2z^2, \quad \frac{dw}{dT} = -5w - b_1wz - b_2w^2,\tag{4.6}$$

which is equivalent to

$$\frac{dz}{dT} = z + a_1zw + a_2z^2, \quad \frac{dw}{dT} = -\frac{5}{4}w - b_1wz - b_2w^2.\tag{4.7}$$

Theorem 4.2. *System* (4.7) *is linearizable if and only if one of the following conditions is satisfied:*

(1) $b_1 = 0$;
(2) $a_1 = 0$;
(3) $3a_2 + 4b_1 = 0$;
(4) $a_2 + b_1 = 0, \quad a_1 + b_2 = 0$;
(5) $a_2 - b_1 = 0, \quad 5a_1 - 3b_2 = 0$;
(6) $a_2 - b_1 = 0, \quad 43a_1^2 - 33a_1b_2 + 6b_2^2 = 0$;
(7) $a_2 + 4b_1 = 0, \quad 5a_1 + 2b_2 = 0$;
(8) $a_2 + 2b_1 = 0, \quad 5a_1 + 3b_2 = 0$;
(9) $a_2 + 2b_1 = 0, \quad 5a_1 + 6b_2 = 0$;
(10) $2a_2 + 3b_1 = 0, \quad 5a_1 + 6b_2 = 0$;
(11) $7a_2 + 4b_1 = 0, \quad a_1 - 2b_2 = 0$;
(12) $3a_2 - 4b_1 = 0, \quad 5a_1 - 2b_2 = 0$;
(13) $a_2 - 2b_1 = 0, \quad 5a_1 - b_2 = 0$.

Proof. The necessity can be proved by the same technique employed in the proof of Theorem 4.1, so we omit it for brevity. Let's turn to the proof of sufficiency.

(1), (2): Under condition (1), system (4.7) satisfies the condition in Lemma 2.4. Under condition (2), after using the transformation $(z, w, \lambda, T) \to (w, z, \frac{1}{\lambda}, -\lambda T)$, system (4.7) also satisfies the condition in Lemma 2.4.

(3): When condition (3) holds, system (4.7) satisfies the condition in Lemma 2.5.

(4): Under condition (4), system (4.7) can be linearized by the transformation

$$u = z(5 - 4a_1w - 5b_1z)^{-1}, \qquad v = w(5 - 4a_1w - 5b_1z)^{-1}.$$

(5), (12): When condition (5) holds, if $b_1 = 0$, it belongs to (1). Else, by means of transformation $(z, w, T) \to (-\frac{3+4a_1u-3v}{3b_1}, u, -4T_1)$, system (4.7) is reduced to

$$\frac{du}{dT_1} = u + \frac{4}{3}a_1u^2 + 4uv, \qquad \frac{dv}{dT_1} = 4v + 12a_1uv - 4v^2. \tag{4.8}$$

When condition (12) holds, by means of transformation $(z, w, T) \to (-\frac{3(1+2a_1u-v)}{4b_1}, u, -2T_1)$, system (4.7) is reduced to

$$\frac{du}{dT_1} = u + 2a_1u^2 + \frac{3}{2}uv, \qquad \frac{dv}{dT_1} = 2v + 9a_1uv - 2v^2. \tag{4.9}$$

By virtue of Corollary 2.12, the node point values of the origin of systems (4.8) and (4.9) are zero, then from Lemma 2.7, they can be linearized.

(6): Condition (6) is equivalent to $a_2 = b_1, b_2 = \frac{1}{12}(33 \pm \sqrt{57})a_1$, by means of transformation $(z, w, T) \to (-\frac{3+4a_1u-3v}{3b_1}, u, -4T_1)$, system (4.7) is reduced to

$$\frac{du}{dT_1} = u + \frac{1}{3}(17 \pm \sqrt{57})a_1u^2 + 4uv, \qquad \frac{dv}{dT_1} = 4v + \frac{4}{9}(13 \pm \sqrt{57})a_1^2u^2 + 12a_1uv - 4v^2. \tag{4.10}$$

Applying recursive formulae (2.12) and (2.13) to compute the node point value of system (4.10), we have

$$\sigma = \frac{1}{4}B_{20}(12A_{20}^2 - 10A_{20}B_{11} + 2B_{11}^2 - 2A_{11}B_{20} + B_{02}B_{20}). \tag{4.11}$$

Substituting $A_{20} = \frac{1}{3}(17 \pm \sqrt{57})a_1, A_{11} = 4, A_{02} = 0, B_{20} = \frac{4}{9}(13 \pm \sqrt{57})a_1^2, B_{11} = 12a_1, B_{02} = -4$ into (4.11), we get $\sigma = 0$.

(7), (8), (13): When condition (7) holds, by means of transformation $(z, w, T) \to (-\frac{-1+u+2a_1v}{4b_1}, v, -T_1)$, system (4.7) is reduced to

$$\frac{du}{dT_1} = u - u^2 - \frac{9}{2}a_1uv, \qquad \frac{dv}{dT_1} = \frac{3}{2}v - \frac{1}{4}uv - 3a_1v^2. \tag{4.12}$$

When condition (8) holds, by means of transformation $(z, w, T) \to (-\frac{-3+3u+4a_1v}{6b_1}, v, -T_1)$, system (4.7) is reduced to

$$\frac{du}{dT_1} = u - u^2 - 3a_1uv, \qquad \frac{dv}{dT_1} = \frac{7}{4}v - \frac{1}{2}uv - \frac{7}{3}a_1v^2. \tag{4.13}$$

When condition (13) holds, by means of transformation $(z, w, T) \to (\frac{-1+u-4a_1v}{2b_1}, v, -T_1)$, system (4.7) is reduced to

$$\frac{du}{dT_1} = u - u^2 + 9a_1uv, \qquad \frac{dv}{dT_1} = \frac{3}{4}v + \frac{1}{2}uv + 3a_1v^2. \tag{4.14}$$

We can note that the origin is transferred into a node and $\frac{3}{2}, \frac{7}{4}, \frac{3}{4} \notin \mathbb{N}$ ensure respectively that the nodes of systems (4.12), (4.13) and (4.14) are linearizable by an analytic change of coordinates (the Poincaré Theorem in [6]), the three systems are therefore always linearizable at the origin.

(9), (10): When condition (9) or (10) holds, we use the transformation $(z, w, T) \to (v, \frac{15+10u-12b_1v}{10a_1}, \frac{4}{5}T_1)$ to bring system (4.7) respectively to

$$\frac{du}{dT_1} = u + \frac{2}{3}u^2 - \frac{36}{25}b_1uv - \frac{144}{125}b_1^2v^2, \qquad \frac{dv}{dT_1} = 2v + \frac{4}{5}uv - \frac{64}{25}b_1v^2 \tag{4.15}$$

or

$$\frac{du}{dT_1} = u + \frac{2}{3}u^2 - \frac{36}{25}b_1uv - \frac{84}{125}b_1^2v^2, \qquad \frac{dv}{dT_1} = 2v + \frac{4}{5}uv - \frac{54}{25}b_1v^2. \tag{4.16}$$

The node point values of the origin of systems (4.15) and (4.16) are zero.

(11): When condition (11) holds, we use the transformation $(z, w, T) \to (\frac{14+8u-7a_1v}{8b_1}, v, -T_1)$ to bring system (4.7) respectively to

$$\frac{du}{dT_1} = u + \frac{4}{7}u^2 - \frac{9}{8}a_1uv + \frac{63}{64}a_1^2v^2, \qquad \frac{dv}{dT_1} = 3v + uv - \frac{3}{8}a_1v^2. \tag{4.17}$$

The node point value of the origin of system (4.17) is zero. □

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