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Reduced-order synchronization of fractional order chaotic systems with fully unknown parameters using modified adaptive control

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Abstract

A novel reduced-order adaptive controller is extended and developed to synchronize two different fractional order chaotic systems with different dimensions. Based upon the parameters modulation and the adaptive control techniques, we show that dynamical evolution of third–order fractional order chaotic system can be synchronized with the projection of a fourth–order fractional order chaotic system even though their parameters are unknown. The techniques are successfully applied to fractional order hyperchaotic Chen (4th-order) and fractional order chaotic Liu (3rd-order) systems. Theoretical analysis and numerical simulations are shown to verify the results.

Keywords: Reduced-order, synchronization, adaptive control, fractional order chaos.
2010 MSC: 3C10, 93C95.

1. Introduction

Fractional calculus, once treated only as a pure theoretical field has fully developed into an applied field of mathematics with applications in a variety of areas such as the theory of control of dynamical systems which is the subject of this paper. The chaotic behavior of fractional order systems is demonstrated by many researchers such as [13, 25, 30, 33]. Synchronization of chaos in fractional order differential systems has applications in several fields such as secure communications, population models and financial systems. Due

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to its importance in wide ranging application areas a number of synchronization schemes and methods have been developed. These include, adaptive control [7, 15, 22, 24, 26, 29, 35], sliding mode control [12, 17, 31, 34, 36], linear active control technique [3, 6, 10, 23], projective synchronization [2, 11, 19, 28] and nonlinear active control [4, 5]. Synchronization between different order chaotic oscillators can be found frequently in nature. For example, synchronization between heart and lung, both systems behave in synchronous ways in the cardiorespiratory system. Another example occurs in the case of thalamic neurons, which is reasonable if their order is different from the one of the hippocampal neurons. Similarly, such synchronization phenomenon can be found in the human brain, in chaotic laser communications and synchronization in the cells of paddlefish, etc. There are few very interesting results in the scientific literature dealing with the synchronization of chaotic systems with different orders [8, 9, 14, 16, 18, 27]. Recently, Agrawal and Das [1] introduced a modification on the adaptive synchronization and parameter identification method with unknown parameters for using in fractional order chaotic systems based on Lyapunov stability method. This work is to further develop the scheme for reduced-order synchronization of chaotic systems with fully unknown parameters. The rest of the paper is organized as follows. In section 2 we briefly describe the problem. In section 3 we describe adaptive reduced- order synchronization strategies with parameter update law for fourth order fractional-order hyperchaotic Chen system and third order fractional-order Liu system to perform the reduced-order synchronization. The results of numerical simulations are given in section 4. Conclusions are given in section 5.

2. Problem formulation and systems description

2.1. Preliminaries of fractional-order calculus

Fractional calculus is a generalization of integration and differentiation to a non-integer order integro differential operator ${}_aD_t^q$, which is defined by

$$ {}_aD_t^q = \begin{cases} \frac{d^q}{dt^q}, & R(q) > 0, \\ 1, & R(q) = 0, \\ \int\limits_a^t (d\tau)^{-q}, & R(q) < 0, \end{cases} \tag{2.1} $$

where $a < t$. Fractional derivatives is defined as follows.

$$ {}_aD_t^q x(t) = \frac{d^n}{dt^n} j_t^{n-q} x(t), \qquad q > 0, \tag{2.2} $$

where $n = \lceil q \rceil$, and

$$ j_t^{\alpha} \varphi(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau, \tag{2.3} $$

where $0 < \alpha \leq 1$ and $\Gamma(.)$ is the gamma function. The Caputo differential operator of fractional order q is defined as

$$ {}^cD_t^q x(t) = j_t^{n-q} x^n(t), \qquad q > 0, \tag{2.4} $$

where $n = \lceil q \rceil$.

Lemma 2.1 ([1, 25])**.** *In Riemann–Liouville derivatives if $p > q \geq 0, m$ and n are integers such that $0 \leq m-1 \leq p < m, 0 \leq n-1 < n$, then we obtain*

$$ {}_aD_t^q \left({}_aD_t^{-q} f(t) \right) = {}_aD_t^{p-q} f(t). \tag{2.5} $$

Lemma 2.2 ([1, 25])**.** *In Riemann–Liouville derivatives if $p > q \geq 0, m$ and n are integers such that $0 \leq m-1 \leq p < m, 0 \leq n-1 \leq q < n$, then we obtain*

$$ {}_aD_t^p \left({}_aD_t^q f(t) \right) = {}_aD_t^{p+q} f(t) - \sum_{j=1}^{n} \left[{}_aD_t^{q-j} f(t) \right]_{t=a} \times \frac{(t-a)^{-p-j}}{\Gamma(1-p-j)}. \tag{2.6}$$

2.2. Modified adaptive reduced-order synchronization

Consider the chaotic master system described by

$$D_t^q x = f(x) + F(x)\alpha, \tag{2.7}$$

where $x \in \Re^m$ is the state vector of the system (2.7), $f : \Re^m \to \Re^m$ is a continuous vector function including nonlinear terms, $F : \Re^m \to \Re^{m\times k}$, and $\alpha \in \Re^k$ are the parameter vectors of the system. Similarly,

$$D_t^q y = g(y) + G(y)\beta + U \tag{2.8}$$

is the slave system; where $y \in \Re^n$ is the state vector, $g : \Re^n \to \Re^n$ is a continuous vector function, $G : \Re^n \to \Re^{n\times \ell}$, and $\beta \in \Re^\ell$ is the parameter vector.

In a situation when $f \neq g$, $F \neq G$ and $n < m$ then the reduced-order synchronization is the only possible type of synchronization. Consider the projection of the master system to be

$$D_t^q x_\imath = f_\imath(x) + F_\imath(x)\alpha, \tag{2.9}$$

where $x_\imath \in \Re^n$, $f_\imath : \Re^m \to \Re^n$, and $F_\imath : \Re^m \to \Re^{n\times k}$. The rest:

$$D_t^q x_\jmath = f_\jmath(x) + F_\jmath(x)\alpha, \tag{2.10}$$

where $x_\jmath \in \Re^u$, $f_\jmath : \Re^m \to \Re^u$, $F_\jmath : \Re^m \to \Re^{u\times k}$ and orders n, u satisfy $u + n = m$. Define the error vector as

$$e(t) = y(t) - x_\imath(t).$$

Then by suitable choice of controller it can be shown that $\lim\limits_{t\to\infty} \|y(t) - x_\imath(t)\| = 0$.

Theorem 2.3. *If the nonlinear control function is selected as*

$$\begin{aligned} U =& f_\imath(x) + F_\imath(x)\alpha - g(y) - G(y)\beta \\ &+ D_t^{q-1} \left[F_\imath(x)(\tilde{\alpha} - \alpha) - G(y)(\tilde{\beta} - \beta) - \left(D_t^{q-1} e(t) \right) \frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))} - ke \right] \end{aligned} \tag{2.11}$$

and adaptive laws of parameters are taken as

$$\dot{\hat{\alpha}} = -[F_\imath(x)]^T e, \qquad \dot{\hat{\beta}} = [G(y)]^T e, \tag{2.12}$$

where $\hat{\alpha} = \tilde{\alpha} - \alpha, \hat{\beta} = \tilde{\beta} - \beta, k > 0$ is a constant and $q \in [0, 1]$ is the order of the derivative and $\tilde{\alpha}, \tilde{\beta}$ are the estimated parameters of α and β respectively.

Proof. From Eqs. (2.8) and (2.9) we get the error dynamical system as follows:

$$D_t^q e(t) = g(y) + G(y)\beta - f(x) - F(x)\alpha + U. \tag{2.13}$$

Inserting (2.11) into (2.13) yields the following:

$$D_t^q e(t) = D_t^{q-1} \left[F_\imath(x)(\tilde{\alpha} - \alpha) - G(y)(\tilde{\beta} - \beta) - (D_t^{q-1} e(t)) \frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))} - ke \right]. \tag{2.14}$$

If a Lyapunov function candidate is chosen as

$$V(e,\hat{\alpha},\hat{\beta}) = \frac{1}{2}\left[e^T e + (\tilde{\alpha}-\alpha)^T(\tilde{\alpha}-\alpha) + (\tilde{\beta}-\beta)^T\left(\tilde{\beta}-\beta\right)\right], \tag{2.15}$$

the time derivative of $V(e,\hat{\alpha},\hat{\beta})$ along the trajectory of the error dynamical system (2.14) is as follows

$$\dot{V}(e,\hat{\alpha},\hat{\beta}) = \left[\dot{e}^T e + (\tilde{\alpha}-\alpha)^T\dot{\tilde{\alpha}} + \left(\tilde{\beta}-\beta\right)^T\dot{\tilde{\beta}}\right] \tag{2.16}$$

Using Lemma 2.2 in Eq. (2.16) we get

$$\begin{aligned}\dot{V}(e,\hat{\alpha},\hat{\beta}) = \Bigg(\Bigg[&D_t^{q-1}\left(D_t^q e(t)\right) + \left(D_t^{q-1} e(t)\right)\frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))}\Bigg]\\ &+(\tilde{\alpha}-\alpha)^T\dot{\tilde{\alpha}} + \left(\tilde{\beta}-\beta\right)^T\dot{\tilde{\beta}}\Bigg).\end{aligned} \tag{2.17}$$

From Eqs. (2.12) and (2.16), we get

$$\begin{aligned}\dot{V}(e,\hat{\alpha},\hat{\beta}) = \Bigg[&D_t^{q-1}\left(D_t^{q-1}\left[F_\imath(x)\;\;(\tilde{\alpha}-\alpha) - G(y)\left(\tilde{\beta}-\beta\right)\right.\right.\\ &-\left(D_t^{q-1}e(t)\right)\frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))}\;-ke]\\ &+\left(D_t^{q-1}e(t)\right)\frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))}\Bigg]^T\\ &+(\tilde{\alpha}-\alpha)^T\dot{\tilde{\alpha}} + \left(\tilde{\beta}-\beta\right)^T\dot{\tilde{\beta}}.\end{aligned} \tag{2.18}$$

Since $\forall q\in[0,1], (1-q)>0$ and $(q-1)<0$. Now using Lemma 2.1 and Eq. (2.12), Eq. (2.18) reduces to

$$\begin{aligned}\dot{V}(e,\hat{\alpha},\hat{\beta}) &= \Bigg[\Big(F_\imath(x)(\tilde{\alpha}-\alpha) - G(y)\left(\tilde{\beta}-\beta\right)\;-\left(D_t^{q-1}e(t)\right)\frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))}\;-ek]\\ &\quad+\left(D_t^{q-1}e(t)\right)\frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))}\Bigg]^T e - (\tilde{\alpha}-\alpha)^T\left([F_\imath(x)]^T e\right)\\ &\quad+\left(\tilde{\beta}-\beta\right)^T\left([G(y)]^T e\right)\\ &=\left[(\tilde{\alpha}-\alpha)^T F_\imath(x)^T - \left(\tilde{\beta}-\beta\right)^T G(y)^T - ke^T\right]e - (\tilde{\alpha}-\alpha)^T[F_\imath(x)]^T e\\ &\quad+\left(\tilde{\beta}-\beta\right)^T[G(y)]^T e\\ &= -ke^T e \le 0.\end{aligned} \tag{2.19}$$

Since V and $\dot{V}$ are positive and negative semi-definite respectively, therefore according to the Lyapunov stability theory [20] the response system (2.8) is both globally and asymptotically synchronized to the drive system (2.9). This completes the proof. □

2.3. Systems description

The fractional-order hyperchaotic Chen system [32] is given by

$$\begin{aligned}\frac{d^\alpha x}{d^\alpha t} &= w + ay - ax, &\qquad \frac{d^\alpha y}{d^\alpha t} &= x(d-z) + cy,\\ \frac{d^\alpha z}{d^\alpha t} &= xy - bz, &\qquad \frac{d^\alpha w}{d^\alpha t} &= yz + rw.\end{aligned} \tag{2.20}$$

where r, a, b, c, d, are positive parameters and x, y, z, and w are state variables. The system (2.20) exhibits hyperchaotic attractor for $a = 35, b = 3, c = 12, d = 7, r = 0.5$ and $0 < \alpha \leq 1$.

The fractional-order chaotic Liu system [10, 21] is described by

$$\begin{aligned}\frac{d^\alpha x_2}{d^\alpha t} &= -a_2 x_2 - g_2 y_2^2,\\ \frac{d^\alpha y_2}{d^\alpha t} &= b_2 y_2 - k_2 x_2 z_2,\\ \frac{d^\alpha z_2}{d^\alpha t} &= -c_2 z_2 + m_2 x_2 y_2,\end{aligned} \tag{2.21}$$

where $a_2 = 1, b_2 = 2.5, c_2 = 5, k_2 = 4, m_2 = 4, g_2 = 1$ and $0 < \alpha \leq 1$.

3. Modified adaptive reduced-order synchronization between the projection $x - y - z$ of the fractional-order hyperchaotic Chen system and the fractional-order Liu system

In order to achieve the behavior of the reduced-order synchronization between the fractional-order hyperchaotic Chen system and fractional-order chaotic Liu system, we take system (2.20) to be the drive system and system (2.21) to be the response system. The variables of the drive system are represented by subscript 1 and the response system by subscript 2. Both the systems are:

$$\begin{aligned}\frac{d^\alpha x_1}{d^\alpha t} &= a_1 y_1 - a_1 x_1 + w_1, & \frac{d^\alpha y_1}{d^\alpha t} &= d_1 x_1 - x_1 z_1 + c_1 y_1,\\ \frac{d^\alpha z_1}{d^\alpha t} &= x_1 y_1 - b_1 z_1, & \frac{d^\alpha w_1}{d^\alpha t} &= y_1 z_1 + r_1 w_1.\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\frac{d^\alpha x_2}{d^\alpha t} &= -a_2 x_2 - g_2 y_2^2 + u_1,\\ \frac{d^\alpha y_2}{d^\alpha t} &= b_2 y_2 - k_2 x_2 z_2 + u_2,\\ \frac{d^\alpha z_2}{d^\alpha t} &= -c_2 z_2 + m_2 x_2 y_2 + u_3.\end{aligned} \tag{3.2}$$

Here the controller is defined as $U = (u_1, u_2, u_3)^T$ which is one of our main objectives. To achieve the intended synchronization between the two systems with unknown parameters we need to design an adaptive controller. The difference of (3.2) and (3.1) gives error dynamical system as below.

$$\begin{aligned}D_t^{q_1} e_1(t) &= -a_2 x_2 - g_2 y_2^2 - a_1(y_1 - x_1) - w_1 + u_1,\\ D_t^{q_2} e_2(t) &= b_2 y_2 - k_2 x_2 z_2 - d_1 x_1 + x_1 z_1 - c_1 y_1 + u_2,\\ D_t^{q_3} e_3(t) &= -c_2 z_2 + m_2 x_2 y_2 - x_1 y_1 + b_1 z_1 + u_3,\end{aligned} \tag{3.3}$$

where $e_1 = x_2 - x_1, e_2 = y_2 - y_1$ and $e_3 = z_2 - z_1$.

Theorem 3.1. *The fractional-order chaotic Liu system* (3.1) *can be synchronized globally and asymptotically for any different initial condition with the projection* $(x - y - z)$ *of the fractional-order hyperchaotic Chen system with the following adaptive controller:*

$$\begin{aligned}u_1 &= a_2 x_2 + y_2^2 + a_1(y_1 - x_1) + w_1 + D_t^{q_1 - 1}\Big[x_2\,\tilde{a}_2 + y_2^2 \tilde{g}_2 + (y_1 - x_1)\,\tilde{a}_1\\ &\quad - \left(D_t^{q_1-1} e_1(t)\right) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))} - e_1\Big],\\ u_2 &= -b_2 y_2 + k_2 x_2 z_2 + d_1 x_1 - x_1 z_1 + c_1 y_1 + D_t^{q_2-1}\Big[-y_2\,\tilde{b}_2 + x_2 z_2 \tilde{k}_2\\ &\quad + x_1 \tilde{d}_1 + y_1 \tilde{c}_1 - \left(D_t^{q_2-1} e_2(t)\right) \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))} - e_2\Big],\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
u_3 = &c_2 z_2 - m_2 x_2 y_2 + x_1 y_1 - b_1 z_1 + D_t^{q_3-1}\left[z_2 \tilde{c}_2 - x_2 y_2 \tilde{m}_2 - z_1 \tilde{b}_1\right. \\
&\left. - \left(D_t^{q_3-1} e_3(t)\right) \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))} - e_3\right],
\end{aligned}
$$

and parameter update rules

$$
\begin{array}{lll}
\dot{\hat{a}}_1 = -(y_1 - x_1)e_1, & \dot{\hat{b}}_1 = z_1 e3, & \dot{\hat{c}}_1 = -y_1 e_2, \\
\dot{\hat{d}}_1 = -x_1 e_2, & \dot{\hat{a}}_2 = -x_2 e_1, & \dot{\hat{b}}_2 = y_2 e_2, \\
\dot{\hat{c}}_2 = -z_2 e_3, & \dot{\hat{m}}_2 = x_2 y_2 e_3, & \dot{\hat{g}}_2 = -y_2^2 e_2, \\
\dot{\hat{k}}_2 = -x_2 z_2 e_2. & &
\end{array} \tag{3.5}
$$

where $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{m}_2, \hat{g}_2, \hat{k}_2$ are estimates of $a_1, b_1, c_1, d_1, a_2, b_2, c_2, m_2, g_2, k_2$, respectively.

Proof. Applying control law equation (3.4) to Eq. (3.3) yields the resulting closed-loop error dynamical system as follows:

$$
\begin{aligned}
D_t^{q_1} e_1(t) &= D_t^{q_1-1}\left[x_2\, \hat{a}_2 + y_2^2 \hat{g}_2 + (y_1 - x_1)\, \hat{a}_1 - \left(D_t^{q_1-1} e_1(t)\right) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))} - e_1\right], \\
D_t^{q_2} e_2(t) &= D_t^{q_2-1}\left[-y_2\, \hat{b}_2 + x_2 z_2 \hat{k}_2 + x_1 \hat{d}_1 + y_1 \hat{c}_1 - \left(D_t^{q_2-1} e_2(t)\right) \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))} - e_2\right], \\
D_t^{q_3} e_3(t) &= D_t^{q_3-1}\left[z_2 \hat{c}_2 - x_2 y_2 \hat{m}_2 - z_1 \hat{b}_1 - \left(D_t^{q_3-1} e_3(t)\right) \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))} - e_3\right],
\end{aligned} \tag{3.6}
$$

where $\hat{a}_1 = \tilde{a}_1 - a_1, \hat{b}_1 = \tilde{b}_1 - b_1, \hat{c}_1 = \tilde{c}_1 - c_1, \hat{d}_1 = \tilde{d}_1 - d_1, \hat{a}_2 = \tilde{a}_2 - a_2, \hat{b}_2 = \tilde{b}_2 - a_2, \hat{c}_2 = \tilde{c}_2 - c_2, \hat{m}_2 = \tilde{m}_2 - m_2, \hat{g}_2 = \tilde{g}_2 - g_2$ and $\hat{k}_2 = \tilde{k}_2 - k_2$.

Consider the following Lyapunov function candidate

$$
V = \frac{1}{2}\left(e^T e + \hat{a}_1 + \hat{b}_1 + \hat{c}_1 + \hat{d}_1 + \hat{a}_2 + \hat{b}_2 + \hat{c}_2 + \hat{m}_2 + \hat{g}_2 + \hat{k}_2\right), \tag{3.7}
$$

then the time derivative of V along the solution of error dynamical system equation (3.6) gives

$$
\dot{V} = (e^T e + \hat{a}_1 \dot{\hat{a}}_1 + \hat{b}_1 \dot{\hat{b}}_1 + \hat{c}_1 \dot{\hat{c}}_1 + \hat{d}_1 \dot{\hat{d}}_1 + \hat{a}_2 \dot{\hat{a}}_2 + \hat{b}_2 \dot{\hat{b}}_2 + \hat{c}_2 \dot{\hat{c}}_2 + \hat{m}_2 \dot{\hat{m}}_2 + \hat{g}_2 \dot{\hat{g}}_2 + \hat{k}_2 \dot{\hat{k}}_2). \tag{3.8}
$$

Using Lemma 2.2 in Eq. (3.7) we get

$$
\begin{aligned}
\dot{V} \quad = \quad & \left(\left[D_t^{1-q_1}\ (D_t^{q_1} e_1(t)) + (D_t^{q_1} e_1(t)) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))}\right] e_1\right. \\
& + \left(\left[D_t^{1-q_2}\ (D_t^{q_2} e_2(t)) + (D_t^{q_2} e_2(t)) \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))}\right] e_2\right. \\
& + \left(\left[D_t^{1-q_3}\ (D_t^{q_3} e_3(t)) + (D_t^{q_3} e_3(t)) \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))}\right] e_3\right. \\
& + \hat{a}_1 \dot{\hat{a}}_1 + \hat{b}_1 \dot{\hat{b}}_1 + \hat{c}_1 \dot{\hat{c}}_1 + \hat{d}_1 \dot{\hat{d}}_1 + \hat{a}_2 \dot{\hat{a}}_2 + \hat{b}_2 \dot{\hat{b}}_2 + \hat{c}_2 \dot{\hat{c}}_2 + \hat{m}_2 \dot{\hat{m}}_2 \\
& + \hat{g}_2 \dot{\hat{g}}_2 + \hat{k}_2 \dot{\hat{k}}_2 \\
= \quad & \left(\left[D_t^{1-q_1}\left(D_t^{q_1-1}\left[x_2\, \hat{a}_2 + y_2^2 \hat{g}_2 + (y_1 - x_1)\, \hat{a}_1 - \left(D_t^{q_1-1} e_1(t)\right) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))} - e_1\right]\right)\right.\right. \\
& \left. + (D_t^{q_1} e_1(t)) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))}\right] e_1
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
&+\left(\left[D_t^{1-q_2}\left(D_t^{q_2-1}\left[-y_2\,\hat{b}_2+x_2z_2\hat{k}_2+x_1\hat{d}_1+y_1\hat{c}_1-\left(D_t^{q_2-1}e_2\left(t\right)\right)\times\frac{(t)^{-(q_2-1)-1}}{\Gamma\left(-\left(q_2-1\right)\right)}-e_2\right]\right)\right.\right.\\
&\left.+\left(D_t^{q_2}e_2(t)\right)\times\frac{(t)^{-(q_2-1)-1}}{\Gamma\left(-\left(q_2-1\right)\right)}\right]e_2\\
&+\left(\left[D_t^{1-q_3}\left(D_t^{q_3-1}\left[z_2\hat{c}_2-x_2y_2\hat{m}_2-z_1\hat{b}_1-\left(D_t^{q_3-1}e_3\left(t\right)\right)\times\frac{(t)^{-(q_3-1)-1}}{\Gamma\left(-\left(q_3-1\right)\right)}-e_3\right]\right)\right.\right.\\
&\left.+\left(D_t^{q_3}e_3(t)\right)\times\frac{(t)^{-(q_3-1)-1}}{\Gamma\left(-\left(q_3-1\right)\right)}\right]e_3+\hat{a}_1\dot{\hat{a}}_1+\hat{b}_1\dot{\hat{b}}_1+\hat{c}_1\dot{\hat{c}}_1+\hat{d}_1\dot{\hat{d}}_1+\hat{a}_2\dot{\hat{a}}_2+\hat{b}_2\dot{\hat{b}}_2+\hat{c}_2\dot{\hat{c}}_2\\
&+\hat{m}_2\dot{\hat{m}}_2+\hat{g}_2\dot{\hat{g}}_2+\hat{k}_2\dot{\hat{k}}_2.
\end{aligned}
$$

Since $\forall q\in[0,1],(1-q)>0$ and $(q-1)<0$. Now using Lemma 2.1 Eq. (3.9) reduces to

$$
\begin{aligned}
\dot{V}=&[x_2\hat{a}_2+y_2^2\hat{g}_2+(y_1-x_1)\hat{a}_1-e_1]e_1+[-y_2\hat{b}_2+x_2z_2\hat{k}_2+x_1\hat{d}_1+y_1\hat{c}_1-e_2]e_2\\
&+[z_2\hat{c}_2-x_2y_2\hat{m}_2-z_1\hat{b}_1-e_3]e_3+\hat{a}_1(-(y_1-x_1)e_1)+\hat{b}_1(z_1e3)+\hat{c}_1(-y_1e_2)\\
&+\hat{d}_1(x_1e_2)+\hat{a}_2(-x_2e_1)+\hat{b}_2(y_2e_2)+\hat{c}_2(-z_2e_3)+\hat{m}_2(x_2y_2e_3)+\hat{g}_2(-y_2^2e_2)\\
&+\hat{k}_2(-x_2z_2e_2),\\
=&-e^Te\leq 0.
\end{aligned}
\tag{3.10}
$$

Since V is positive definite and $\dot{V}$ is negative definite in the neighborhood of zero solution of system equation (3.6), it follows $\lim\limits_{t\to\infty}\|e\left(t\right)\|=0$. Therefore system (3.2) can synchronize system (3.1) asymptotically. This completes the proof. □

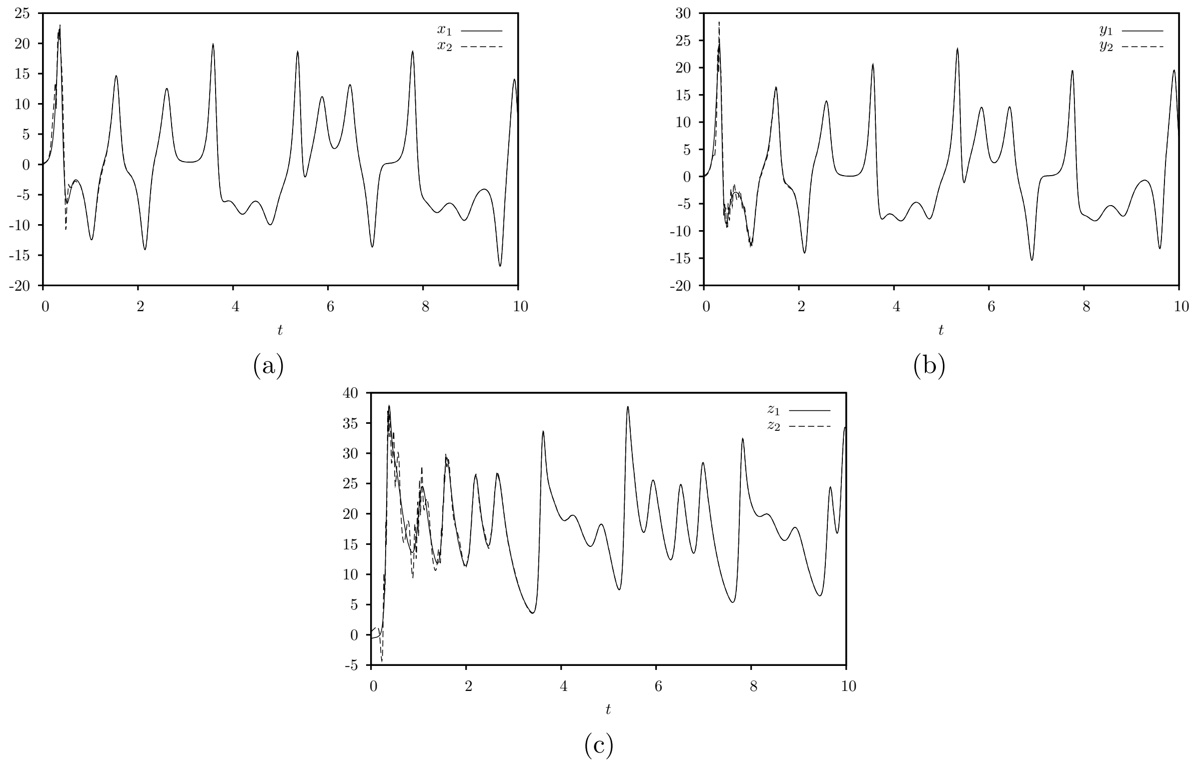

Figure 1: State trajectories of drive system (3.1) and response system (3.1): (a) Signals x_1 and x_2; (b) signals y_1 and y_2; and (c) signals z_1 and z_2.

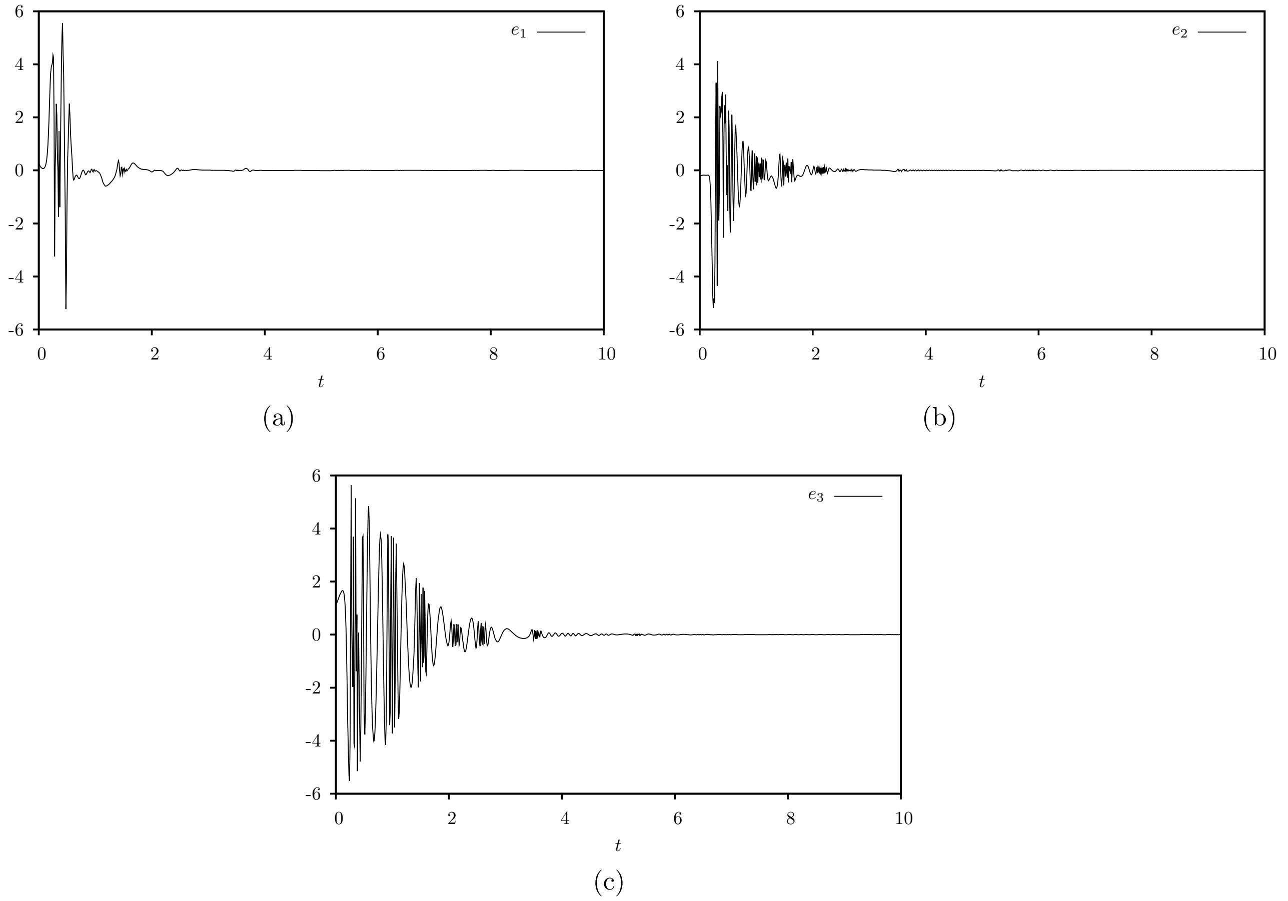

Figure 2: The error signals $e_1; e_2; e_3$ of the hyperchaotic Chen and Liu systems under the controller (3.4) and the parameters update law (3.5).

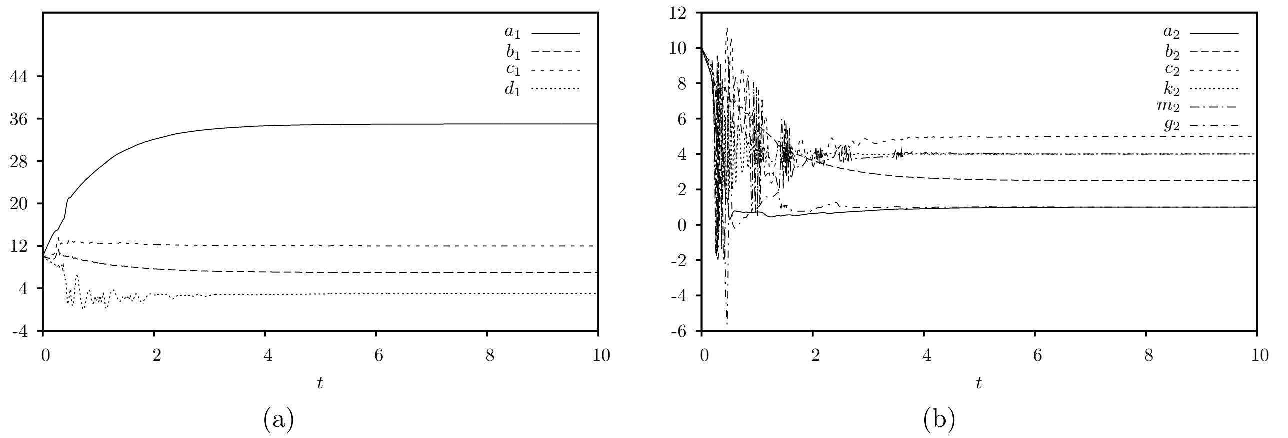

Figure 3: Changing parameters $a_1; b_1; c_1; d_1$ and $a_2; b_2; c_2; k_2; m_2; g_2$ of the hyperchaotic Chen and Liu systems with time t.

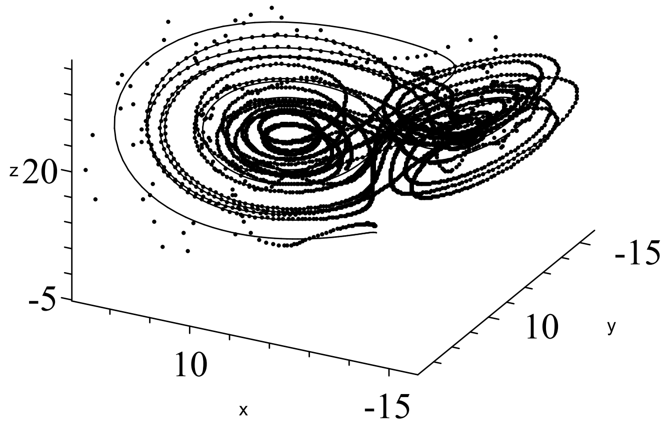

Figure 4: Hyperchaotic Chen system (solid line) and the controlled Liu system (dotted line).

4. Numerical simulations

To verify and demonstrate the effectiveness of the proposed method, we discuss the simulation results for the reduced–order synchronization problem between the $(x - y - z)$ projective of the hyperchaotic Chen and the Liu systems. In the numerical simulations, the fourth-order Runge-Kutta method is used to solve both systems with time step size 0.001. Assume that the initial conditions, $(x_1(0) = -0.1, y_1(0) = 0.2, z_1(0) = -0.6, w_1(0) = 0.4)$ and $(x_2(0) = 0.2, y_2(0) = 0, z_2(0) = 0.5)$ are employed. Hence the error system has the initial values $(e_1(0) = 0.3, e_2(0) = -0.2, e_3(0) = 1.1)$. The unknown parameters are chosen as $a_1 = 35, b_1 = 3, c_1 = 12, d_1 = 7, r_1 = 0.5$ and $a_2 = 1, b_2 = 2.5, c_2 = 5, k_2 = 4, m_2 = 4, g_2 = 1$, the fractional order is chosen as $\alpha = 0.95$, in simulations so that the both systems exhibits a chaotic behavior. Reduced-order synchronization of the systems (3.1) and (3.2) via adaptive control law (3.4) and (3.5) with the initial estimated parameters $\hat{a}_1(0) = 10, \hat{b}_1(0) = 10, \hat{c}_1(0) = 10, \hat{d}_1(0) = 10$ and $\hat{a}_2(0) = 10, \hat{b}_2(0) = 10, \hat{c}_2(0) = 10, \hat{m}_2(0) = 10, \hat{g}_2(0) = 10, \hat{k}_2(0) = 10$ are shown in Figs. (1)–(2). Figs. (1) and (2) display the state response and the reduced-order synchronization errors of systems (3.1) and (3.2). Fig. (3) shows that the estimates $\hat{a}_1(t), \hat{b}_1(t), \hat{c}_1(t), \hat{d}_1(t)$ and $\hat{a}_2(t), \hat{b}_2(t), \hat{c}_2(t), \hat{m}_2(t), \hat{g}_2(t), \hat{k}_2(t)$ of the unknown parameters converge to $a_1 = 35, b_1 = 3, c_1 = 12, d_1 = 7, r_1 = 0.5$ and $a_2 = 1, b_2 = 2.5, c_2 = 5, k_2 = 4, m_2 = 4, g_2 = 1$ as $t \to \infty$. Fig. (4) shows that the Liu system is control to the $(x - y - z)$ projective of the hyperchaotic Chen system.

5. Conclusion

In this paper we study the reduced-order synchronization of fractional-order chaotic systems with uncertain parameters. The reduced-order synchronization problem is demonstrated and proved using rigorous analytical and numerical procedures. The synchronization of the dynamical evolution of a 3rd-order fractional chaotic system was realized with the canonical projection of a 4th-order fractional chaotic system even though their parameters were unknown. This was based upon the parameters modulation and the adaptive control techniques. The proven techniques was applied to the fractional order hyperchaotic Chen system (4th-order) with fractional order Liu system (3rd order). The theoretical analysis and numerical simulations have verified and supported our assumptions.

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