Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 1858–1871
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed points and quadratic ρ-functional equations

Choonkil Park[a], Sang Og Kim[b,*]

[a]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea.
[b]Department of Mathematics, Hallym University, Chuncheon 24252, Korea.

Communicated by R. Saadati

Abstract

In this paper, we solve the quadratic ρ-functional equations

$$f(x+y)+f(x-y)-2f(x)-2f(y)=\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right), \tag{1}$$

where ρ is a fixed non-Archimedean number or a fixed real or complex number with $\rho \neq -1, 2$, and

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)=\rho(f(x+y)+f(x-y)-2f(x)-2f(y)), \tag{2}$$

where ρ is a fixed non-Archimedean number or a fixed real or complex number with $\rho \neq -1, \frac{1}{2}$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional equations (1) and (2) in non-Archimedean Banach spaces and in Banach spaces.

Keywords: Hyers-Ulam stability, non-Archimedean normed space, fixed point, quadratic ρ-functional equation.
2010 MSC: 46S10, 39B62, 39B52, 47H10, 47S10, 12J25.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [15] gave a first affirmative partial

*Corresponding author
Email addresses: baak@hanyang.ac.kr (Choonkil Park), sokim@hallym.ac.kr (Sang Og Kim)

Received 2015-10-01

answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. Gajda [11] following the same approach as in Rassias [23], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [11], as well as by Rassias and Šemrl [24] that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [11], as well as of Rassias and Šemrl [24] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. Găvruta [12], who among others studied the Hyers-Ulam stability of functional equations (cf. the books of Czerwik [8, 9], Hyers, Isac and Th. M. Rassias [16]). The hyperstability of the Cauchy equation was proved by Brzdek [3].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [25] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. See [13, 14, 26, 27] for more functional equations. The survey on the Hyers-Ulam stability of functional equations was given by Brillouet-Bulluot, Brzdek and Cieplinski [2].

The functional equation

$$2f\left(\frac{x+y}{2}\right)+2\left(\frac{x-y}{2}\right)=f(x)+f(y)$$

is called a *Jensen type quadratic equation*.

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \le |r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1 ([19])**.** Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\| \qquad (r\in K, x\in X)$;
(iii) the strong triangle inequality

$$\|x+y\| \le \max\{\|x\|,\|y\|\}, \qquad \forall x,y\in X$$

holds. Then $(X,\|\cdot\|)$ is called a *non-Archimedean normed space*.

We recall a fundamental result in fixed point theory.

Theorem 1.2 ([4, 10])**.** *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th. M. Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 20, 21, 22]).

In Section 2, we solve the quadratic functional equation (1) in vector spaces and prove the Hyers-Ulam stability of the quadratic functional equation (1) in non-Archimedean Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic functional equation (2) in vector spaces and prove the Hyers-Ulam stability of the quadratic functional equation (2) in non-Archimedean Banach spaces by using the fixed point method.

In Section 4, we prove the Hyers-Ulam stability of the quadratic functional equation (1) in Banach spaces by using the fixed point method.

In Section 5, we prove the Hyers-Ulam stability of the quadratic functional equation (2) in Banach spaces by using the fixed point method.

2. Quadratic ρ-functional equation (1) in non-Archimedean Banach spaces

Throughout Sections 2 and 3, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$ and let ρ be a fixed non-Archimedean number with $\rho \neq -1, 2$.

Lemma 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0 \tag{2.1}$$

for all $x, y \in X$ if and only if the mapping $f : X \to Y$ satisfies

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) = 0 \tag{2.2}$$

for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (2.1). Letting $x = y = 0$ in (2.1), we get $f(0) = 0$. Letting $y = x$ in (2.1), we get $f(2x) - 4f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus $f\left(\frac{x}{2}\right) = \frac{1}{4} f(x)$ for all $x \in X$. So $f : X \to Y$ satisfies (2.2). Assume that $f : X \to Y$ satisfies (2.2). Letting $x = y = 0$ in (2.2), we get $f(0) = 0$. Letting $y = 0$ in (2.2), we get $4f\left(\frac{x}{2}\right) = f(x)$ for all $x \in X$, and so $f(2x) = 4f(x)$ for all $x \in X$. So $f : X \to Y$ satisfies (2.1). □

We solve the quadratic ρ-functional equation (1) in vector spaces.

Lemma 2.2. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \tag{2.3}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.3).

Letting $x = y = 0$ in (2.3), we get $-2f(0) = 2\rho f(0)$. So $f(0) = 0$. Letting $y = x$ in (2.3), we get $f(2x) - 4f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.4}$$

for all $x \in X$.

It follows from (2.3) and (2.4) that

$$\begin{aligned} f(x+y) + f(x-y) - 2f(x) - 2f(y) =& \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \\ =& \frac{\rho}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

Now we prove the Hyers-Ulam stability of the quadratic ρ-functional equation (2.3) in non-Archimedean Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{|4|}\varphi(x, y) \tag{2.5}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\|f(x+y) + f(x-y) - 2f(x) - 2f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \le \varphi(x, y) \tag{2.6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{L}{|4|(1-L)}\varphi(x, x) \tag{2.7}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.6), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x, x) \tag{2.8}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x, x), \quad \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [18]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \le |4|\varepsilon\varphi\left(\frac{x}{2},\frac{x}{2}\right) \le |4|\varepsilon\frac{L}{|4|}\varphi(x,x) \le L\varepsilon\varphi(x,x)$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.8) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2},\frac{x}{2}\right) \le \frac{L}{|4|}\varphi(x,x)$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{|4|}$.

By Theorem 1.2, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{2.9}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.9) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - Q(x)\| \le \mu\varphi(x,x)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality $\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$ for all $x \in X$;

(3) $d(f,Q) \le \frac{1}{1-L}d(f,Jf)$, which implies

$$\|f(x) - Q(x)\| \le \frac{L}{|4|(1-L)}\varphi(x,x)$$

for all $x \in X$.

It follows from (2.5) and (2.6) that

$$\begin{aligned}
&\left\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) - \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\
&= \lim_{n\to\infty} |4|^n \left\|f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right. \\
&\quad \left.-\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\
&\le \lim_{n\to\infty} |4|^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic. □

Corollary 2.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\| f(x+y) + f(x-y) - 2f(x) - 2f(y) - \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| \le \theta(\|x\|^r + \|y\|^r) \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2\theta}{|2|^r - |4|}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = |2|^{2-r}$ and we get desired result. □

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le |4|L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$ Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{|4|(1-L)}\varphi(x, x)$$

for all $x \in X$.

Proof. It follows from (2.8) that

$$\left\| f(x) - \frac{1}{4}f(2x) \right\| \le \frac{1}{|4|}\varphi(x, x) \tag{2.11}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (2.11) that $d(f, Jf) \le \frac{1}{|4|}$. So

$$\|f(x) - Q(x)\| \le \frac{1}{|4|(1-L)}\varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.10). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{|4| - |2|^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = |2|^{r-2}$ and we get desired result. □

3. Quadratic ρ-functional equation (2) in non-Archimedean Banach spaces

Let $|2| \neq 1$ and let ρ be a fixed non-Archimedean number with $\rho \neq -1, \frac{1}{2}$.
We solve the quadratic ρ-functional equation (2) in vector spaces.

Lemma 3.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1). Letting $x = y = 0$ in (3.1), we get $2f(0) = -2\rho f(0)$. So $f(0) = 0$. Letting $y = 0$ in (3.1), we get

$$4f\left(\frac{x}{2}\right) - f(x) = 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\frac{1}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) =& 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ =& \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{|4|}\varphi(x, y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \leq \varphi(x, y) \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{1-L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x, 0) \tag{3.4}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x,0), \quad \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [18]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.4) that $d(f, Jf) \le 1$. By Theorem 1.2, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{3.5}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.5) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - Q(x)\| \le \mu\varphi(x,0)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality $\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$ for all $x \in X$;

(3) $d(f,Q) \le \frac{1}{1-L} d(f, Jf)$, which implies $\|f(x) - Q(x)\| \le \frac{1}{1-L}\varphi(x,0)$ for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \le \theta(\|x\|^r + \|y\|^r) \tag{3.6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{|2|^r\theta}{|2|^r - |4|}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = |2|^{2-r}$ and we get desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le |4|L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$ Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{L}{1-L}\varphi(x,0)$$

for all $x \in X$.

Proof. It follows from (3.4) that

$$\left\|f(x)-\frac{1}{4}f(2x)\right\| \le \frac{1}{|4|}\varphi(2x,0) \le L\varphi(x,0) \tag{3.7}$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 3.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (3.7) that $d(f,Jf) \le L$. So $d(f,Q) \le \frac{L}{1-L}d(f,Jf)$, which implies

$$\|f(x)-Q(x)\| \le \frac{L}{1-L}\varphi(x,0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x)-Q(x)\| \le \frac{|2|^r\theta}{|4|-|2|^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = |2|^{r-2}$ and we get desired result. □

4. Quadratic ρ-functional equation (1) in Banach spaces

Throughout Sections 4 and 5, assume that X is a normed space and that Y is a Banach space. Let ρ be a fixed real or complex number with $\rho \ne -1, 2$.

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (2.3) in Banach spaces.

Theorem 4.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \le \frac{L}{4}\varphi(x,y) \tag{4.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\| \le \varphi(x,y) \tag{4.2}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \le \frac{L}{4(1-L)}\varphi(x,x) \tag{4.3}$$

for all $x \in X$.

Proof. Letting $x = y$ in (4.2), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x, x) \tag{4.4}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x, x), \quad \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [18]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (4.4) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{4}\varphi(x, x),$$

for all $x \in X$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.2, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{4.5}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (4.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \leq \mu\varphi(x, x)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality $\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$ for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies $\|f(x) - Q(x)\| \leq \frac{L}{4(1-L)}\varphi(x, x)$ for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 4.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y) - \rho\left(2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) - f(x) - f(y)\right)\| \leq \theta(\|x\|^r + \|y\|^r) \tag{4.6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.1 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{2-r}$ and we get desired result. □

Theorem 4.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$ Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.2)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{1}{4(1-L)}\varphi(x,x)$$

for all $x \in X$.

Proof. It follows from (4.4) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(x,x) \tag{4.7}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 4.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (4.7) that $d(f, Jf) \le \frac{1}{4}$. So

$$\|f(x) - Q(x)\| \le \frac{1}{4(1-L)}\varphi(x,x)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.3 and 4.1. □

Corollary 4.4. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (4.6)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.3 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-2}$ and we get desired result. □

5. Quadratic ρ-functional equation (2) in Banach spaces

Let ρ be a fixed real or complex number with $\rho \ne -1, \frac{1}{2}$.

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional equation (3.1) in Banach spaces.

Theorem 5.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x,y)$$

for all $x, y \in X$.

Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \le \varphi(x,y) \tag{5.1}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{1-L}\varphi(x,0) \tag{5.2}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (5.1), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x,0) \tag{5.3}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x,0), \quad \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [18]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (5.3) that $d(f, Jf) \le 1$.

By Theorem 1.2, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{5.4}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (5.4) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - Q(x)\| \le \mu\varphi(x,0)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality $\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$ for all $x \in X$;

(3) $d(f,Q) \le \frac{1}{1-L} d(f,Jf)$, which implies $\|f(x) - Q(x)\| \le \frac{1}{1-L}\varphi(x,0)$ for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.3 and 4.1. □

Corollary 5.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \le \theta(\|x\|^r + \|y\|^r) \tag{5.5}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.1 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{2-r}$ and we get desired result. □

Theorem 5.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$ Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (5.1). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{L}{1-L}\varphi(x,0)$$

for all $x \in X$.

Proof. It follows from (5.3) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(2x,0) \le L\varphi(x,0) \tag{5.6}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 5.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (5.6) that $d(f, Jf) \le L$. So

$$\|f(x) - Q(x)\| \le \frac{L}{1-L}\varphi(x,0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 5.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (5.5). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2^r\theta}{4-2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.3 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-2}$ and we get desired result. □

Acknowledgement

S. O. Kim was supported by Hallym University Research Fund, 2015 (HRF-201510-009).

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66. 1

[2] N. Brillouet-Belluot, J. Brzdek, K. Cieplinski, *On some recent developments in Ulam's type stability*, Abstr. Appl. Anal., **2012** (2012), 41 pages. 1

[3] J. Brzdek, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar., **141** (2013), 58–67. 1
[4] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math., **2003** (2003), 7 pages. 1.2
[5] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber., **346** (2004), 43–52. 1
[6] L. Cădariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl., **2008** (2008), 15 pages. 1
[7] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math., **27** (1984), 76–86. 1
[8] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Co., Inc., River Edge, NJ, (2002). 1
[9] S. Czerwik, *Stability of Functional Equations of Ulam-Hyers-Rassias Type*, Hadronic Press, Palm Harbor, Florida, (2003). 1
[10] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74** (1968), 305–309. 1.2
[11] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci., **14** (1991), 431–434. 1
[12] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436. 1
[13] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math., **62** (2001), 303–309. 1
[14] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl., **5** (2002), 707–710. 1
[15] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224. 1
[16] D. H. Hyers, G. Isac, Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, (1998). 1
[17] G. Isac, Th. M. Rassias, *Stability of ψ-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci., **19** (1996), 219–228. 1
[18] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl., **343** (2008), 567–572. 2, 3, 4, 5
[19] M. S. Moslehian, G. Sadeghi, *A Mazur-Ulam theorem in non-Archimedean normed spaces*, Nonlinear Anal., **69** (2008), 3405–3408. 1.1
[20] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl., **2007** (2007), 15 pages. 1
[21] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory Appl., **2008** (2008), 9 pages. 1
[22] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, **4** (2003), 91–96. 1
[23] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300. 1
[24] Th. M. Rassias, P. Šemrl, *On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl., **173** (1993), 325–338. 1
[25] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano, **53** (1983), 113–129. 1
[26] N. E. Tovmasian, V. H. Iritsyan, *Some functional equations with shifts in a class of analytic functions*, J. Contemp. Math. Anal., **35** (2000), 15–25. 1
[27] N. E. Tovmasian, V. S. Zakarian, *Some singular functional equations in a class of analytic functions*, J. Contemp. Math. Anal., **35** (2000), 40–56. 1
[28] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ., New York, (1960). 1