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Iterative common solutions of fixed point and variational inequality problems

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Abstract

In this paper, fixed point and variational inequality problems are investigated based on a viscosity approximation method. Strong convergence theorems are established in the framework of Hilbert spaces.

Keywords: Inverse-strongly monotone operator, nonexpansive mapping, variational inequality, fixed point.
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1. Introduction and Preliminaries

Monotone variational inequality theory, which was introduced in sixties, has emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in finance, economics, optimization, engineering and medicine see, for example, [1], [8], [9]-[11], [17], [25], [26] and the references therein. This field is dynamic and is experiencing an explosive growth in both theory and applications; as a consequence, research techniques and problems are drawn from various fields. The ideas and techniques of monotone variational inequalities are being applied in a variety of diverse areas of sciences and prove to be productive and innovative. It has been shown that this theory provides the most natural, direct, simple, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems, see, for example, [2], [5]-[7], [18]-[21], [23], [24], [29] and the references therein. Recently, fixed-point methods have been extensively investigated for solving monotone variational inequalities. Among the fixed-point algorithms, Mann-like iterative algorithms are efficient for solving several nonlinear problems.

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However, Mann-like iterative algorithms are only weakly convergent even in Hilbert spaces; see [12] for more details and the references therein. In many disciplines, including economics [17], quantum physics [10], image recovery [8] and control theory [11], problems arises in infinite dimension spaces. In such problems, norm convergence (strong convergence) is often much more desirable than weak convergence, for it translates the physically tangible property that the energy $\|x_n - x\|$ of the error between the iterate x_n and the solution x eventually becomes arbitrarily small. Halpern-like iterative algorithms, which are strongly convergent, have been extensively investigated. Recently, Moudafi [22] introduced a viscosity method for solving fixed points of nonlinear operators in the framework of Hilbert spaces. He showed that the convergence point is not only a fixed point of nonlinear operators but an unique solution to some monotone variational inequality; see [22] for more details and the references therein. In this paper, we consider a Moudafi's viscosity iterative method for solving common solutions of monotone variational inequality and fixed point problems. Strong convergence theorems of common solutions are established in the framework of Hilbert spaces. The results presented in this paper mainly improve the corresponding results in [13], [15], [16], [30]-[33].

Let H be a real Hilbert space with inner product $\langle x, y\rangle$ and induced norm $\|x\| = \sqrt{\langle x, x\rangle}$ for $x, y \in H$. Let C be a nonempty closed and convex subset of H. Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle \geq 0 \quad \forall x, y \in C.$$

A is said to be inverse-strongly monotone iff there exists a positive constant $L > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq L\|Ax - Ay\|^2 \quad \forall x, y \in C.$$

From the definition, we see that every inverse-strongly monotone mapping is also monotone and Lipschitz continuous.

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \quad \forall y \in C.$$

The solution set of the variational inequality is denoted by $VI(C, A)$ in this paper. One of classical methods of solving the variational inequality, is the gradient algorithm $P_C(I - r_nA)x_n$, $n = 0, 1, \cdots$, where $r_n > 0$.

Let $S : C \to C$ be a mapping. Recall that S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C.$$

S is said to be α-contractive iff there exists a constant $0 \leq \alpha < 1$ such that

$$\|Sx - Sy\| \leq \alpha\|x - y\| \quad \forall x, y \in C.$$

In this paper, we use $F(S)$ to stand for the set of fixed points of S. For the class of nonexpansive mappings, we know that $F(S)$ is nonempty if C is a weakly compact subset of reflexive Banach spaces; see [3] and the references therein.

Lemma 1.1 ([4])**.** *Let C be a closed convex subset of a Hilbert space H. Let $\{T_i\}_{i=1}^r$, where r is some positive integer, be a sequence of nonexpansive mappings on C. Suppose $\cap_{i=1}^r F(T_i)$ is nonempty. Let $\{\mu_i\}$ be a sequence of positive numbers with $\sum_{i=1}^r = 1$. Then a mapping S on C defined by $Sx = \sum_{i=1}^r \mu_i T_i x$ for $x \in C$ is well defined, nonexpansive and $F(S) = \cap_{i=1}^r F(T_i)$ holds.*

Lemma 1.2 ([28])**.** *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^\infty \gamma_n = \infty$ *and* $\lim_{n\to\infty} \gamma_n = 0$*;*

(ii) $\sum_{n=1}^{\infty}|\delta_n| < \infty$ *or* $\limsup_{n\to\infty}\delta_n/\gamma_n \leq 0$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 1.3 ([27])**.** *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and let* $\{\beta_n\}$ *be a sequence in* $(0,1)$ *with*

$$0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1.$$

Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_n - y_{n+1}\| - \|x_n - x_{n+1}\|) \leq 0.$$

Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 1.4 ([3])**.** *Let* H *be a real Hilbert space,* C *be a nonempty closed convex subset of* H *and* $S : C \to C$ *be a nonexpansive mapping. Then* $I - S$ *is demiclosed at zero, that is,* $\{x_n\}$ *converges weakly to some point* x *and* $\{x_n - Tx_n\}$ *converges in norm to* 0. *Then* $x = Tx$.

2. Main results

Theorem 2.1. *Let* H *be a real Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* $A_i : C \to H$ *be a* μ_i*-inverse-strongly monotone mapping for each* $1 \leq i \leq r$, *where* r *is some positive integer. Let* $S : C \to C$ *be a nonexpansive mapping with a fixed point and let* $f : C \to C$ *be a fixed* α*-contractive mapping. Assume that* $\mathcal{F} := \cap_{i=1}^{r} VI(C, A_i) \cap F(S) \neq \emptyset$. *Let* $\{\lambda_i\}$ *be real numbers in* $(0, 2\mu_i)$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be real sequences in* $(0,1)$. *Let* $\{x_n\}$ *be a sequence defined by the following manner:*

$$\begin{cases} x_1 \in C, \\ y_{n,i} \approx P_C(x_n - \lambda_i A_i x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S\sum_{i=1}^{r}\eta_i y_{n,i}, \quad n \geq 1, \end{cases} \tag{2.1}$$

where the criterion for the approximate computation of $y_{n,i}$ *in* C *is* $\|y_{n,i} - P_C(x_n - \lambda_i A_i x_n)\| \leq e_{n,i}$, *where* $\lim_{n\to\infty}\|e_{n,i}\| = 0$ *for each* $1 \leq i \leq r$. *Assume that the above control sequences satisfies the following conditions:*

(a) $\alpha_n + \beta_n + \gamma_n = \sum_{i=1}^{r}\eta_i = 1\ \forall n \geq 1$;

(b) $1 > \limsup_{n\to\infty}\beta_n \geq \liminf_{n\to\infty}\beta_n > 0$;

(c) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$.

Then sequence $\{x_n\}$ *converges in norm to a common solution* p, *which is also the unique solution to the following variational inequality:*

$$\langle f(p) - p, p - q\rangle \geq 0\ \forall q \in \mathcal{F}.$$

Proof. First, we show sequences $\{x_n\}$ is bounded. For any $x, y \in C$, we see

$$\begin{aligned} \|(I-\lambda_i A_i)x - (I-\lambda_i A_i)y\|^2 &= \|x-y\|^2 - 2\lambda_i\langle A_i x - A_i y, x-y\rangle + \lambda_i^2\|A_i x - A_i y\|^2 \\ &\leq \|x-y\|^2 - \lambda_i(2\mu_i - \lambda_i)\|A_i x - A_i y\|^2. \end{aligned}$$

Using restriction $\lambda_i \in (0, 2\mu_i)$, we find that $I - \lambda_i A_i$ is nonexpansive. Fixing $x^* \in \mathcal{F}$, we have from Lemma 1.1 that

$$\|x_{n+1} - x^*\| \leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|S\sum_{i=1}^{r}\eta_i y_{n,i} - x^*\|$$

$$\begin{aligned}
&\le \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\|+\beta_n\|x_n-x^*\| \\
&\quad +\gamma_n\sum_{i=1}^{r}\eta_i\|y_{n,i}-P_C(x^*-\lambda_iA_ix^*)\| \\
&\le \alpha_n\alpha\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|+\beta_n\|x_n-x^*\| \\
&\quad +\gamma_n\sum_{i=1}^{r}\eta_i\|e_{n,i}\|+\gamma_n\sum_{i=1}^{r}\eta_i\|P_C(x_n-\lambda_iA_ix_n)-x^*\| \\
&\le \big(1-\alpha_n(1-\alpha)\big)\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|+\gamma_n\sum_{i=1}^{r}\eta_i\|e_{n,i}\| \\
&\le \big(1-\alpha_n(1-\alpha)\big)\|x_n-x^*\|+\alpha_n(1-\alpha)\frac{\|f(x^*)-x^*\|}{1-\alpha}+\sum_{i=1}^{r}\eta_i\|e_{n,i}\| \\
&\le \max\{\|x_n-x^*\|,\frac{\|f(x^*)-x^*\|}{1-\alpha}\}+\sum_{i=1}^{r}\eta_i\|e_{n,i}\|.
\end{aligned}$$

By mathematical induction, we have

$$\|x_{n+1}-x^*\|\le \max\{\|x_n-x^*\|,\frac{\|f(x^*)-x^*\|}{1-\alpha}\}+\sum_{i=1}^{r}\eta_i(\sum_{n=0}^{\infty}\|e_{n,i}\|)<\infty.$$

This shows that sequence $\{x_n\}$ is bounded. Note that

$$\begin{aligned}
\|y_{n+1,i}-y_{n,i}\| &\le \|y_{n+1,i}-P_C(x_{n+1}-\lambda_iA_ix_{n+1})\|+\|P_C(x_{n+1}-\lambda_iA_ix_{n+1})-P_C(x_n-\lambda_iA_ix_n)\| \\
&\quad +\|P_C(x_n-\lambda_iA_ix_n)-y_{n,i}\| \\
&\le \|e_{n+1,i}\|+\|x_{n+1}-x_n\|+\|e_{n,i}\|.
\end{aligned}$$

Putting $y_n=\sum_{i=1}^{r}\eta_iy_{n,i}$, we have

$$\begin{aligned}
\|y_{n+1}-y_n\| &\le \sum_{i=1}^{r}\eta_i\|y_{n+1,i}-y_{n,i}\| \\
&\le \sum_{i=1}^{r}\eta_i(\|e_{n+1,i}\|+\|e_{n,i}\|)+\|x_{n+1}-x_n\|.
\end{aligned}$$

Put $\kappa_n=\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}$ for all $n\ge 1$. That is, $x_{n+1}=(1-\beta_n)\kappa_n+\beta_nx_n\ \forall n\ge 1$. Note that

$$\begin{aligned}
\kappa_{n+1}-\kappa_n &= \frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}Sy_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nf(x_n)+\gamma_nSy_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})+\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}Sy_{n+1}-\frac{\alpha_n}{1-\beta_n}f(x_n)-\frac{1-\beta_n-\alpha_n}{1-\beta_n}Sy_n \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(f(x_{n+1})-Sy_{n+1}\big)+\frac{\alpha_n}{1-\beta_n}\big(Sy_n-f(x_n)\big)+Sy_{n+1}-Sy_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\kappa_{n+1}-\kappa_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|Sy_n-f(x_n)\|+\|Sy_{n+1}-Sy_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|Sy_n-f(x_n)\|+\|y_{n+1}-y_n\|.
\end{aligned}$$

This implies

$$\begin{aligned}\|\kappa_{n+1}-\kappa_n\|-\|x_{n+1}-x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|Sy_n-f(x_n)\|\\ +\sum_{i=1}^{r}\eta_i(\|e_{n+1,i}\|+\|e_{n,i}\|).\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty}(\|\kappa_{n+1}-\kappa_n\|-\|x_{n+1}-x_{n+1}\|)<0.$$

Using Lemma 1.3, one has $\lim_{n\to\infty}\|\kappa_n-x_n\|=0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.2}$$

On the other hand, since P_C is firmly nonexpansive, one has

$$\begin{aligned}\|P_C(I-\lambda_iA_i)x_n-x^*\|^2 &\le \langle (I-\lambda_iA_i)x_n-(I-\lambda_iA_i)x^*, P_C(I-\lambda_iA_i)x_n-x^*\rangle\\ &=\frac{1}{2}\big(\|(I-\lambda_iA_i)x_n-(I-\lambda_iA_i)x^*\|^2+\|P_C(I-\lambda_iA_i)x_n-x^*\|^2\\ &\quad-\|(I-\lambda_iA_i)x_n-(I-\lambda_iA_i)x^*-(P_C(I-\lambda_iA_i)x_n-x^*)\|^2\big)\\ &\le\frac{1}{2}\big(\|x_n-x^*\|^2+\|P_C(I-\lambda_iA_i)x_n-x^*\|^2\\ &\quad-\|x_n-P_C(I-\lambda_iA_i)x_n-\lambda_i(A_ix_n-A_ix^*)\|^2\big)\\ &=\frac{1}{2}\big(\|x_n-x^*\|^2+\|P_C(I-\lambda_iA_i)x_n-x^*\|^2-\|x_n-P_C(I-\lambda_iA_i)x_n\|^2\\ &\quad+2\lambda_i\langle A_ix_n-A_ix^*, x_n-P_C(I-\lambda_iA_i)x_n\rangle-\lambda_i^2\|A_ix_n-A_ix^*\|^2\big).\end{aligned}$$

It follows that

$$\|P_C(I-\lambda_iA_i)x_n-x^*\|^2\le\|x_n-x^*\|^2-\|x_n-P_C(I-\lambda_iA_i)x_n\|^2+M_i\|A_ix_n-A_ix^*\|, \tag{2.3}$$

where M_i is an appropriate constant such that

$$M_i=\max\{2\lambda_i\|x_n-P_C(I-\lambda_iA_i)x_n\|:\forall n\ge 1\}.$$

From the nonexpansivity of S, one has

$$\begin{aligned}\|x_{n+1}-x^*\|^2&\le\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|Sy_n-x^*\|^2\\ &\le\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|\sum_{i=1}^{r}\eta_iy_{n,i}-x^*\|^2\\ &\le\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\sum_{i=1}^{r}\eta_i\|e_{n,i}\|+\gamma_n\sum_{i=1}^{r}\eta_i\|P_C(x_n-\lambda_iA_ix_n)-x^*\|^2\\ &\le\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\sum_{i=1}^{r}\eta_i\|e_{n,i}\|+\gamma_n\sum_{i=1}^{r}\eta_i(\|x_n-x^*\|^2\\ &\quad-2\lambda_i\langle A_ix_n-A_ix^*,x_n-x^*\rangle+\lambda_i^2\|A_ix_n-A_ix^*\|^2)\\ &\le\alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2+\gamma_n\sum_{i=1}^{r}\eta_i\|e_{n,i}\|-\gamma_n\sum_{i=1}^{r}\eta_i\lambda_i(2\mu_i-\lambda_i)\|A_ix_n-A_ix^*\|^2.\end{aligned}$$

It follows that

$$\gamma_n\sum_{i=1}^{r}\eta_i\lambda_i(2\mu_i-\lambda_i)\|A_ix_n-A_ix^*\|^2\le\alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\gamma_n\sum_{i=1}^{r}\eta_i\|e_{n,i}\|$$

$$\begin{aligned}&\leq \alpha_n\|f(x_n)-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|\\&\quad+\gamma_n\sum_{i=1}^r\eta_i\|e_{n,i}\|.\end{aligned}$$

From 2.2, one obtains $\lim_{n\to\infty}\|A_ix_n-A_ix^*\|=0$ $\forall 1\leq i\leq r$. Note that

$$\begin{aligned}\|y_n-x_n\|&\leq\|\sum_{i=1}^r\eta_iy_{n,i}-\sum_{i=1}^r\eta_iP_C(I-\lambda_iA_i)x_n\|+\|\sum_{i=1}^r\eta_iP_C(I-\lambda_iA_i)x_n-x_n\|\\&\leq\sum_{i=1}^r\eta_i\|e_{n,i}\|+\sum_{i=1}^r\eta_i\|P_C(I-\lambda_iA_i)x_n-x_n\|^2.\end{aligned}$$

It follows from 2.3 that

$$\sum_{i=1}^r\eta_i\|P_C(I-\lambda_iA_i)x_n-x^*\|^2\leq\|x_n-x^*\|^2-\|y_n-x_n\|+\sum_{i=1}^r\eta_i\|e_{n,i}\|+\sum_{i=1}^r\eta_iM_i\|A_ix_n-A_ix^*\|.$$

Hence, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2&\leq\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|\sum_{i=1}^r\eta_iy_{n,i}-x^*\|^2\\&\leq\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\sum_{i=1}^r\eta_i\|e_{n,i}\|+\gamma_n\sum_{i=1}^r\eta_i\|P_C(x_n-\lambda_iA_ix_n)-x^*\|^2\\&\leq\alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2+\gamma_n\sum_{i=1}^r\eta_i\|e_{n,i}\|-\gamma_n\|y_n-x_n\|+\gamma_n\sum_{i=1}^r\eta_i\|e_{n,i}\|\\&\quad+\gamma_n\sum_{i=1}^r\eta_iM_i\|A_ix_n-A_ix^*\|.\end{aligned}$$

This implies

$$\begin{aligned}\gamma_n\|y_n-x_n\|^2&\leq\alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\&\quad+\gamma_n\sum_{i=1}^r\eta_iM_i\|A_ix_n-A_ix^*\|+2\gamma_n\sum_{i=1}^r\eta_i\|e_{n,i}\|\\&\leq\alpha_n\|f(x_n)-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|\\&\quad+\gamma_n\sum_{i=1}^r\eta_iM_i\|A_ix_n-A_ix^*\|+2\gamma_n\sum_{i=1}^r\eta_i\|e_{n,i}\|.\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0.$$

Since

$$\|Sy_n-x_n\|\leq\frac{\alpha_n}{\gamma_n}\|f(x_n)-x_n\|+\frac{1}{\gamma_n}\|x_{n+1}-x_n\|,$$

we find

$$\lim_{n\to\infty}\|Sy_n-x_n\|=0.$$

From

$$\begin{aligned}\|Sx_n-x_n\|&\leq\|x_n-Sy_n\|+\|Sy_n-Sx_n\|\\&\leq\|x_n-Sy_n\|+\|y_n-x_n\|,\end{aligned}$$

we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

Since $P_{\mathcal{F}}f$ is α-contractive, we have it has an unique fixed point. Let use p to denote the unique fixed point, that is, $p = P_{\mathcal{F}}f(p)$.

Next, we show

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle \le 0.$$

To show it, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle = \lim_{i\to\infty} \langle f(p) - p, x_{n_i} - p\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $\bar{x}$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup \bar{x}$. Define a mapping $W : C \to C$ by

$$Wx = \sum_{i=1}^{r} \eta_i P_C(I - \lambda_i A_i)x \quad \forall x \in C.$$

Using Lemma 1.1, we see that W is nonexpansive with

$$F(W) = \cap_{i=1}^{r} F(P_C(I - \lambda_i A_i)) = \cap_{i=1}^{r} VI(C, A_i).$$

Since $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$, we can obtain that $\bar{x} \in F(W)$. Using Lemma 1.4, we see that $\bar{x} \in F(S)$. This proves that

$$\bar{x} \in F(W) \cap F(S) = \cap_{i=1}^{r} VI(C, A_i) \cap F(S).$$

It follows that

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle \le 0.$$

Since

$$\|y_n - p\| \le \sum_{i=1}^{r} \eta_i \|e_{n,i}\| + \|x_n - p\|,$$

one has

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle + \beta_n \|x_n - p\|\|x_{n+1} - p\| + \gamma_n \|Sy_n - p\|\|x_{n+1} - p\| \\
&\le \alpha_n \langle f(p) - p, x_{n+1} - p\rangle + \alpha_n \alpha \|x_n - p\|\|x_{n+1} - p\| + \beta_n \|x_n - p\|\|x_{n+1} - p\| \\
&\quad + \gamma_n \|y_n - p\|\|x_{n+1} - p\| \\
&\le \alpha_n \langle f(p) - p, x_{n+1} - p\rangle + \frac{1 - \alpha_n(1-\alpha)}{2} (\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\
&\quad + \gamma_n \|x_{n+1} - p\| \sum_{i=1}^{r} \eta_i \|e_{n,i}\|.
\end{aligned}$$

It follows that

$$\|x_{n+1} - p\|^2 \le \big(1 - \alpha_n(1-\alpha)\big)\|x_n - p\|^2 + 2\big(\alpha_n \langle f(p) - p, x_{n+1} - p\rangle + \|x_{n+1} - p\| \sum_{i=1}^{r} \eta_i \|e_{n,i}\|\big).$$

Using Lemma 1.2, one has $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof. □

If S is the identity operator, one has the following result.

Corollary 2.2. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A_i : C \to H$ be a μ_i-inverse-strongly monotone mapping for each $1 \leq i \leq r$, where r is some positive integer. Let $f : C \to C$ be a fixed α-contractive mapping. Assume that $\mathcal{F} := \cap_{i=1}^{r} VI(C, A_i) \neq \emptyset$. Let $\{\lambda_i\}$ be real numbers in $(0, 2\mu_i)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence defined by the following manner:*

$$\begin{cases} x_1 \in C, \\ y_{n,i} \approx P_C(x_n - \lambda_i A_i x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{r} \eta_i y_{n,i}, \quad n \geq 1, \end{cases}$$

where the criterion for the approximate computation of $y_{n,i}$ in C is $\|y_{n,i} - P_C(x_n - \lambda_i A_i x_n)\| \leq e_{n,i}$, where $\lim_{n\to\infty} \|e_{n,i}\| = 0$ for each $1 \leq i \leq r$. Assume that the above control sequences satisfies the following conditions:

(a) $\alpha_n + \beta_n + \gamma_n = \sum_{i=1}^{r} \eta_i = 1 \ \forall n \geq 1$;

(b) $1 > \limsup_{n\to\infty} \beta_n \geq \liminf_{n\to\infty} \beta_n > 0$;

(c) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then sequence $\{x_n\}$ converges in norm to a common solution p, which is also the unique solution to the following variational inequality: $\langle f(p) - p, p - q\rangle \geq 0 \ \forall q \in \mathcal{F}$.

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