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Best proximity point theorems for multivalued mappings on partially ordered metric spaces

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Abstract

In this paper, we prove some best proximity point theorems for multivalued mappings in the setting of complete partially ordered metric spaces. As an application, we infer best proximity point and fixed point results for single valued mappings in partially ordered metric spaces. The results presented generalize and improve various known results from best proximity and fixed point theory.

Keywords: Partially ordered set, optimal approximate solution, proximally increasing mapping, fixed point, best proximity point.
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1. Introduction and preliminaries

Let A be a nonempty subset of a metric space (X, d). A mapping $T : A \to X$ is said to have a fixed point in A, if the fixed point equation $Tx = x$ has at least one solution. That is, $x \in A$ is a fixed point of T if $d(x, Tx) = 0$. Suppose the fixed point equation $Tx = x$ does not have a solution, then $d(x, Tx) > 0$ for all $x \in A$. In such a situation, it is our mission to find an element $x \in A$ such that $d(x, Tx)$ is minimum in some sense. The best approximation theory and best proximity pair theorems are studied in this direction. Here we state the best approximation theorem due to Ky Fan [8].

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Theorem 1.1 ([8]). *Let A be a nonempty compact convex subset of a normed linear space X and $T : A \to X$ be a continuous function. Then there exists $x \in A$ such that $\|x - Tx\| = d(Tx, A) := \inf\{\|Tx - a\| : a \in A\}$.*

Such an element $x \in A$ in Theorem 1.1 is called a best approximant of T in A. Note that if $x \in A$ is a best approximant, then $\|x - Tx\|$ need not be the optimum. Best proximity point theorems have been explored to find sufficient conditions so that the minimization problem $\min_{x\in A} \|x - Tx\|$ has at least one solution. Now, let us consider two nonempty subsets A, B of a metric space X and a mapping $T : A \to B$. The natural question is whether one can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}$. Since $d(x, Tx) \geq d(A, B)$, the optimal solution to the problem of minimizing the real valued function $x \to d(x, Tx)$ over the domain A of the mapping T will be the one for which the value $d(A, B)$ is attained. A point $x_0 \in A$ is called a best proximity point of T if $d(x_0, Tx_0) = d(A, B)$. Note that if $d(A, B) = 0$, then the best proximity point is nothing but a fixed point of T. The existence and convergence of best proximity points is an interesting topic of optimization theory which recently attracted the attention of many authors [4, 7, 11, 12, 16, 19, 20]. For the existence of best proximity point in the setting of partially order metric space, one can go through to [1, 2, 13, 14, 15, 17, 18].

Let X be a non-empty set such that $(X, d, \preceq)$ is a partially ordered metric space. Consider A and B are non-empty subsets of the metric space (X, d). We denote by $CB(X)$ the class of all nonempty closed and bounded subsets of X and $B(X)$ the class of all nonempty bounded subsets of X. The following notions are used subsequently:

$$\begin{aligned} D(x, B) &= \inf\{d(x, y) : y \in B\} \text{ for all } x \in X, \\ \delta(A, B) &:= \sup\{d(x, y) : x \in A \text{ and } y \in B\}, \\ d(A, B) &:= \inf\{d(x, y) : x \in A \text{ and } y \in B\}, \\ A_0 &= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \\ B_0 &= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned}$$

In [12], the authors discussed sufficient conditions which guarantee the non-emptiness of A_0 and B_0. Also, in [19], the authors proved that A_0 is contained in the boundary of A.

Suppose that $T : A \to 2^B$ is a multivalued non-self-mappings. An element $x_0 \in A$ is said to be best proximity point for T if $x_0 \in Tx_0$. On the other hand if $A \cap B = \emptyset$, the multifunction T has no fixed point. Then $D(x, Tx) > 0$ for all $x \in A$. Analogously, one can investigate to find necessary conditions so that minimization problem $\min_{x\in A} D(x, Tx)$ has at least one solution. Since $D(x, Tx) \geq d(A, B)$ for all $x \in A$, the optimal solution to the problem of minimizing the real valued function $x \to D(x, Tx)$ over the domain A of the mapping T will be the one for which the value $d(A, B)$ is attained. A point $x_0 \in A$ is called a best proximity point of a multivalued non-self-mapping T, if $D(x_0, Tx_0) = d(A, B)$. Note that if $d(A, B) = 0$, then the best proximity point is nothing but a fixed point of T.

For the existence and convergence of best proximity point for multivalued non-self mappings, we refer the reader to [3, 9].

The following notion of an altering distance function was introduced by Khan et al. in [10].

Definition 1.2. A function $\psi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function if it satisfies the following conditions.

(i) ψ is continuous and non-decreasing.
(ii) $\psi(t) = 0$ if and only if $t = 0$.

To define the multivalued nondecreasing map, in [5], Beg and Butt presented the following relations between two subsets of X.

Definition 1.3 ([5]). Let A and B be two nonempty subsets of a partially ordered set $(X, \preceq)$. The relation between A and B is denoted and defined as follows: $A \prec_1 B$, if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$.

In [6], Choudhury and Metiya proved the existence of fixed point for multivalued self mappings in partially ordered metric spaces.

Theorem 1.4. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A be a non-empty closed subset X and $T : A \to B(A)$ is a multivalued mapping such that the following conditions are satisfied:*

(i) there exists $x_0 \in A$ such that $\{x_0\} \prec_1 Tx_0$,

(ii) T satisfies $\psi(\delta(Tx, Ty)) \leq k\psi(M(x, y))$ for all $x \preceq y$ in A, where $k \in (0, 1)$,
$M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \dfrac{D(x, Ty) + D(y, Tx)}{2}\}$ *and ψ is altering distance function,*

(iii) for $x, y \in A$, $x \preceq y$ implies $Tx \prec_1 Ty$,

(iv) if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \to x$, then $x_n \preceq x$ for all n.

Then T has a fixed point.

In this article, we attempt to give a partial generalization of Theorem 1.4 by considering a non-self multivalued map T.

The notion of P-property was introduced by Sankar Raj as follows.

Definition 1.5 ([16])**.** Let (A, B) be a pair of non-empty subsets of a metric space X with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if

$$\left.\begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned}\right\} \implies d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 1.6. Let A, B be two non-empty closed convex subsets of a Hilbert space X. Then (A, B) satisfies the P-property.

Example 1.7. Let A, B be two non-empty subsets of a metric space (X, d) such that $A_0 \neq \emptyset$ and $d(A, B) = 0$. Thus (A, B) has the P-property.

Definition 1.8 ([17])**.** A mapping $T : A \to B$ is said to be proximally increasing if it satisfies the condition that

$$\left.\begin{aligned} y_1 \leq y_2 \\ d(x_1, Ty_1) = d(A, B) \\ d(x_2, Ty_2) = d(A, B) \end{aligned}\right\} \implies x_1 \leq x_2,$$

where $x_1, x_2, y_1, y_2 \in A$.

One can see that, for a self-mapping, the notion of proximally increasing mapping reduces to that of increasing mapping.

Here we defined the notion of proximal relation between two subsets of X.

Definition 1.9. Let A and B be two nonempty subsets of a partially ordered metric space $(X, d, \preceq)$ such that $A_0 \neq \emptyset$. Let B_1 and B_2 be two non-empty subsets of B_0. The proximal relation between B_1 and B_2 is denoted and defined as follows: $B_1 \prec_{(1)} B_2$, if for every $b_1 \in B_1$ with $d(a_1, b_1) = d(A, B)$ there exists $b_2 \in B_2$ with $d(a_2, b_2) = d(A, B)$ such that $a_1 \preceq a_2$.

One can see that, when $A = B$, $B_1 \prec_{(1)} B_2$ reduces to $B_1 \prec_1 B_2$.

Example 1.10. Let $X = \mathbb{R}^2$ and consider the order $(x, y) \preceq (z, t) \Leftrightarrow x \leq z$ and $y \leq t$, where $\leq$ is usual order in $\mathbb{R}$. Thus, $(X, \preceq)$ is a partially ordered set. Besides, (X, d) is a complete metric space where d is usual metric. Let $A = \{(0, x) : x \in [0, \infty)\}$ and $B = \{(1, x) : x \in [0, \infty)\}$ be a closed subset of X. Note that, $d(A, B) = 1, A = A_0$ and $B = B_0$. Take $B_1 = \{(1, x) : x \in [0, 1]\}$ and $B_2 = \{(1, x) : x \in [2, 3]\}$. Then $B_1 \prec_{(1)} B_2$, that is, for every $b_1 \in B_1$ with $d(a_1, b_1) = d(A, B)$ there exists $b_2 \in B_2$ with $d(a_2, b_2) = d(A, B)$ such that $a_1 \preceq a_2$.

2. Main Results

Now, let us state our main result.

Theorem 2.1. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A and B be non-empty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) satisfies the P-property. Let $T : A \to CB(B)$ be a multivalued mapping such that the following conditions are satisfied:*

(i) there exist two elements x_0, x_1 in A_0 and $y_0 \in Tx_0$ such that

$$d(x_1, y_0) = d(A, B) \text{ and } x_0 \preceq x_1,$$

(ii) Tx_0 is included in B_0 for all $x_0 \in A_0$ and

$$\psi(\delta(Tx, Ty)) \leq k\psi(M(x, y)) - k\psi(d(A, B)) \text{ for all } x \preceq y \text{ in } A, \tag{2.1}$$

where $k \in (0, 1)$, $M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\}$ and ψ is altering distance function also it satisfies $\psi(s + t) \leq \psi(s) + \psi(t)$, for all $s, t \in [0, \infty)$,

(iii) for $x, y \in A_0$, $x \preceq y$ implies $Tx \prec_{(1)} Ty$,

(iv) if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \to x$, then $x_n \preceq x$ for all n.

Then, there exists an element x in A such that

$$D(x, Tx) = d(A, B).$$

Proof. By the assumption *(i)*, there exist two elements x_0, x_1 in A_0 and $y_0 \in Tx_0$ such that $d(x_1, y_0) = d(A, B)$ and $x_0 \preceq x_1$. By the assumption *(iii)*, $Tx_0 \prec_{(1)} Tx_1$, there exists $y_1 \in Tx_1$ with $d(x_2, y_1) = d(A, B)$ such that $x_1 \preceq x_2$. In general, for each $n \in \mathbb{N}$, there exists $x_{n+1} \in A_0$ and $y_n \in Tx_n$ such that $d(x_{n+1}, y_n) = d(A, B)$. Hence, we obtain

$$d(x_{n+1}, y_n) = D(x_{n+1}, Tx_n) = d(A, B) \text{ for all } n \in \mathbb{N} \tag{2.2}$$

$$\text{with } x_0 \preceq x_1 \preceq x_2 \preceq \cdots x_n \preceq x_{n+1} \cdots .$$

If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then $D(x_{n_0+1}, Tx_{n_0}) = D(x_{n_0}, Tx_{n_0}) = d(A, B)$. This means that x_{n_0} is a best proximity point of T and the proof is finished. Thus, we can suppose that $x_n \neq x_{n+1}$ for all n. Since $d(x_{n+1}, y_n) = d(A, B)$ and $d(x_n, y_{n-1}) = d(A, B)$ and (A, B) has the P-property,

$$d(x_n, x_{n+1}) = d(y_{n-1}, y_n) \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

Since $x_{n-1} \prec x_n$, by inequality

$$\psi(d(x_n, x_{n+1})) = \psi(d(y_{n-1}, y_n)) \leq \psi(\delta(Tx_{n-1}, Tx_n)) \leq k\psi(M(x_{n-1}, x_n)) - k\psi(d(A, B)). \tag{2.4}$$

Now from the triangle inequality for d, we have

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), \frac{D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})}{2}\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_{n-1}, y_{n-1}), d(x_n, y_n), \frac{d(x_{n-1}, y_n) + d(x_n, y_{n-1})}{2}\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_{n-1}, y_{n-2}) + d(y_{n-2}, y_{n-1}), d(x_n, y_{n-1}) + d(y_{n-1}, y_n), \\ &\qquad \frac{d(x_{n-1}, y_{n-2}) + d(y_{n-2}, y_{n-1}) + d(y_{n-1}, y_n) + d(x_n, y_{n-1})}{2}\} \\ &\leq \max\{d(x_{n-1}, x_n), d(A, B) + d(x_{n-1}, x_n), d(A, B) + d(x_n, x_{n+1}), \end{aligned}$$

$$\frac{d(A,B)+d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(A,B)}{2}\}$$
$$=\max\{d(A,B)+d(x_{n-1},x_n),d(A,B)+d(x_n,x_{n+1})\}.$$

From (2.4) and above inequality, we get

$$\psi(d(x_n,x_{n+1}))\leq k\psi(\max\{d(A,B)+d(x_{n-1},x_n),d(A,B)+d(x_n,x_{n+1})\})-k\psi(d(A,B)). \tag{2.5}$$

If $d(x_n,x_{n+1})>d(x_{n-1},x_n)$. From (2.5), we obtain

$$\begin{aligned}\psi(d(x_n,x_{n+1}))&\leq k\psi(d(A,B)+d(x_n,x_{n+1}))-k\psi(d(A,B))\\&\leq k[\psi(d(A,B)+\psi(d(x_n,x_{n+1}))]-k\psi(d(A,B))\\&=k\psi(d(x_n,x_{n+1}))<\psi(d(x_n,x_{n+1})),\end{aligned}$$

which is a contradiction. So, we have

$$d(x_n,x_{n+1})\leq d(x_{n-1},x_n). \tag{2.6}$$

Hence, the sequence $\{d(x_n,x_{n+1})\}$ is monotone non-increasing and bounded below. Thus, there exists $r\geq 0$ such that

$$\lim_{n\to\infty}d(x_n,x_{n+1})=r\geq 0. \tag{2.7}$$

Suppose that $\lim_{n\to\infty}d(x_n,x_{n+1}))=r>0$. Using (2.6) then the inequality (2.5) becomes

$$\begin{aligned}\psi(d(x_n,x_{n+1}))&\leq k\psi(d(A,B)+d(x_{n-1},x_n))-k\psi(d(A,B))\\&\leq k[\psi(d(A,B))+\psi(d(x_{n-1},x_n))]-k\psi(d(A,B))\\&=k\psi(d(x_{n-1},x_n)),\end{aligned}$$

implies that $\psi(r)\leq k\psi(r)$, which is a contradiction unless $r=0$. Hence,

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0. \tag{2.8}$$

Now to prove that $\{x_n\}$ is a Cauchy sequence. In contrary case, suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon>0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is smallest index for which $n(k)>m(k)>k$, $d(x_{m(k)},x_{n(k)})\geq\epsilon$. This means that

$$d(x_{m(k)},x_{n(k)-1})<\epsilon. \tag{2.9}$$

and

$$\begin{aligned}\epsilon&\leq d(x_{m(k)},x_{n(k)})\\&\leq d(x_{m(k)},x_{n(k)-1})+d(x_{n(k)-1},x_{n(k)})\\&<\epsilon+d(x_{n(k)-1},x_{n(k)}).\end{aligned}$$

Letting $k\to\infty$ and using (2.8) we can conclude that

$$\lim_{k\to\infty}d(x_{m(k)},x_{n(k)})=\epsilon. \tag{2.10}$$

Again,

$$d(x_{m(k)},x_{n(k)-1})\leq d(x_{m(k)},x_{n(k)})+d(x_{n(k)},x_{n(k)-1})$$

and

$$d(x_{m(k)},x_{n(k)})\leq d(x_{m(k)},x_{n(k)-1})+d(x_{n(k)},x_{n(k)-1}).$$

Therefore,

$$|d(x_{m(k)}, x_{n(k)-1}) - d(x_{m(k)}, x_{n(k)})| \leq d(x_{n(k)}, x_{n(k)-1}).$$

Taking $k \to \infty$ and using (2.10) and (2.8) it follows that

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \epsilon. \tag{2.11}$$

Similarly, we can prove that

$$\begin{aligned}\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)})\\ = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon.\end{aligned} \tag{2.12}$$

Since $m(k) < n(k)$, $x_{m(k)-1} \preceq x_{n(k)-1}$, from (2.3) and (2.1), we have

$$\begin{aligned}\psi(d(x_{m(k)}, x_{n(k)})) &\leq \psi(\delta(Tx_{m(k)-1}, Tx_{n(k)-1}))\\ &\leq k\psi(M(x_{m(k)-1}, x_{n(k)-1})) - k\psi(d(A, B)),\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}&M(x_{m(k)-1}, x_{n(k)-1})\\ &\quad= \max\{d(x_{m(k)-1}, x_{n(k)-1}), D(x_{m(k)-1}, Tx_{m(k)-1}), D(x_{n(k)-1}, Tx_{n(k)-1}),\\ &\qquad \frac{D(x_{m(k)-1}, Tx_{n(k)-1}) + D(x_{n(k)-1}, Tx_{m(k)-1})}{2}\}\\ &\quad\leq \max\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, y_{m(k)-1}), d(x_{n(k)-1}, y_{n(k)-1}),\\ &\qquad \frac{d(x_{m(k)-1}, y_{n(k)-1}) + d(x_{n(k)-1}, y_{m(k)-1})}{2}\}\\ &\quad\leq \max\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, y_{m(k)-2}) + d(y_{m(k)-2}, y_{m(k)-1}),\\ &\qquad d(x_{n(k)-1}, y_{n(k)-2}) + d(y_{n(k)-2}, y_{n(k)-1}),\\ &\qquad \frac{d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, y_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, y_{m(k)-1})}{2}\}.\end{aligned}$$

Using (2.3) and $d(x_{n+1}, y_n) = d(A, B)$ in the above inequality, we get

$$\begin{aligned}&M(x_{m(k)-1}, x_{n(k)-1})\\ &\quad\leq \max\{d(x_{m(k)-1}, x_{n(k)-1}), d(A, B) + d(x_{m(k)-1}, x_{m(k)}), d(A, B) + d(x_{n(k)-1}, x_{n(k)}),\\ &\qquad \frac{2d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) + d(x_{m(k)-1}, x_{m(k)}) + 2d(A, B)}{2}\}.\end{aligned} \tag{2.14}$$

Using (2.14) in (2.13) and taking $k \to \infty$, from (2.8),(2.10) and (2.12), we get

$$\psi(\epsilon) \leq k\psi(\max\{\epsilon, d(A, B), d(A, B), \epsilon + d(A, B)\}) - k\psi(d(A, B)) \tag{2.15}$$

$$\leq k\psi(\epsilon) + k\psi(d(A, B)) - k\psi(d(A, B)) = k\psi(\epsilon), \tag{2.16}$$

which is a contradiction to the property of ψ. Thus, $\{x_n\}$ is a Cauchy sequence in A and hence converges to some element x in A. Since $d(x_n, x_{n+1}) = d(y_{n-1}, y_n)$, the sequence $\{y_n\}$ in B is Cauchy and then is convergent. Suppose that $y_n \to y$. By the relation $d(x_{n+1}, y_n) = d(A, B)$ for all n, we conclude that $d(x, y) = d(A, B)$. We now claim that $y \in Tx$.

Since $\{x_n\}$ is a increasing sequence in A and $x_n \to x$, by the hypothesis (iii), $x_n \preceq x$ $\forall n$.

$$\begin{aligned}\psi(D(y_n,Tx)) &\le \psi(\delta(Tx_n,Tx))\\ &\le k\psi(\max\{d(x_n,x),D(x_n,Tx_n),D(x,Tx),\frac{D(x_n,Tx)+D(x,Tx_n)}{2}\})-k\psi(d(A,B))\\ &\le k\psi(\max\{d(x_n,x),d(x_n,y_n),D(x,Tx),\frac{D(x_n,Tx)+d(x,y_n)}{2}\})-k\psi(d(A,B)).\end{aligned}$$

Taking $n\to\infty$ in the above inequality, using $x_n\to x$, $y_n\to y$ and $d(x,y)=d(A,B)$, we get

$$\begin{aligned}\psi(D(y,Tx)) &\le k\psi(\max\{0,d(A,B),D(x,Tx),\frac{D(x,Tx)+d(A,B)}{2}\})-k\psi(d(A,B))\\ &\le k\psi(D(x,Tx))-k\psi(d(A,B))\\ &\le k\psi(d(x,y)+D(y,Tx))-k\psi(d(A,B))\\ &\le k\psi(d(x,y))+k\psi(D(y,Tx))-k\psi(d(A,B))\\ &\le k\psi(d(A,B))+k\psi(D(y,Tx))-k\psi(d(A,B))\\ &\le k\psi(D(y,Tx)),\end{aligned}$$

which is contradiction unless $D(y,Tx)=0$.

This implies that $y\in Tx$ and hence $D(x,Tx)=d(A,B)$. That is x is a best proximity point of the mapping T. □

Let us illustrate the Theorem 2.1 with the following example.

Example 2.2. Let $X=\mathbb{R}^2$ and consider the order $(x,y)\preceq(z,t)\Leftrightarrow x\le z$ and $y\le t$, where $\le$ is usual order in $\mathbb{R}$.

Thus, $(X,\preceq)$ is a partially ordered set. Besides, (X,d_1) is a complete metric space where the metric is defined as $d_1((x_1,y_1),(x_2,y_2))=|x_1-x_2|+|y_1-y_2|$. Let $A=\{(-6,0),(0,-6),(0,5)\}$ and $B=\{(-1,0),(0,-1),(0,0),(-1,1),(1,1)\}$ be a closed subset of X. Then, $d(A,B)=5, A=A_0$ and $B=B_0$. Let $T:A\to CB(B)$ be defined as

$$T(x,y)=\begin{cases}\{(0,-1),(0,0)\} & \text{if } (x,y)=(-6,0),\\ \{(-1,1),(0,0),(-1,0)\} & \text{if } (x,y)=(0,-6),\\ \{(1,1),(-1,1)\} & \text{if } (x,y)=(0,5).\end{cases}$$

It can be seen that condition (i) is true. Since there exist two elements $(-6,0),(0,5)$ in A_0 and $(0,0)\in T(-6,0)$ such that

$$d((0,5),(0,0))=d(A,B)=5 \text{ and } (-6,0)\preceq(0,5).$$

Now, we have to prove the condition (ii). It is easy to prove that Tx_0 is included in B_0 for all $x_0\in A_0$.

Note that the only comparable elements in A are $(-6,0)\preceq(0,5)$ and $(0,-6)\preceq(0,5)$. In both cases, $\delta(Tx,Ty)=3$, $M(x,y)=11$ and $d(A,B)=5$.

For $\psi(t)=2t$, we get, $\psi(\delta(Tx,Ty))=6$ and $\psi(M(x,y))-\psi(d(A,B))=12$. Hence T satisfies the condition (ii) when $k\in[\frac{1}{2},1)$, that is

$$\psi(\delta(Tx,Ty))\le k\psi(M(x,y))-k\psi(d(A,B)) \text{ for all } x\preceq y \text{ in } A. \tag{2.17}$$

Also, it can be easily prove the conditions (iii) and (iv). Hence all the hypotheses of the Theorem 2.1 are satisfied. Also, it can be observed that $(0,5)$ is the best proximity point of the mapping T. That is , $D((0,5),T(0,5))=d(A,B)=5$.

3. Applications

As an application of our results, we infer new best proximity point and fixed point results for multivalued and single valued mappings in the set up of partially ordered metric spaces.

If we take ψ an identity function in Theorem 2.1, then we have the following result.

Corollary 3.1. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A and B be non-empty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) satisfies the P-property. Let $T : A \to CB(B)$ be a multivalued mapping such that the following conditions are satisfied:*

(i) there exist two elements x_0, x_1 in A_0 and $y_0 \in Tx_0$ such that

$$d(x_1, y_0) = d(A, B) \text{ and } x_0 \preceq x_1,$$

(ii) Tx_0 is included in B_0 for all $x_0 \in A_0$ and

$$\delta(Tx, Ty) \leq k[M(x, y) - d(A, B)] \text{ for all } x \preceq y \text{ in } A, \tag{3.1}$$

where $k \in (0, 1)$ and $M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \dfrac{D(x, Ty) + D(y, Tx)}{2}\}$,

(iii) for $x, y \in A_0$, $x \preceq y$ implies $Tx \prec_{(1)} Ty$,

(iv) if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \to x$, then $x_n \preceq x$ for all n.

Then, there exists an element x in A such that

$$D(x, Tx) = d(A, B).$$

That is x is a best proximity point of the mapping T.

Theorem 3.2. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A and B be non-empty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) satisfies the P-property. Let $T : A \to CB(B)$ be a multivalued mapping such that the following conditions are satisfied:*

(i) there exist elements x_0, x_1 in A_0 and $y_0 \in Tx_0$ such that

$$d(x_1, y_0) = d(A, B) \text{ and } x_0 \preceq x_1,$$

(ii) Tx_0 is included in B_0 for all $x_0 \in A_0$ and

$$\psi(\delta(Tx, Ty)) \leq k\psi(M(x, y)) \text{ for all } x \preceq y \text{ in } A, \tag{3.2}$$

where $k \in (0, 1)$, ψ is altering distance function and

$$M(x, y) = \max\{d(x, y), D(x, Tx) - d(A, B), D(y, Ty) - d(A, B), \frac{D(x, Ty) + D(y, Tx)}{2} - d(A, B)\},$$

(iii) for $x, y \in A_0$, $x \preceq y$ implies $Tx \prec_{(1)} Ty$,

(iv) if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \to x$, then $x_n \preceq x$ for all n.

Then, there exists an element x in A such that

$$D(x, Tx) = d(A, B).$$

Proof. The proof is similar to Theorem 2.1. □

As a consequence of Theorem 3.2, by taking T is single valued map, we derive the following result.

Corollary 3.3. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A and B be non-empty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) satisfies the P-property. Let $T : A \to B$ be a single valued mapping such that the following conditions are satisfied:*

(i) there exist elements x_0, x_1 in A_0 such that

$$d(x_1, Tx_0) = d(A, B) \text{ and } x_0 \preceq x_1,$$

(ii) $T(A_0) \subseteq B_0$ and

$$\psi(d(Tx, Ty)) \leq k\psi(M(x, y)) \text{ for all } x \preceq y \text{ in } A, \tag{3.3}$$

where $k \in (0, 1)$, ψ is altering distance function and

$$M(x, y) = \max\{d(x, y), d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B)\},$$

(iii) for $x, y \in A_0$, $x \preceq y$ implies $\{Tx\} \prec_{(1)} \{Ty\}$,

(iv) if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \to x$, then $x_n \preceq x$ for all n.

Then, there exists an element x in A such that

$$d(x, Tx) = d(A, B).$$

If we take $A = B$ in Theorem 2.1 and Theorem 3.2, then we conclude to the following result.

Corollary 3.4. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A be a non-empty closed subset X and $T : A \to CB(A)$ is a multivalued mapping such that the following conditions are satisfied:*

(i) there exist two elements x_0, x_1 in A and $y_0 \in Tx_0$ such that

$$d(x_1, y_0) = 0 \text{ and } x_0 \preceq x_1 = y_0,$$

(ii) T satisfies

$$\psi(\delta(Tx, Ty)) \leq k\psi(M(x, y)) \quad \text{for all } x \preceq y \text{ in } A, \tag{3.4}$$

where $k \in (0, 1)$, $M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\}$ and ψ is altering distance function,

(iii) for $x, y \in A$, $x \preceq y$ implies $Tx \prec_1 Ty$,

(iv) if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \to x$, then $x_n \preceq x$ for all n.

Then, there exists an element x in A such that

$$D(x, Tx) = 0.$$

That is x is a fixed point of the mapping T.

The following corollary is a special case of Theorem 2.1 when T is a single valued self mapping.

Corollary 3.5. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A be non-empty closed subset of the metric space (X, d). Let $T : A \to A$ be a single valued mapping such that the following conditions are satisfied:*

(i) there exist elements x_0 and x_1 in A such that

$$d(x_1, Tx_0) = 0 \quad \text{and } x_0 \preceq x_1,$$

(ii) T *satisfies*

$$\psi(d(Tx,Ty)) \leq k\psi(M(x,y)) \text{ for all } x \preceq y \text{ in } A, \tag{3.5}$$

where $k \in (0,1)$, $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$ *and* ψ *is altering distance function,*

(iii) for $x, y \in A$, $x \preceq y$ *implies* $\{Tx\} \prec_1 \{Ty\}$,

(iv) if $\{x_n\}$ *is a nondecreasing sequence in* A *such that* $x_n \to x$, *then* $x_n \preceq x$ *for all* n.

Then, there exists an element x *in* A *such that*

$$d(x,Tx) = 0.$$

That is x *is a fixed point of the mapping* T.

Furthermore, if we take T is single valued self mapping and ψ is identity function in Theorem 2.1,then we deduce the following result.

Corollary 3.6. *Let* $(X, \preceq, d)$ *be a partially ordered complete metric space. Let* A *be non-empty closed subset of the metric space* (X,d). *Let* $T : A \to A$ *be a single valued mapping such that the following conditions are satisfied:*

(i) there exist elements x_0 *and* x_1 *in* A *such that*

$$d(x_1, Tx_0) = 0 \quad \text{and } x_0 \preceq x_1,$$

(ii) T *satisfies*

$$d(Tx,Ty) \leq kM(x,y) \text{ for all } x \preceq y \text{ in } A, \tag{3.6}$$

where $k \in (0,1)$ *and* $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

(iii) for $x, y \in A$, $x \preceq y$ *implies* $\{Tx\} \prec_1 \{Ty\}$,

(iv) if $\{x_n\}$ *is a nondecreasing sequence in* A *such that* $x_n \to x$, *then* $x_n \preceq x$ *for all* n.

Then, there exists an element x *in* A *such that*

$$d(x,Tx) = 0.$$

That is x *is a fixed point of the mapping* T.

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