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Sharp upper bound involving circuit layout system

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Abstract

In this paper, the circuit layout system in a Euclidean space is defined. By means of the algebraic, analytic, geometry and inequality theories, a sharp upper bound involving circuit layout system is obtained as follows:

$$\frac{1}{2}\sum_{1\leqslant j-i\leqslant N-1,1\leqslant i\leqslant N}\left\|A_j^*-A_i^*\right\|\leqslant\frac{1}{4}\sqrt{N\left(1+\max_{1\leqslant i\leqslant n}|\cos\angle A_i|\right)}\csc^2\frac{\pi}{2N}\sqrt{\sum_{i=1}^{n}\|A_{i+1}-A_i\|^2}.$$

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2010 MSC: 26D15, 51K05.

1. Introduction

We first introduce a **passage layout problem** as follows.

Let Γ be a polygon road. Assume that we need to build N factories $A_1^*, A_2^*, \ldots, A_N^*$ on the road Γ which are interdependent, and there is a constant $\delta>0$ such that

$$\left\|A_{j+1}^*-A_j^*\right\|\geqslant\delta>0,\ \forall\ j:1\leqslant j\leqslant N,\ N\geqslant 3.$$

Then, in order to facilitate the work, for any $i,j:1\leqslant i\neq j\leqslant N$, we need to build an underground passage (such as the **subway**) $\left[A_i^*A_j^*\right]$ which connect the factories A_i^* and A_j^*. As well as we need to estimate the

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building cost of the underground passages. That is to say, we need to find among all possible locations of $A_1^*, A_2^*, \ldots, A_N^*$ such that the total length

$$\frac{1}{2}\sum_{1\leqslant j-i\leqslant N-1\,,\,1\leqslant i\leqslant N}\left\|A_j^*-A_i^*\right\| \tag{1.1}$$

of the underground passages is the maximal one, where

$$A_i^*=A_j^*\Leftrightarrow i\equiv j \pmod N,\ i,j=0,\pm1,\pm2,\ldots\ . \tag{1.2}$$

In order to study the above problem, we need to recall some basic concepts [2, 6, 7].

Let $\mathbb{E}$ be a Euclidean space, and let $\alpha,\beta\in\mathbb{E}$. The inner product of α and β is denoted by $\langle\alpha,\beta\rangle$ and the norm of α is denoted by $\|\alpha\|\triangleq\sqrt{\alpha^2}$, where $\alpha^2\triangleq\langle\alpha,\alpha\rangle$. The angle between two nonzero vectors α and β is defined to be

$$\angle(\alpha,\beta)\triangleq\arccos\frac{\langle\alpha,\beta\rangle}{\|\alpha\|\,\|\beta\|}\in[0,\pi].$$

The dimension $\dim\mathbb{E}$ of $\mathbb{E}$ satisfies $\dim\mathbb{E}\geqslant n$ if and only if there exist n linearly independent vectors $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$ in $\mathbb{E}$ [6].

Let $B,C\in\mathbb{E}$ where $\mathbb{E}$ is a Euclidean space. Then the closed, open and closed-open segments joining them will respectively be denoted by

$$[BC]\triangleq\{\chi_{B,C}(t)|\ t\in[0,1]\}\,,\ \ (BC)\triangleq\{\chi_{B,C}(t)|\ t\in(0,1)\}\,,$$

$$[BC)\triangleq\{\chi_{B,C}(t)|\ t\in[0,1)\}\ \text{ and }\ (BC]\triangleq\{\chi_{B,C}(t)|\ t\in(0,1]\}\,,$$

where

$$\chi_{B,C}(t)\triangleq(1-t)B+tC.$$

Let $\mathbf{A}=(A_1,\cdots,A_n)$, where

$$A_i\neq A_{i+1},\ i=1,2,\ldots,n,\ n\geqslant3,$$

be a sequence of points in $\mathbb{E}$ with the dimension $\dim\mathbb{E}\geqslant2$, where

$$A_i=A_j\Leftrightarrow i\equiv j \pmod n,\ i,j=0,\pm1,\pm2,\ldots\ . \tag{1.3}$$

We say that the set

$$\Gamma_n(\mathbf{A})\triangleq\bigcup_{i=1}^{n}[A_iA_{i+1})$$

is an n**-polygon**, or a polygon if no confusion is caused. The angle of $\Gamma_n(\mathbf{A})$ at A_i , where $i=1,2,\ldots,n$, are defined as

$$\angle A_i\triangleq\angle(A_i-A_{i-1},A_{i+1}-A_i)\,.$$

We also denote the total length (or perimeter) of an n-polygon $\Gamma_n(\mathbf{A})$ by

$$|\Gamma_n(\mathbf{A})|\triangleq\sum_{i=1}^{n}\|A_{i+1}-A_i\|\,,$$

and we say that

$$\|\Gamma_n(\mathbf{A})\|\triangleq\frac{1}{2}\sum_{1\leqslant j-i\leqslant n-1,1\leqslant i\leqslant n}\|A_j-A_i\|$$

is the **norm** of the n-polygon $\Gamma_n(\mathbf{A})$.

The circuit layout system CLS $\{\Gamma_n(\mathbf{A}),\Gamma_N(\mathbf{A}^*),\delta\}_{\mathbb{E}}$ is defined as follows [6].

Definition 1.1. Let $\Gamma_n(\mathbf{A})$ and $\Gamma_N(\mathbf{A}^*)$, where $N \geqslant n \geqslant 3$, be two polygons in $\mathbb{E}$ with the dimension $\dim \mathbb{E} \geqslant 2$. We say the set

$$\mathrm{CLS}\left\{\Gamma_n(\mathbf{A}), \Gamma_N(\mathbf{A}^*), \delta\right\}_{\mathbb{E}} \triangleq \left\{\Gamma_n(\mathbf{A}), \Gamma_N(\mathbf{A}^*), \delta\right\}$$

is a **circuit layout system** (or CLS for short) if the set is non-empty and the following conditions are satisfied:
(H1.1) $\angle A_i \in (0, \pi),\ i = 1, 2, \ldots, n.$
(H1.2) $A_j^* \in \Gamma_n(\mathbf{A})$ for $j \in \{1, 2, \ldots, N\}$ and $A_1^* \in [A_1A_2)$.
(H1.3) If $A_j^*, A_{j+1}^* \in [A_iA_{i+1})$, then $A_{j+1}^* \in (A_j^*A_{i+1})$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, N$.
(H1.4) If $A_j^* \in [A_iA_{i+1})$ and $A_k^* \in [A_{i+1}A_{i+2})$ for $j, k \in \{1, 2, \ldots, N\}$ and $i \in \{1, 2, \cdots, n\}$, then $j < k$.
(H1.5) For any $i \in \{1, 2, \ldots, n\}$, there exists $j \in \{1, 2, \cdots, N\}$ such that $A_j^* \in [A_iA_{i+1})$.
(H1.6) For any $j \in \{1, 2, \ldots, N\}$, there is $\delta > 0$ such that

$$\left\|A_{j+1}^* - A_j^*\right\| \leqslant \delta.$$

Obviously, for the circuit layout system $\mathrm{CLS}\left\{\Gamma_n(\mathbf{A}), \Gamma_N(\mathbf{A}^*), \delta\right\}_{\mathbb{E}}$, we have

$$\left|\Gamma_N\left(\mathbf{A}^*\right)\right| \leqslant \left|\Gamma_n\left(\mathbf{A}\right)\right|. \tag{1.4}$$

But in [6], the authors obtained several sharp lower bounds of $\left|\Gamma_N\left(\mathbf{A}^*\right)\right|$ as follows.

Assertion 1.2. *Let* $\mathrm{CLS}\left\{\Gamma_n(\mathbf{A}), \Gamma_N(\mathbf{A}^*), \delta\right\}_{\mathbb{E}}$ *be a* CLS*, where n is an odd number. Then we have the following inequality:*

$$\left|\Gamma_N\left(\mathbf{A}^*\right)\right| \geqslant \left|\Gamma_n\left(\mathbf{A}\right)\right| \sin\frac{\angle A}{2} + \left(1 - \sin\frac{\angle A}{2}\right)(N-n)\,\delta. \tag{1.5}$$

Assertion 1.3. *Let* $\mathrm{CLS}\left\{\Gamma_n(\mathbf{A}), \Gamma_N(\mathbf{A}^*), \delta\right\}_{\mathbb{E}}$ *be a* CLS*, where n is an even number, and let*

$$\sum_{j=1}^{n}(-1)^{j+1}a_j \geqslant 0.$$

Then we have the following two assertions:
(I) If

$$\delta\,(N-n) > \sum_{j=1}^{n}(-1)^{j+1}a_j,$$

then we have

$$\begin{aligned}\left|\Gamma_N\left(\mathbf{A}^*\right)\right| \geqslant \Bigg\{&\sin^2\frac{\angle A}{2}\left[\left|\Gamma_n\left(\mathbf{A}\right)\right| - \delta\,(N-n)\right]^2 \\ &+ 4\delta^2\cos^2\frac{\angle A}{2}\min{}^2\left\{\{\omega\}, 1-\{\omega\}\right\}\Bigg\}^{1/2} + \delta\,(N-n),\end{aligned} \tag{1.6}$$

where

$$\omega = \frac{\sum_{j=1}^{n}(-1)^{j+1}a_j + \delta\,(N-n)}{2\delta},\quad \{\omega\} = \omega - [\omega] \in [0,1),$$

and $[\omega]$ is the Gaussian function.
(II) If

$$\delta\,(N-n) \leqslant \sum_{j=1}^{n}(-1)^{j+1}a_j,$$

then we have

$$|\Gamma_N(\mathbf{A}^*)| \geqslant \left\{\sin^2\frac{\angle A}{2}[|\Gamma_n(\mathbf{A})| - \delta(N-n)]^2 + \cos^2\frac{\angle A}{2}\left[\sum_{j=1}^{n}(-1)^{j+1}a_j - \delta(N-n)\right]^2\right\}^{1/2} + \delta(N-n). \tag{1.7}$$

For Assertion 1.3, one of the interesting examples is as follows.

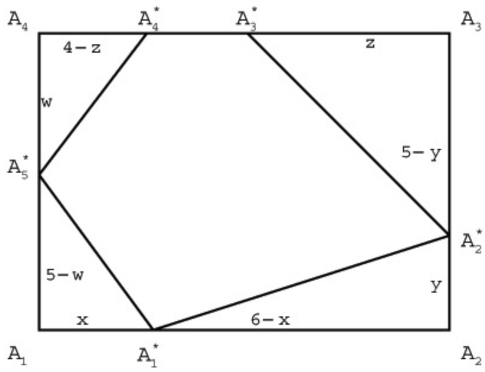

Figure 1: The graph of the CLS $\{\Gamma_4(\mathbf{A}), \Gamma_5(\mathbf{A}^*), 2\}_{\mathbb{R}^2}$.

Example 1.4. (see Example 4.3 in [6]) Consider the CLS $\{\Gamma_4(\mathbf{A}), \Gamma_5(\mathbf{A}^*), 2\}_{\mathbb{R}^2}$, see Figure 1, where $\Gamma_4(\mathbf{A})$ is a rectangle, and

$$\|A_2 - A_1\| = \|A_4 - A_3\| = 6, \ \ \|A_3 - A_2\| = \|A_1 - A_4\| = 5,$$

and

$$A_1^* \in [A_1A_2), \ A_2^* \in [A_2A_3), \ A_3^*, A_4^* \in [A_3A_4), \ A_5^* \in [A_4A_1).$$

Then we have

$$\inf\{|\Gamma_5(\mathbf{A}^*)|\} = 10\sqrt{2} + 2. \tag{1.8}$$

In this paper, we will study the sharp upper bounds of

$$\|\Gamma_N(\mathbf{A}^*)\| \triangleq \frac{1}{2}\sum_{1\leqslant j-i\leqslant N-1, 1\leqslant i\leqslant N}\left\|A_j^* - A_i^*\right\|.$$

Our purpose is to estimate the building cost of the underground passages in the above passage layout problem.

Our main result is the following Theorem 1.5.

Theorem 1.5. *Let* CLS $\{\Gamma_n(\mathbf{A}), \Gamma_N(\mathbf{A}^*), \delta\}_{\mathbb{E}}$ *be a* CLS, *and let* $n \geqslant 4$. *Then we have*

$$\frac{1}{2}\sum_{1\leqslant j-i\leqslant N-1, 1\leqslant i\leqslant N}\left\|A_j^* - A_i^*\right\| \leqslant \frac{1}{4}\sqrt{N\left(1 + \max_{1\leqslant i\leqslant n}|\cos\angle A_i|\right)\csc^2\frac{\pi}{2N}}\sqrt{\sum_{i=1}^{n}\|A_{i+1} - A_i\|^2}. \tag{1.9}$$

Equality in (1.9) *holds if* $\mathbb{E} = \mathbb{R}^2$, $n = N = 4$ *and* $\Gamma_n(\mathbf{A}) = \Gamma_N(\mathbf{A}^*)$ *is a regular* 4*-polygon.*

The connotation of Euclidean space is very rich.

Let $\mathbb{E}$ be an abstract n-dimensional linear space in the real number field $\mathbb{R}$, and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the base of $\mathbb{E}$, as well as let $P \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Then, for any

$$\alpha = x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_n\varepsilon_n \in \mathbb{E},\ \beta = y_1\varepsilon_1 + y_2\varepsilon_2 + \cdots + y_n\varepsilon_n \in \mathbb{E},$$

we can define the inner product $\langle \alpha, \beta \rangle$ as follows,

$$\langle \alpha, \beta \rangle \triangleq (x_1, x_2, \ldots, x_n)\, P\, (y_1, y_2, \ldots, y_n)^T, \tag{1.10}$$

which satisfies the following conditions of inner product:
(i) $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle,\ \forall\ \alpha, \beta \in \mathbb{E}$;
(ii) $\langle \lambda\alpha, \beta \rangle = \lambda \langle \alpha, \beta \rangle,\ \forall\ \alpha \in \mathbb{E},\ \forall\ \lambda \in \mathbb{R}$;
(iii) $\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle,\ \forall\ \alpha, \beta, \gamma \in \mathbb{E}$;
(iv) $\langle \alpha, \alpha \rangle \geqslant 0,\ \forall\ \alpha \in \mathbb{E}$ and $\langle \alpha, \alpha \rangle = 0 \Leftrightarrow \alpha = 0$.
Hence for the above inner product $\langle \alpha, \beta \rangle$, the $\mathbb{E}$ is a Euclidean space where $\dim \mathbb{E} = n$.

Let $S(\mathbb{R}^{n\times n})$ be a set of real symmetric matrices which are defined on $\mathbb{R}^{n\times n}$. Then, for any $A, B \in S(\mathbb{R}^{n\times n})$, we can define the inner product $\langle A, B \rangle$ as follows,

$$\langle A, B \rangle \triangleq \mathrm{tr}(AB), \tag{1.11}$$

and we can easily prove that which satisfies the conditions of inner product, where $\mathrm{tr}(A)$ is the trace of the matrix A. So, for the above inner product $\langle A, B \rangle$, the $S(\mathbb{R}^{n\times n})$ is a Euclidean space where $\dim S(\mathbb{R}^{n\times n}) = n^2$.

Let $C[a,b]$ be a set of continuous functions which are defined on the interval $[a,b]$. Then, for any $f, g \in C[a,b]$, we can define the inner product $\langle f, g \rangle$ as follows,

$$\langle f, g \rangle \triangleq \int_a^b f(t)g(t)\mathrm{d}t. \tag{1.12}$$

Therefore, for the above inner product $\langle f, g \rangle$, the $C[a,b]$ is a Euclidean space where $\dim C[a,b] = \infty$.

Based on the above analysis, we know that Theorem 1.5 is of great theoretical significance and extensive application value.

2. Preliminaries

In order to prove Theorem 1.5, we need seven lemmas as follows.

According to the assumptions (H1.2)–(H1.5), we may easily get the following Lemmas 2.1 and 2.2.

Lemma 2.1 (see Lemma 2.4 in [6])**.** *Let $B, C \in \mathbb{E}$. If $B \neq C$ and $D \in [BC]$, then*

$$\|C - B\| = \|C - D\| + \|D - B\|. \tag{2.1}$$

Lemma 2.2 (see Lemma 2.5 in [6])**.** *Let* CLS $\{\Gamma_n(\mathbf{A}), \Gamma_N(\mathbf{A}^*), \delta\}_{\mathbb{E}}$ *be a* CLS. *Then for any $i \in \{1, 2, \ldots, n\}$, there exist*

$$\sigma(i) \in \{1, 2, \ldots, N\} \quad \textit{and } \tau(i) \in \{0, 1, \cdots, N-n\}$$

such that

$$A^*_{\sigma(i)+k} \in [A_i A_{i+1}),\ k = 0, 1, \ldots,\ \tau(i),$$

and

$$\sum_{i=1}^{n} \tau(i) = N - n. \tag{2.2}$$

Lemma 2.3. *If $N \geqslant 4$, then*

$$\sum_{k=1}^{N-1} \sin \frac{k\pi}{N} = \cot \frac{\pi}{2N}. \tag{2.3}$$

Proof. According to the Euler's formula:

$$\exp(\theta \mathbf{j}) = \cos\theta + \mathbf{j}\sin\theta,$$

where $\mathbf{j}^2 = -1$, we see that

$$\begin{aligned}\sum_{k=1}^{N-1} \sin \frac{k\pi}{N} &= \sum_{k=0}^{N-1} \sin \frac{k\pi}{N} = \operatorname{Im}\left(\sum_{k=0}^{N-1} \exp \frac{k\pi \mathbf{j}}{N}\right) \\ &= \operatorname{Im}\left[\frac{1-\exp(\pi\mathbf{j})}{1-\exp\frac{\pi\mathbf{j}}{N}}\right] = \operatorname{Im}\left[\frac{2}{\exp\frac{\pi\mathbf{j}}{2N}\left(\exp\frac{-\pi\mathbf{j}}{2N} - \exp\frac{\pi\mathbf{j}}{2N}\right)}\right] \\ &= \operatorname{Im}\left(\frac{2\exp\frac{-\pi\mathbf{j}}{2N}}{-2\mathbf{j}\sin\frac{\pi}{2N}}\right) = \cot\frac{\pi}{2N}.\end{aligned}$$

That is to say, (2.3) holds. The proof is completed. □

Lemma 2.4. *For any 4-polygon $\Gamma_4(A,B,C,D)$ in $\mathbb{E}$, we have*

$$\|C-A\|^2 + \|D-B\|^2 \leqslant \|C-B\|^2 + \|A-D\|^2 + 2\|B-A\| \times \|D-C\|. \tag{2.4}$$

Equality in (2.4) holds if and only if $\angle(B-A, D-C) = \pi$.

Proof. Set

$$(B-A, C-B, D-C, A-D, C-A, D-B) = (a,b,c,d,e,f).$$

Then (2.4) can be rewritten as

$$e^2 + f^2 \leqslant b^2 + d^2 + 2\|a\| \cdot \|c\|. \tag{2.5}$$

Since

$$a+b=e,\ c+d=-e,\ b+c=f,\ d+a=-f,$$

we have

$$\begin{aligned} 2\langle a,b\rangle &= e^2 - a^2 - b^2, \\ 2\langle c,d\rangle &= e^2 - c^2 - d^2, \\ 2\langle b,c\rangle &= f^2 - b^2 - c^2, \\ 2\langle d,a\rangle &= f^2 - d^2 - a^2, \\ a+b+c+d &= 0. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= (a+b+c+d)^2 \\ &= a^2+b^2+c^2+d^2 + 2(\langle a,b\rangle + \langle c,d\rangle + \langle b,c\rangle + \langle d,a\rangle) + 2(\langle a,c\rangle + \langle b,d\rangle) \\ &= -(a^2+b^2+c^2+d^2) + 2(e^2+f^2) + 2(\langle a,c\rangle + \langle b,d\rangle) \\ &= -2(a^2+b^2+c^2+d^2) + 2(e^2+f^2) + (a+c)^2 + (b+d)^2 \\ &= -2(a^2+b^2+c^2+d^2) + 2(e^2+f^2) + 2(a+c)^2 \\ &= -2(b^2+d^2) + 2(e^2+f^2) + 4\langle a,c\rangle \end{aligned}$$

$$\begin{aligned}&\geqslant -2(b^2+d^2)+2(e^2+f^2)-4\|a\|\cdot\|c\|\\&\Rightarrow e^2+f^2\leqslant b^2+d^2+2\|a\|\cdot\|c\|.\end{aligned}$$

That is to say, (2.5) holds. Equality in (2.5) holds if and only if,

$$-2\langle a,c\rangle = 2\|a\|\cdot\|c\| \Leftrightarrow \angle(B-A,D-C)\triangleq\angle(a,c)=\pi.$$

The proof is completed. □

Lemma 2.5. *Let $\Gamma_N(A)(N\geqslant 4)$ be a polygon in $\mathbb{E}$, and let*

$$S_k\triangleq\sin\frac{k\pi}{N},\;\; L_k\triangleq\frac{1}{N(2S_k)^2}\sum_{i=1}^{N}\|A_{i+k}-A_i\|^2,\;\; k=1,2,\ldots,N-1.$$

Then we have

$$L_k\leqslant\left(\frac{S_1}{S_k}\right)^2 L_1+\frac{S_{k-1}S_{k+1}}{2S_k^2}\left(L_{k+1}+L_{k-1}\right),\;\; k=2,3,\ldots,N-2, \tag{2.6}$$

and equalities in (2.6) *hold if, and only if,*

$$\frac{A_{i-1+k}-A_i}{S_{k-1}}=\frac{A_{i+k}-A_{i-1}}{S_{k+1}},\;\; i=1,2,\ldots,N, \tag{2.7}$$

and a sufficient condition that the equalities in (2.6) *hold is that $\mathbb{E}=\mathbb{R}^2$ and $\Gamma_N(A)$ is a regular N-polygon in $\mathbb{R}^2$.*

Proof. Consider the quadrilateral $\Gamma_4\left(A_{i-1},A_i,A_{i-1+k},A_{i+k}\right)$. From

$$\left(\frac{\|A_{i-1+k}-A_i\|}{S_{k-1}}-\frac{\|A_{i+k}-A_{i-1}\|}{S_{k+1}}\right)^2\geqslant 0, \tag{2.8}$$

we obtain that

$$2\|A_{i-1+k}-A_i\|\cdot\|A_{i+k}-A_{i-1}\|\leqslant\frac{S_{k+1}}{S_{k-1}}\|A_{i-1+k}-A_i\|^2+\frac{S_{k-1}}{S_{k+1}}\|A_{i+k}-A_{i-1}\|^2. \tag{2.9}$$

It follows from Lemma 2.4 and (2.9) that

$$\begin{aligned}&\|A_{i+k}-A_i\|^2+\|A_{i-1+k}-A_{i-1}\|^2\\&\leqslant\|A_i-A_{i-1}\|^2+\|A_{i+k}-A_{i-1+k}\|^2+2\|A_{i-1+k}-A_i\|\cdot\|A_{i+k}-A_{i-1}\|\\&\leqslant\|A_i-A_{i-1}\|^2+\|A_{i+k}-A_{i-1+k}\|^2+\frac{S_{k+1}}{S_{k-1}}\|A_{i-1+k}-A_i\|^2\\&\quad+\frac{S_{k-1}}{S_{k+1}}\|A_{i+k}-A_{i-1}\|^2,\end{aligned} \tag{2.10}$$

which implies that

$$\sum_{i=1}^{N}\left(\|A_{i+k}-A_i\|^2+\|A_{i-1+k}-A_{i-1}\|^2\right)\leqslant M, \tag{2.11}$$

where

$$M\triangleq\sum_{i=1}^{N}\left(\|A_i-A_{i-1}\|^2+\|A_{i+k}-A_{i-1+k}\|^2+\frac{S_{k+1}}{S_{k-1}}\|A_{i-1+k}-A_i\|^2+\frac{S_{k-1}}{S_{k+1}}\|A_{i+k}-A_{i-1}\|^2\right).$$

Since

$$\begin{aligned}
\sum_{i=1}^{N}\|A_{i+k}-A_i\|^2 &= \sum_{i=1}^{N}\|A_{i-1+k}-A_{i-1}\|^2 = 4NS_k^2L_k,\\
\sum_{i=1}^{N}\|A_i-A_{i-1}\|^2 &= \sum_{i=1}^{N}\|A_{i+k}-A_{i-1+k}\|^2 = 4NS_1^2L_1,\\
\sum_{i=1}^{N}\|A_{i-1+k}-A_i\|^2 &= 4NS_{k-1}^2L_{k-1},
\end{aligned}$$

and

$$\sum_{i=1}^{N}\|A_{i+k}-A_{i-1}\|^2 = 4NS_{k+1}^2L_{k+1},$$

the inequality (2.11) is equivalent to

$$8NS_k^2L_k \leqslant 8NS_1^2L_1 + 4NS_{k+1}S_{k-1}L_{k-1} + 4NS_{k-1}S_{k+1}L_{k+1},$$

that is

$$L_k \leqslant \left(\frac{S_1}{S_k}\right)^2 L_1 + \frac{S_{k-1}S_{k+1}}{2S_k^2}(L_{k+1}+L_{k-1}).$$

According to Lemma 2.4, equalities in (2.6) hold if and only if (2.7) holds. Furthermore, as can be checked easily, a sufficient condition that the equalities in (2.6) hold is that $\mathbb{E}=\mathbb{R}^2$ and $\Gamma_N(A)$ is a regular N-polygon in $\mathbb{R}^2$. The proof is completed. □

Remark 2.6. We remark here that the sufficient condition of equalities in (2.6) is not necessary. For example, when $\mathbb{E}=\mathbb{R}^2,\ N=4$, the equality in (2.6) holds if and only if $\Gamma_4(A)$ is a parallelogram in $\mathbb{R}^2$.

Indeed, if $\mathbb{E}=\mathbb{R}^2,\ N=4$ and $k=2$, then

$$\begin{aligned}
\frac{A_{i-1+k}-A_i}{S_{k-1}} &= \frac{A_{i+k}-A_{i-1}}{S_{k+1}} \Leftrightarrow\\
\frac{A_{i+1}-A_i}{S_1} &= \frac{A_{i+2}-A_{i-1}}{S_3} \Leftrightarrow\\
A_{i+1}-A_i &= A_{i+2}-A_{i-1},\ i=1,2, \Leftrightarrow\\
A_2-A_1 &= A_3-A_4,\ A_3-A_2=A_4-A_1.
\end{aligned} \tag{2.12}$$

Remark 2.7. If $\Gamma_N(A)$ is a regular N-polygon, then

$$L_k = R_0^2,\ k=1,2,\ldots,N-1, \tag{2.13}$$

where R_0 denotes the radius of the circumcircle of $\Gamma_N(A)$.

Lemma 2.8. *Let $\Gamma_N(A)$ be a polygon in $\mathbb{E}$ with $\dim\mathbb{E}\geqslant 2$, where $N\geqslant 4$, and let L_k be defined in* Lemma 2.5. *Then for any positive integers*

$$k,j : k\geqslant 2,\ k+j\leqslant N-1,$$

there exist positive constants $C_{k+j,j},\ C_{k-1,j},\ C_{1,j}$, which depend only on k,j,N, such that

$$L_k \leqslant C_{k+j,j}L_{k+j} + C_{k-1,j}L_{k-1} + C_{1,j}L_1, \tag{2.14}$$

and

$$C_{k+j,j}+C_{k-1,j}+C_{1,j}=1. \tag{2.15}$$

A sufficient condition that equalities in (2.14) *hold is that $\mathbb{E}=\mathbb{R}^2$ and $\Gamma_N(A)$ is a regular N-polygon in $\mathbb{R}^2$.*

Proof. The proof is based on the mathematical induction method for j.

(I) When $j = 1$, let

$$C_{k+1,1} = C_{k-1,1} = \frac{S_{k-1}S_{k+1}}{2S_k^2} > 0 \quad \text{and } C_{1,1} = \left(\frac{S_1}{S_k}\right)^2 > 0.$$

From Lemma 2.5, we have

$$L_k \leqslant C_{k+1,1}L_{k+1} + C_{k-1,1}L_{k-1} + C_{1,1}L_1. \tag{2.16}$$

Let $\Gamma_N(A)$ be a regular N-polygon in $\mathbb{R}^2$. In view of Remark 2.7, we know that

$$L_k = L_{k+1} = L_{k-1} = L_1 = R_0^2 > 0.$$

It follows from Lemma 2.5 that equality in (2.16) holds. Thus,

$$C_{k+1,1} + C_{k-1,1} + C_{1,1} = 1.$$

(II) Suppose that (2.14) and (2.15) hold for $j = n \geqslant 1$. Then there exist positive constants $C_{k+n,n}$, $C_{k-1,n}$, $C_{1,n}$ such that

$$C_{k+n,n} + C_{k-1,n} + C_{1,n} = 1, \tag{2.17}$$

and

$$L_k \leqslant C_{k+n,n}L_{k+n} + C_{k-1,n}L_{k-1} + C_{1,n}L_1, \tag{2.18}$$

and a sufficient condition that the equalities in (2.18) hold is that $\Gamma_N(A)$ is a regular N-polygon.

Since $k+1 \geqslant 3 > 2$ and $(k+1)+n \leqslant N-1$, by the inductive assumption, there exist positive constants $C^*_{k+1+n,n}$, $C^*_{k,n}$, $C^*_{1,n}$ such that

$$C^*_{k+1+n,n} + C^*_{k,n} + C^*_{1,n} = 1, \tag{2.19}$$

and

$$L_{k+1} \leqslant C^*_{k+1+n,n}L_{k+1+n} + C^*_{k,n}L_k + C^*_{1,n}L_1. \tag{2.20}$$

Substituting (2.20) into (2.16), we see that

$$L_k \leqslant C_{k+1,1}\left(C^*_{k+1+n,n}L_{k+1+n} + C^*_{k,n}L_k + C^*_{1,n}L_1\right) + C_{k-1,1}L_{k-1} + C_{1,1}L_1. \tag{2.21}$$

Note that

$$0 < C_{k+1,1} < 1, 0 < C^*_{k,n} < 1 \quad \text{and } 1 - C_{k+1,1}C^*_{k,n} > 0.$$

Solving the inequality (2.21) with respect to L_k, we obtain that

$$L_k \leqslant C^{**}_{k+n+1,n+1}L_{k+n+1} + C^{**}_{k-1,n+1}L_{k-1} + C^{**}_{1,n+1}L_1, \tag{2.22}$$

where

$$C^{**}_{k+n+1,n+1} = \frac{C_{k+1,1}C^*_{k+n+1,n}}{1 - C_{k+1,1}C^*_{k,n}} > 0,$$

$$C^{**}_{k-1,n+1} = \frac{C_{k-1,1}}{1 - C_{k+1,1}C^*_{k,n}} > 0,$$

$$C^{**}_{1,n+1} = \frac{C_{k+1,1}C^*_{1,n} + C_{1,1}}{1 - C_{k+1,1}C^*_{k,n}} > 0.$$

Let $\Gamma_N(A)$ be a regular N-polygon in $\mathbb{R}^2$. In view of Remark 2.7, we know that

$$L_k = L_{k+n+1} = L_{k-1} = L_1 = R_0^2 > 0.$$

It follows from Lemma 2.5 and our induction hypothesis that the equality in (2.22) holds. Thus,

$$C^{**}_{k+n+1,n+1} + C^{**}_{k-1,n+1} + C^{**}_{1,n+1} = 1.$$

This ends the proof. □

Lemma 2.9. *Let $\Gamma_N(A)$ be a polygon in $\mathbb{E}$ with $\dim \mathbb{E} \geqslant 2$, where $N \geqslant 4$, and let L_k be defined in* Lemma 2.5. *Then $L_k \leqslant L_1$, i.e.,*

$$\sum_{i=1}^{N} \|A_{i+k} - A_i\|^2 \leqslant \left(\frac{\sin \frac{k\pi}{N}}{\sin \frac{\pi}{N}}\right)^2 \sum_{i=1}^{N} \|A_{i+1} - A_i\|^2, \; k = 2, 3, \ldots, N-2. \tag{2.23}$$

A sufficient condition that the equalities in (2.23) *hold is that $\mathbb{E} = \mathbb{R}^2$ and $\Gamma_N(A)$ is a regular N-polygon in $\mathbb{R}^2$.*

Proof. Set $k + j = N - 1$ in (2.14). Then

$$L_k \leqslant C_{N-1,N-1-k} L_{N-1} + C_{k-1,N-1-k} L_{k-1} + C_{1,N-1-k} L_1. \tag{2.24}$$

Since

$$A_i = A_j \Leftrightarrow i \equiv j (\mathrm{mod} N),$$

we have

$$L_{N-1} \triangleq \frac{1}{N(2S_{N-1})^2} \sum_{i=1}^{N-1} \|A_{i+N-1} - A_i\|^2 = \frac{1}{N(2S_1)^2} \sum_{i=1}^{N-1} \|A_{i-1} - A_i\|^2 = L_1. \tag{2.25}$$

It follows from (2.24) and (2.25) that, for any $k \in \{2, 3, \ldots, N-2\}$, there exist positive constants C_{k-1} and C_1 such that

$$C_{k-1} + C_1 = 1, \; L_k \leqslant C_{k-1} L_{k-1} + C_1 L_1, \tag{2.26}$$

as well as

$$C_{k-1} = C_{k-1,N-1-k} > 0, \; C_1 = C_{N-1,N-1-k} + C_{1,N-1-k} > 0.$$

Repeated use (2.26), we get

$$\begin{aligned} L_k &\leqslant C_{k-1} L_{k-1} + C_1 L_1 \leqslant C_{k-1} \left(C^*_{k-2} L_{k-2} + C^*_1 L_1\right) + C_1 L_1 \\ &= C^{**}_{k-2} L_{k-2} + C^{**}_1 L_1 \leqslant C^{***}_{k-3} L_{k-3} + C^{***}_1 L_1 \leqslant \cdots \leqslant C L_1. \end{aligned}$$

Hence,

$$L_k \leqslant C L_1. \tag{2.27}$$

Set $\Gamma_N(A)$ is a regular polygon in $\mathbb{R}^2$, by Remark 2.7, we know that

$$L_k = L_1 = R_0^2 > 0.$$

By Lemma 2.8, equality in (2.27) holds, which implies that $C = 1$. Hence (2.23) holds. This completes the proof. □

3. Proof of Theorem 1.5

Proof. Set that

$$x_i \triangleq \left\| A^*_{\sigma(i)} - A_i \right\|, \qquad y_i \triangleq \left\| A^*_{\sigma(i-1)+\tau(i-1)} - A_i \right\|,$$

$$z_{i,k} \triangleq \left\| A^*_{\sigma(i)+k} - A^*_{\sigma(i)+k-1} \right\|, \qquad \rho \triangleq \max_{1 \leqslant i \leqslant n} \left|\cos \angle A_i\right|.$$

By Lemmas 2.1 and 2.2, we have

$$x_i + y_{i+1} + \sum_{k=1}^{\tau(i)} z_{i,k} = \|A_{i+1} - A_i\|, i = 1, 2, \ldots, n. \tag{3.1}$$

By Lemma 2.2, we obtain that

$$\mathbf{A}^* = \left(\ldots, A^*_{\sigma(i-1)}, \ldots, A^*_{\sigma(i-1)+\tau(i-1)}, A^*_{\sigma(i)}, \ldots, A^*_{\sigma(i)+\tau(i)}, \ldots\right). \tag{3.2}$$

According to the Jensen's inequality [7, Lemma 2.6]:

$$\sum_{k=1}^{n} x_k^{\gamma} \leqslant \left(\sum_{k=1}^{n} x_k\right)^{\gamma}, \ \forall\ x \in [0,\infty)^n,\ \forall\ \gamma \in (1,\infty),$$

(3.1), (3.2) and

$$\|\alpha - \beta\| = \sqrt{\|\alpha\|^2 + \|\beta\|^2 - 2\|\alpha\| \cdot \|\beta\| \cos\angle(\alpha,\beta)},$$

we see that

$$\begin{aligned}
\sum_{i=1}^{N} \left\|A^*_{i+1} - A^*_i\right\|^2 &= \sum_{i=1}^{n} \left(\left\|A^*_{\sigma(i)} - A^*_{\sigma(i-1)+\tau(i-1)}\right\|^2 + \sum_{k=1}^{\tau(i)} z_{i,k}^2\right) \\
&= \sum_{i=1}^{n} \left[\left\|\left(A^*_{\sigma(i)} - A_i\right) - \left(A^*_{\sigma(i-1)+\tau(i-1)} - A_j\right)\right\|^2 + \sum_{k=1}^{\tau(i)} z_{i,k}^2\right] \\
&= \sum_{i=1}^{n} \left(x_i^2 + y_i^2 - 2x_i y_i \cos\angle A_i + \sum_{k=1}^{\tau(i)} z_{i,k}^2\right) \\
&\leqslant \sum_{i=1}^{n} \left(x_i^2 + y_i^2 + 2\rho x_i y_i + \sum_{k=1}^{\tau(i)} z_{i,k}^2\right) \\
&\leqslant \sum_{i=1}^{n} \left[x_i^2 + y_i^2 + \rho\left(x_i^2 + y_i^2\right) + \sum_{k=1}^{\tau(i)} z_{i,k}^2\right] \\
&= (1+\rho) \sum_{i=1}^{n} \left(x_i^2 + y_i^2\right) + \sum_{i=1}^{n} \sum_{k=1}^{\tau(i)} z_{i,k}^2 \\
&= (1+\rho) \sum_{j=1}^{n} \left(x_i^2 + y_{i+1}^2\right) + \sum_{i=1}^{n} \sum_{k=1}^{\tau(i)} z_{i,k}^2 \\
&\leqslant (1+\rho) \sum_{i=1}^{n} \left(x_i + y_{i+1}\right)^2 + \sum_{i=1}^{n} \sum_{k=1}^{\tau(i)} z_{i,k}^2 \\
&= (1+\rho) \sum_{i=1}^{n} \left(\|A_{i+1} - A_i\| - \sum_{k=1}^{\tau(i)} z_{i,k}\right)^2 + \sum_{i=1}^{n} \sum_{k=1}^{\tau(i)} z_{i,k}^2 \\
&\leqslant (1+\rho) \left[\sum_{i=1}^{n} \left(\|A_{i+1} - A_i\| - \sum_{k=1}^{\tau(i)} z_{i,k}\right)^2 + \sum_{i=1}^{n} \sum_{k=1}^{\tau(i)} z_{i,k}^2\right] \\
&= (1+\rho) \sum_{i=1}^{n} \left[\left(\|A_{i+1} - A_i\| - \sum_{k=1}^{\tau(i)} z_{i,k}\right)^2 + \sum_{k=1}^{\tau(i)} z_{i,k}^2\right] \\
&\leqslant (1+\rho) \sum_{i=1}^{n} \left(\|A_{i+1} - A_i\| - \sum_{k=1}^{\tau(i)} z_{i,k} + \sum_{k=1}^{\tau(i)} z_{i,k}\right)^2
\end{aligned}$$

$$= (1+\rho)\sum_{i=1}^{n}\|A_{i+1}-A_i\|^2,$$

i.e.,

$$\sum_{i=1}^{N}\left\|A_{i+1}^*-A_i^*\right\|^2 \leqslant (1+\rho)\sum_{i=1}^{n}\|A_{i+1}-A_i\|^2. \tag{3.3}$$

According to the power mean inequality (see [7, Lemma 2.3] and [1, 2, 3, 4, 5]), we have that

$$\sum_{k=1}^{n}\mu_k x_k^{\gamma} \geqslant \left(\sum_{k=1}^{n}\mu_k x_k\right)^{\gamma},\ \forall\ x,\mu\in[0,\infty)^n,\ \forall\ \gamma\in(1,\infty),$$

where μ satisfies the condition

$$\sum_{k=1}^{n}\mu_k = 1.$$

According to Lemmas 2.9, 2.3 and (3.1), we obtain that

$$\begin{aligned}
\frac{1}{2}\sum_{1\leqslant j-i\leqslant N-1,1\leqslant i\leqslant N}\left\|A_j^*-A_i^*\right\| &= \frac{1}{2}\sum_{k=1}^{N-1}\sum_{i=1}^{N}\left\|A_{i+k}^*-A_i^*\right\| \\
&= \frac{N}{2}\sum_{k=1}^{N-1}\frac{1}{N}\sum_{i=1}^{N}\left\|A_{i+k}^*-A_i^*\right\| \\
&\leqslant \frac{N}{2}\sum_{k=1}^{N-1}\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|A_{i+k}^*-A_i^*\right\|^2} \\
&\leqslant \frac{N}{2}\sum_{k=1}^{N-1}\sqrt{\frac{1}{N}\left(\frac{\sin\frac{k\pi}{N}}{\sin\frac{\pi}{N}}\right)^2\sum_{i=1}^{N}\left\|A_{i+1}^*-A_i^*\right\|^2} \\
&= \frac{1}{2}\sqrt{N}\csc\frac{\pi}{N}\left(\sum_{k=1}^{N-1}\sin\frac{k\pi}{N}\right)\sqrt{\sum_{i=1}^{N}\left\|A_{i+1}^*-A_i^*\right\|^2} \\
&\leqslant \frac{1}{2}\sqrt{N}\csc\frac{\pi}{N}\left(\sum_{k=1}^{N-1}\sin\frac{k\pi}{N}\right)\sqrt{(1+\rho)\sum_{i=1}^{n}\|A_{i+1}-A_i\|^2} \\
&= \frac{1}{2}\sqrt{N(1+\rho)}\csc\frac{\pi}{N}\cot\frac{\pi}{2N}\sqrt{\sum_{i=1}^{n}\|A_{i+1}-A_i\|^2} \\
&= \frac{1}{4}\sqrt{N(1+\rho)}\csc^2\frac{\pi}{2N}\sqrt{\sum_{i=1}^{n}\|A_{i+1}-A_i\|^2} \\
&= \frac{1}{4}\sqrt{N\left(1+\max_{1\leqslant i\leqslant n}|\cos\angle A_i|\right)}\csc^2\frac{\pi}{2N}\sqrt{\sum_{i=1}^{n}\|A_{i+1}-A_i\|^2}.
\end{aligned}$$

This shows that inequality (1.9) holds.

Based on the above proof, we may see that if $\mathbb{E}=\mathbb{R}^2,\ n=N=4,\ \Gamma_n(\mathbf{A})=\Gamma_N(\mathbf{A}^*)$, and $\Gamma_n(\mathbf{A})$ is a regular 4-gon, then the equality in (1.9) holds, see Example 4.1. This completes the proof of Theorem 1.5. □

A large number of algebraic, analytic, geometry and inequality theories are used in the proof of our results. In order to prove Theorem 1.5, we need Lemmas 2.1, 2.2, 2.3, 2.4, 2.5, 2.8 and 2.9. Indeed, the proof of Theorem 1.5 is both interesting and difficult. Some techniques related to the proof of Theorem 1.5 can also be found in the references [1]–[3] cited in this paper.

4. An example for Theorem 1.5

We give here an example to illustrate the applications of Theorem 1.5.

Example 4.1. Consider the CLS $\{\Gamma_4(\mathbf{A}), \Gamma_4(\mathbf{A}^*), l\}_{\mathbb{R}^2}$, here $\mathbb{E} = \mathbb{R}^2$, $n = N = 4$, $0 < l < 1/2$, and $\Gamma_4(\mathbf{A})$ is a regular 4-polygon where

$$\|A_{i+1} - A_i\| = 1, \ i = 1, 2, 3, 4,$$

see Figure 2.

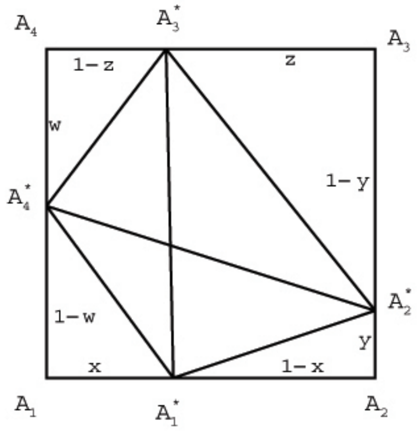

Figure 2: The graph of the CLS $\{\Gamma_4(\mathbf{A}), \Gamma_4(\mathbf{A}^*), l\}_{\mathbb{R}^2}$ where $0 < l < 1/2$.

If $\Gamma_n(\mathbf{A}) = \Gamma_N(\mathbf{A}^*) \Leftrightarrow (A_1, A_2, A_3, A_4) = (A_1^*, A_2^*, A_3^*, A_4^*)$, then, by (1.2), we have

$$\begin{aligned}\frac{1}{2}\sum_{1\leqslant j-i\leqslant N-1, 1\leqslant i\leqslant N}\left\|A_j^* - A_i^*\right\| &= \frac{1}{2}\sum_{1\leqslant j-i\leqslant 3, 1\leqslant i\leqslant 4}\|A_j - A_i\| \\ &= \|A_2 - A_1\| + \|A_3 - A_2\| + \|A_4 - A_3\| \\ &\quad + \|A_1 - A_4\| + \|A_1 - A_3\| + \|A_2 - A_4\| \\ &= 4 + 2\sqrt{2},\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{4}\sqrt{N\left(1 + \max_{1\leqslant i\leqslant n}|\cos\angle A_i|\right)\csc^2\frac{\pi}{2N}}\sqrt{\sum_{i=1}^{n}\|A_{i+1} - A_i\|^2} \\ &= \frac{1}{4}\sqrt{4\left(1 + \max_{1\leqslant i\leqslant 4}\left|\cos\frac{\pi}{2}\right|\right)\csc^2\frac{\pi}{8}} \times \sqrt{4} \\ &= \csc^2\frac{\pi}{8} = \frac{1}{\sin^2\frac{\pi}{8}} = \frac{2}{1 - \cos\frac{\pi}{4}} = \frac{2}{1 - \sqrt{2}/2} \\ &= 4 + 2\sqrt{2}.\end{aligned}$$

Therefore, equality in (1.9) holds for this case. According to Theorem 1.5, we have

$$\sup\{\|\Gamma_4(\mathbf{A}^*)\|\} = 4 + 2\sqrt{2}. \tag{4.1}$$

On the other hand, by means of the Mathematica software, we know that

$$\begin{aligned}
\|\Gamma_4(\mathbf{A}^*)\| &= \|A_2^* - A_1^*\| + \|A_3^* - A_2^*\| + \|A_{4^*} - A_3^*\| + \|A_1^* - A_4^*\| + \|A_1^* - A_3^*\| + \|A_2^* - A_4^*\| \\
&= \sqrt{(1-x)^2 + y^2} + \sqrt{(1-y)^2 + z^2} + \sqrt{(1-z)^2 + w^2} + \sqrt{(1-w)^2 + x^2} \\
&\quad + \sqrt{(1-x-z)^2 + 1} + \sqrt{(1-y-w)^2 + 1} \\
&\geqslant 2 + 2\sqrt{2},
\end{aligned}$$

where $(x, y, z, w) \in [0,1]^4$, and the equality holds if and only if

$$x = y = z = w = \frac{1}{2},$$

which is the solution of the equation group

$$\frac{\partial\|\Gamma_4(\mathbf{A}^*)\|}{\partial x} = \frac{\partial\|\Gamma_4(\mathbf{A}^*)\|}{\partial y} = \frac{\partial\|\Gamma_4(\mathbf{A}^*)\|}{\partial z} = \frac{\partial\|\Gamma_4(\mathbf{A}^*)\|}{\partial w} = 0.$$

Therefore,

$$\inf\{\|\Gamma_4(\mathbf{A}^*)\|\} = 2 + 2\sqrt{2}. \tag{4.2}$$

We remark here that, for the infimum of $F(x, y, z, w) \triangleq \|\Gamma_4(\mathbf{A}^*)\|$, by Mathematica software, a direct calculation gives

$$\inf\{F(x, y, z, w)\} = F(0.49999\cdots, 0.49998\cdots, 0.50001\cdots, 0.50003\cdots) = 4.82842712474619\cdots. \tag{4.3}$$

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