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Solutions of fractional differential equations by Sumudu transform and variational iteration method

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Abstract

With the help of the Sumudu transform and the variational iteration method, we solve differential equations and fractional differential equations related to entropy, wavelets etc. The methods which produce solutions in terms of convergent series is explained. Some examples are provided to show the accuracy and easy implementation and to show the methodology. ©2016 All rights reserved.

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1. Introduction

Differential calculus has great importance in the fields of science and technology. In recent years, fractional calculus has been used in various fields like in modeling physical and engineering processes. It is worth noting that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases [5]. In the recent years, fractional calculus has played a very important role in various fields such as chemistry, biology, mechanics, electricity, signal and image processing and notably control theory, wavelets, entropy, etc [4, 5, 6, 21, 22, 27, 28, 31]. In this paper we use Lagrange's multiplier technique (see [16]) which was widely used to solve a number of problems which arise in mathematical physics and other related areas. This technique was developed into a powerful analytical method, (Variational iteration method – VIM) [12] for solving differential equations. The method was applied to initial boundary value problems [1, 13, 26, 36, 37] and fuzzy equations [17, 18, 30], etc. Generally,

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in solving initial value problems of differential equations by VIM, the crucial point is identifying the Lagrange multipliers. To solve fractional differential equations (FDEs) by this method, we may directly use the Lagrange multipliers in ordinary differential equations (ODEs), which often results in poor convergence. To improve this method, a few ideas are suggested. The Riemann–Liouville integral emerges in the constructed correctional functional, but the integration by parts is difficult to apply. To avoid this problem, the Riemann–Liouville integral is replaced by an integral which allows the integration by parts. This is a very strong simplification but it affects the whole process after that step. Therefore, the Lagrange multiplier is determined by a simplification, not reasonably explained in the literature, so far. The technique can be universally extended to solve both ordinary differential equations and fractional differential equations with initial value conditions.

2. Basics of the Variational Iteration Method

To illustrate the basic idea of this method, we consider the following general nonlinear system

$$\frac{d^m u(t)}{dt^m} + L[u(t)] + N[u(t)] = g(t), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, $g(t)$ is a known continuous function and m is the order of the highest-order derivative of the function. The basic characteristic of the method is construction of the following correctional functional for (2.1):

$$u_{n+1} = u_n + \int_0^t \lambda\,(t,\tau) \left[\frac{d^m u}{d\tau^m} + L[u] + N[u] - g(\tau) \right] d\tau,$$

where $\lambda\,(t,\tau)$ is called the general Lagrange multiplier [16] and u_n is an approximate solution of n^{th} order.

Concerning the VIM [12], we know that the integration by parts plays an important role in the determination of Lagrange multipliers. But in fractional calculus, generally, the following integration by parts cannot be done

$${}_0I_t^{\alpha} v_o^C D_t^{\alpha} u = [uv]\,|_0^t - {}_0I_t^{\alpha} u_o^C D_t^{\alpha} v,$$

where ${}_0^C D_t^{\alpha}$ is the well known Caputo derivative, ${}_0I_t^{\alpha}$ is the Riemann–Liouville fractional integral and $v = v(t)$. This is a particular case of VIM, for which this method is not so successful as other analytical methods like the Adomian decomposition method (ADM) [10, 29, 32] and the homotopy perturbation method (HPM) [19, 24, 33] in fractional calculus. To remove this obstacle, we consider the following reconstruction of the method, using the Sumudu transform.

3. Sumudu Transform and the Mittag-Leffler Function

In early 90's, Watugala [2] introduced a new integral transform named the Sumudu transform and applied it to solve ordinary differential equations in engineering control problems. The Sumudu transform is defined over the set of functions:

$$A = \{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{|\,t\,|\tau_j},\ \text{if } t \in (-1)^j \times [0,\infty) \}$$

by

$$\tilde{G}(u) = S[f(t)] = \int_0^{\infty} f(ut)e^{-t} dt, \quad u \in (-\tau_1, \tau_2)$$

see [3, 7, 20, 23].

By using Sumudu transform of multiple differentiation, we obtain

$$\begin{aligned} S\left[{}_0^C D_t^{\alpha} f(t)\right] &= u^{m-\alpha} \left[\frac{\tilde{G}(u)}{u^m} - \sum_{k=0}^{m-1} \frac{f^k(0)}{u^{m-k}} \right], \\ &= \left[\frac{\tilde{G}(u)}{u^{\alpha}} - \sum_{k=0}^{m-1} \frac{f^k(0+)}{u^{\alpha-k}} \right], \qquad (m-1 < \alpha \le m)\,, \end{aligned} \tag{3.1}$$

where $\tilde{G}(u) = S\left[f(t)\right]$.

The Mittag-Leffler function, which is a generalization of the exponential function, is defined by [11]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (\alpha \in C, \;\; Re(\alpha) > 0)\,.$$

4. Identification of the Lagrange Multipliers and Basic idea of VIM by Sumudu Transform

We will see the whole process of the Lagrange multipliers in the case of an algebraic equation. The solution of the algebraic equation $f(x) = 0$ can be obtained by an iteration formula

$$x_{n+1} = x_n + \lambda f(x_n)\,. \tag{4.1}$$

The optimality condition for the extreme $\frac{\delta x_{n+1}}{\delta x_n} = 0$, leads to

$$\lambda = -\frac{1}{f'(x_n)}, \tag{4.2}$$

where δ is the classical variational operator. By using given initial value x_0, we can find the approximate solution x_{n+1} by the following iterative scheme, using (4.1) and (4.2)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \;\; f'(x_0) \neq 0, \; n = 0, 1, 2, \ldots\,. \tag{4.3}$$

The above defined formula (4.3) is well known as the Newton–Raphson formula and has quadratic convergence.

In this paper we extend this idea to find the unknown Lagrange multiplier. In this process, first we apply the Sumudu transform to (2.1), and get

$$u^{-m}\overline{v}(s) - u^{-m}v(0) - \cdots - u^{-1}v^{m-1}(0) + S\left(R[v] \;\; + N[v]\right) = S\left(g(t)\right). \tag{4.4}$$

Using (4.1), the iteration formula (4.4) can be written as

$$\bar{v}_{n+1}(u) = \bar{v}_n(u) + \lambda(u)\left[u^{-m}\overline{v}(u) - u^{-m}v(0) - \cdots - u^{-1}v^{m-1}(0) + S\left(R[v] \;\; + N[v]\right) - S\left(g(t)\right)\right], \tag{4.5}$$

where $\lambda(u) = -u^m$.

By applying the inverse Sumudu transform, S^{-1} to (4.5), after putting the value of $\lambda(u)$, we get

$$\begin{aligned} v_{n+1}(t) &= v_n(t) - S^{-1}\left[u^m\left\{u^{-m}\overline{v}(u) - u^{-m}v(0) - \cdots - u^{-1}v^{m-1}(0) + S\left(R[v] \;\; + N[v]\right) - S\left(g(t)\right)\right\}\right] \\ &= S^{-1}\left[\left(v(0) + \cdots + u^{m-1}\, v^{m-1}(0)\right) + u^m\left(S\left(R[v] \;\; + N[v]\right) - S\left(g(t)\right)\right)\right]. \end{aligned}$$

where the initial iteration $v_0(t)$ is

$$\begin{aligned} v_0(t) &= S^{-1}\left(v(0) + \cdots + u^{m-1}v^{m-1}(0)\right), \\ &= v(0) + v'(0)\,t + \cdots + \frac{t^{m-1}v^{m-1}(0)}{(m-1)!}. \end{aligned} \tag{4.6}$$

The formula (4.6) shows why the initial iteration in the classical VIM is determined by Taylor series.

5. Application to ODEs

In this section, we apply the above defined method to solve both ordinary differential equations and fractional differential equations.

Example 5.1. Consider the following radioactive-decay differential equation:

$$\frac{dN}{dt} = -\alpha N, \qquad N(0) = N_0$$

where $N(t)$ is the number of radioactive nuclei at time t and N_0 the number at $t = 0$, and if their rate of decay $(-dN/dt)$ is proportional to the number of undecayed nuclei, with the constant of proportionality α, then we can obtain successive approximate solutions by

$$\begin{aligned}
&N_0(t) = N(0) = N_0,\\
&N_1(t) = S^{-1}\left(1 - u\alpha\right) N_0 = N_0\left(1 - t\alpha\right),\\
&N_2(t) = S^{-1}\left(1 - \alpha u + \alpha^2 u^2\right) N_0 = N_0\left(1 - \alpha t + \frac{\alpha^2 t^2}{2!}\right),\\
&N_n(t) = S^{-1}\left(1 - \alpha t + \cdots + (-1)^{n+1}\alpha^n u^n\right) N_0 = N_0\left(1 - \alpha t + \cdots + (-1)^{n+1}\frac{\alpha^n t^n}{n!}\right).
\end{aligned}$$

As $n \to \infty$, $N_n(t)$ tends to the exact solution $N_0 e^{-\alpha t}$.

Example 5.2. Consider the following differential equation:

$$\frac{d^2 v}{dt^2} + v = 0, \qquad v(0) = 2, \quad v'(0) = 3,$$

We can obtain successive approximate solutions

$$\begin{aligned}
&v_0(t) = v(0) = 2,\\
&v_1(t) = S^{-1}\left(2 + 3u - 2u^2\right) = \left(2 + 3t - \frac{2t^2}{2!}\right),\\
&v_2(t) = S^{-1}\left(2 + 3u - 2u^2 - 3u^3 + 2u^4\right) = \left(2 + 3t - \frac{2t^2}{2!} - \frac{3t^3}{3!} + \frac{2t^4}{4!}\right),
\end{aligned}$$

As $n \to \infty$, $v_n(t)$ tends to the exact solution $2\cos t + 3\sin t$.

There exist a lot of choices of $v_0(t)$ and $\lambda(u)$ which affect the speed of the convergence.

We note that the integration by parts is not used and the calculation of the Lagrange multiplier here is much simpler. The VIM can be easily extended to FDEs and this is important and the main purpose of our work.

6. Application to FDEs

To illustrate the basic idea of this method for fractional differential equations, we consider a general nonlinear nonhomogeneous fractional differential equation with initial conditions of the following form

$${}_0^C D_t^{\alpha} v + L(v) + N(v) = g(t), \tag{6.1}$$

subject to the initial conditions

$$v^{(k)}\left(0^+\right) = d_k, \;\; 0 < t, \, 0 < \alpha, \, m = [\alpha] + 1, k = 0, 1, \ldots, m-1,$$

where ${}_0^C D_t^{\alpha} v$ is the Caputo derivative [6].

The first step of this method is an application of the Sumudu transform to both sides of (6.1) (the Sumudu transform of ${}_0^C D_t^\alpha v$ is defined by (3.1)) we get

$$\overline{v}_{n+1}(u) = \overline{v}_n(u) + \lambda(u)\left[\left\{u^{-\alpha}\overline{v}(u) - \sum_{k=0}^{m-1}\frac{v^k(0+)}{u^{\alpha-k}} + S\left(R[v] \ + N[v]\right) - S\left(g(t)\right)\right\}\right]. \tag{6.2}$$

Puting $\lambda = -u^\alpha$, the Lagrange multiplier in (6.2) and applying the inverse Sumudu transform, we get

$$\begin{aligned} v_{n+1}(t) &= v_n(u) - S^{-1}\left[u^\alpha\left\{u^{-\alpha}\overline{v}(u) - \sum_{k=0}^{m-1}\frac{v^k(0+)}{u^{\alpha-k}} + S\left(R[v] \ + N[v] - g(t)\right)\right\}\right] \\ &= S^{-1}\left[\left\{\sum_{k=0}^{m-1}\frac{v^k(0+)}{u^{-k}} - u^\alpha S\left(R[v] \ + N[v] - g(t)\right)\right\}\right], \end{aligned}$$

where

$$v_0(t) = S^{-1}\left(\sum_{k=0}^{m-1}\frac{v^k(0+)}{u^{-k}}\right),$$

or

$$v_0(t) \ = v(0) + v'(0)\,t + \cdots + \frac{t^{m-1}v^{-m+1}(0)}{(m-1)!}.$$

Example 6.1. Consider the following fractional-order logistic differential equation see [8, 25]:

$${}_0^C D_t^\alpha v(t) = rv(t)(1-v(t)), \qquad t>0, r>0,\ 0<\alpha\le 1, \tag{6.3}$$

with the initial condition

$$v(0) = v_0.$$

We get the first two successive approximate solutions by

$$\begin{aligned} v_0(t) &= v(0) = v_0, \\ v_1(t) &= S^{-1}\left(v_0 + \frac{u^\alpha\left(v_0 - v_0^2\right)}{2}\right) = \frac{v_0}{2}\left(2 + \frac{t^\alpha\left(1-v_0\right)}{\Gamma(\alpha+1)}\right). \end{aligned}$$

Particular Case. If we take $r = 1/2$, and the initial condition $v(0) = 1/2$, we can obtain successive approximate solutions

$$\begin{aligned} v_0(t) &= v(0) = 1/2, \\ v_1(t) &= S^{-1}\left(\frac{1}{2} + \frac{u^\alpha}{8}\right) = \left(\frac{1}{2} + \frac{0.125\,t^\alpha}{\Gamma(\alpha+1)}\right), \\ v_2(t) &= S^{-1}\left(\frac{1}{2} + \frac{u^\alpha}{8} - \frac{u^{3\alpha}}{128}\right) = \left(\frac{1}{2} + \frac{0.125\,t^\alpha}{\Gamma(\alpha+1)} - \frac{0.0078125\,t^{3\alpha}}{\Gamma(3\alpha+1)}\right). \end{aligned}$$

As $n\to\infty$, $v_n(t)$ tends to the exact solution of (6.3). For $\alpha = 1$, we have $v(t) = \frac{e^{t/2}}{1+e^{t/2}}$.

Example 6.2. Consider the relaxation oscillator equation

$${}_0^C D_t^\alpha v + \omega^\alpha v = 0, \quad v(0) = 1, \quad v'(0) = 0, \qquad 0<t,\ 0<\alpha<2,\ \omega>0. \tag{6.4}$$

Applying the Sumudu transform to (6.4), we get the following iteration formula

$$\overline{v}_{n+1}(u) = \overline{v}_n(u) + \lambda(u)\left[\frac{\overline{v}(u)}{u^\alpha} - \frac{v(0)}{u^\alpha} - \frac{v'(0^+)}{u^{\alpha-1}} + \omega^\alpha S\left(v_n\right)\right],$$

as a result, after identifying a Lagrange multiplier $\lambda = -u^{\alpha}$, approximate solutions of (6.4) can be found by

$$v_{n+1}(t) = v_n(u) - S^{-1}\left[u^{\alpha}\left\{\frac{\overline{v}\,(u)}{u^{\alpha}} - \frac{v(0)}{u^{\alpha}} - \frac{v'(0^+)}{u^{\alpha-1}} + \omega^{\alpha} S\,(v_n)\right\}\right],$$

or

$$v_{n+1}(t) \;= S^{-1}\left[u^{\alpha}\left\{\frac{v(0)}{u^{\alpha}} - \frac{v'(0^+)}{u^{\alpha-1}} + \omega^{\alpha} S\,(v_n)\right\}\right],$$

wherefrom

$$\begin{aligned}
v_0(t) &= v(0) = 1,\\
v_1(t) &= S^{-1}\left(1 - \omega^{\alpha}u^{\alpha}\right) = \left(1 - \frac{\omega^{\alpha}t^{\alpha}}{\Gamma\,(1+\alpha)}\right),\\
v_2(t) &= S^{-1}\left(1 - \omega^{\alpha}u^{\alpha} + \omega^{2\alpha}u^{2\alpha}\right) = \left(1 - \frac{\omega^{\alpha}t^{\alpha}}{\Gamma\,(1+\alpha)} + \frac{\omega^{2\alpha}t^{2\alpha}}{\Gamma\,(1+2\alpha)}\right).
\end{aligned}$$

As $n \to \infty$, $v_n(t)$ tends to the exact solution $E_{\alpha}((-\omega t)^{\alpha})$ [25], where $E_{\alpha}((-\omega t)^{\alpha})$ denotes the Mittag-Leffler function, see [7].

Example 6.3. Consider the time-fractional diffusion equation

$${}_{0}^{C}D_t^{\alpha} v = \frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial\,(x\,v(x,t))}{\partial x}, \quad 0 < \alpha < 1, \tag{6.5}$$

with

$$v(x,0) = x^2.$$

Das [14] found the VIM solution of the fractional semi-derivative equation. Other methods applied to this equation are available in [9] and in the monographs [4, 34] on fractional calculus.

As a result, after identifying a Lagrange multiplier $\lambda = -u^{\alpha}$, approximate solutions of (6.5) can be found by

$$v_{n+1}(t) = S^{-1}\left[x^2 + u^{\alpha} S\left(\frac{\partial^2 v_n(x,t)}{\partial x^2} + \frac{\partial\,(x\,v_n(x,t))}{\partial x}\right)\right],$$

wherefrom

$$\begin{aligned}
v_0(t) &= v(0) = x^2,\\
v_1(t) &= S^{-1}\left[x^2 + \left(2+3x^2\right)u^{\alpha}\right] = \left(x^2 - \frac{\left(2+3x^2\right)t^{\alpha}}{\Gamma\,(1+\alpha)}\right),\\
v_2(t) &= S^{-1}\left(x^2 + \left(2+3x^2\right)u^{\alpha} + \left(8+9x^2\right)u^{2\alpha}\right) = \left(x^2 + \frac{\left(2+3x^2\right)t^{\alpha}}{\Gamma\,(1+\alpha)} + \frac{\left(8+9x^2\right)t^{2\alpha}}{\Gamma\,(1+2\alpha)}\right).
\end{aligned}$$

As $n \to \infty$, $v_n(t)$ tends to the exact solution $E_{\alpha}(k^i t^{\alpha})$ [25], where $E_{\alpha}(k^i t^{\alpha})$ denotes the Mittag-Leffler function and $k^i = x^2 + (1+x^2)(3^i - 1)$.

The efficiency of the method for a nonlinear differential equation with variable coefficients is illustrated in [15]. For other applications of a new modified VIM to ODEs and FDEs, readers are referred to [4, 15, 34, 35].

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