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Sharp bounds for Neuman means with applications

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Abstract

In the article, we present the sharp bounds for the Neuman mean $N_{AG}(a,b)$, $N_{GA}(a,b)$, $N_{QA}(a,b)$ and $N_{AQ}(a,b)$ in terms of the convex combinations of the arithmetic and one-parameter harmonic and contraharmonic means. As applications, we find several sharp inequalities for the first Seiffert, second Seiffert, Neuman-Sándor and logarithmic means.

Keywords: Neuman mean, Schwab-Borchardt mean, harmonic mean, geometric mean, arithmetic mean, quadratic mean, contra-harmonic mean.
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1. Introduction

For $a,b>0$ with $a\neq b$, the Schwab-Borchardt mean $SB(a,b)$ [6, 7] of a and b is defined by

$$SB(a,b)=\begin{cases}\frac{\sqrt{b^2-a^2}}{\arccos{(a/b)}}, & a<b,\\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}{(a/b)}}, & a>b,\end{cases}$$

where $\cosh^{-1}(x)=\log(x+\sqrt{x^2-1})$ is the inverse hyperbolic cosine function.

It is well known that the Schwab-Borchardt mean $SB(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean. For example, $SB[G(a,b),A(a,b)]=(a-b)/[2\arcsin((a-b)/(a+b))]=P(a,b)$ is the first Seiffert mean, $SB[A(a,b),Q(a,b)]=(a-b)/[2\arctan((a-b)/(a+b))]=T(a,b)$ is the second Seiffert mean, $SB[Q(a,b),A(a,b)]=(a-b)/[2\sinh^{-1}((a-b)/(a+b))]=M(a,b)$ is the Neuman-Sándor

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mean, $SB[A(a,b),G(a,b)]=(a-b)/[2\tanh^{-1}((a-b)/(a+b))]=L(a,b)$ is the logarithmic mean, where $G(a,b)=\sqrt{ab}$ is the geometric mean, $A(a,b)=(a+b)/2$ is the arithmetic mean, $Q(a,b)=\sqrt{(a^2+b^2)/2}$ is quadratic mean, $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$ is the inverse hyperbolic sine function and $\tanh^{-1}(x)=\log[(1+x)/(1-x)]/2$ is the inverse hyperbolic tangent function.

Let $X(a,b)$ and $Y(a,b)$ be the symmetric bivariate means of a and b. Then the Neuman mean $N_{XY}(a,b)$ [5] is given by

$$N_{XY}(a,b)=\frac{1}{2}\left(X(a,b)+\frac{Y^2(a,b)}{SB(X(a,b),Y(a,b))}\right). \tag{1.1}$$

Let $a>b>0$, $v=(a-b)/(a+b)\in(0,1)$. Then the following explicit formulas and inequalities can be found in the literature [5].

$$N_{AG}(a,b)=\frac{A(a,b)}{2}\left[1+(1-v^2)\frac{\tanh^{-1}(v)}{v}\right], \tag{1.2}$$

$$N_{GA}(a,b)=\frac{A(a,b)}{2}\left[\sqrt{1-v^2}+\frac{\arcsin(v)}{v}\right], \tag{1.3}$$

$$N_{AQ}(a,b)=\frac{A(a,b)}{2}\left[1+(1+v^2)\frac{\arctan(v)}{v}\right], \tag{1.4}$$

$$N_{QA}(a,b)=\frac{A(a,b)}{2}\left[\sqrt{1+v^2}+\frac{\sinh^{-1}}{v}\right], \tag{1.5}$$

$$\begin{aligned}H(a,b)<G(a,b)<L(a,b)<N_{AG}(a,b)<P(a,b)<N_{GA}(a,b)<A(a,b)\\ <M(a,b)<N_{QA}(a,b)<T(a,b)<N_{AQ}(a,b)<Q(a,b)<C(a,b),\end{aligned}$$

where $H(a,b)=2ab/(a+b)$ is the harmonic mean and $C(a,b)=(a^2+b^2)/(a+b)$ is the contra-harmonic mean.

Recently, the bounds for Neuman means $N_{AG}(a,b)$, $N_{GA}(a,b)$, $N_{AQ}(a,b)$ and $N_{QA}(a,b)$ have attracted the attention of several researchers.

Neuman [5] proved that the double inequalities

$$\alpha_1A(a,b)+(1-\alpha_1)G(a,b)<N_{GA}(a,b)<\beta_1A(a,b)+(1-\beta_1)G(a,b),$$

$$\alpha_2Q(a,b)+(1-\alpha_2)A(a,b)<N_{AQ}(a,b)<\beta_2Q(a,b)+(1-\beta_2)A(a,b),$$

$$\alpha_3A(a,b)+(1-\alpha_3)G(a,b)<N_{AG}(a,b)<\beta_3A(a,b)+(1-\beta_3)G(a,b),$$

$$\alpha_4Q(a,b)+(1-\alpha_4)A(a,b)<N_{QA}(a,b)<\beta_4Q(a,b)+(1-\beta_4)A(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 2/3$, $\beta_1\geq\pi/4$, $\alpha_2\leq 2/3$, $\beta_2\geq(\pi-2)/[4(\sqrt{2}-1)]=0.689\ldots$, $\alpha_3\leq 1/3$, $\beta_3\geq 1/2$, $\alpha_4\leq 1/3$ and $\beta_4\geq[\log(1+\sqrt{2})+\sqrt{2}-2]/[2(\sqrt{2}-1)]=0.356\ldots$.

In [10], Zhang et al. presented the best possible parameters $\alpha_1,\alpha_2,\beta_1,\beta_2\in[0,1/2]$ and $\alpha_3,\alpha_4,\beta_3,\beta_4\in[1/2,1]$ such that the double inequalities

$$G(\alpha_1a+(1-\alpha_1)b,\alpha_1b+(1-\alpha_1)a)<N_{AG}(a,b)<G(\beta_1a+(1-\beta_1)b,\beta_1b+(1-\beta_1)a),$$

$$G(\alpha_2a+(1-\alpha_2)b,\alpha_2b+(1-\alpha_2)a)<N_{GA}(a,b)<G(\beta_2a+(1-\beta_2)b,\beta_2b+(1-\beta_2)a),$$

$$Q(\alpha_3a+(1-\alpha_3)b,\alpha_3b+(1-\alpha_3)a)<N_{QA}(a,b)<Q(\beta_3a+(1-\beta_3)b,\beta_3b+(1-\beta_3)a),$$

$$Q(\alpha_4a+(1-\alpha_4)b,\alpha_4b+(1-\alpha_4)a)<N_{AQ}(a,b)<Q(\beta_4a+(1-\beta_4)b,\beta_4b+(1-\beta_4)a)$$

hold for all $a,b>0$ with $a\neq b$.

Qian et al. [9] proved that the double inequalities

$$\alpha_1A(a,b)+(1-\alpha_1)L(a,b)<N_{AG}(a,b)<\beta_1A(a,b)+(1-\beta_1)L(a,b),$$

$$\alpha_2 A(a,b) + (1-\alpha_2)P(a,b) < N_{GA}(a,b) < \beta_2 A(a,b) + (1-\beta_2)P(a,b),$$
$$\alpha_3 Q(a,b) + (1-\alpha_3)M(a,b) < N_{QA}(a,b) < \beta_3 Q(a,b) + (1-\beta_3)M(a,b),$$
$$\alpha_4 Q(a,b) + (1-\alpha_4)T(a,b) < N_{AQ}(a,b) < \beta_4 Q(a,b) + (1-\beta_4)T(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 0$, $\beta_1\geq 1/2$, $\alpha_2\leq 0$, $\beta_2\geq(\pi^2-8)/(4\pi-8)$, $\alpha_3\leq 0$, $\beta_3\geq[\sqrt{2}\log^2(1+\sqrt{2})+2\log(1+\sqrt{2})-2\sqrt{2}]/[4\log(1+\sqrt{2})-2\sqrt{2}]$, $\alpha_4\leq 0$ and $\beta_4\geq(\pi^2+2\pi-16)/(4\sqrt{2}\pi-16)$.

Let $a,b>0$, $p\in[0,1]$ and $\mathcal{N}$ be a symmetric bivariate mean, then the one-parameter mean $\mathcal{N}_p(a,b)$ was defined by Neuman [3] as follows

$$\mathcal{N}_p(a,b) = \mathcal{N}\left[\frac{1+p}{2}a + \frac{1-p}{2}b, \frac{1-p}{2}a + \frac{1+p}{2}b\right]. \tag{1.6}$$

Neuman [4] proved that the double inequalities

$$H_{p_1(a,b)} < N_{AG}(a,b) < H_{q_1}(a,b), \quad H_{p_2(a,b)} < N_{GA}(a,b) < H_{q_2}(a,b), \tag{1.7}$$

$$C_{p_3(a,b)} < N_{QA}(a,b) < C_{q_3}(a,b), \quad C_{p_4(a,b)} < N_{AQ}(a,b) < C_{q_4}(a,b) \tag{1.8}$$

hold for all $a,b>0$ with $a\neq b$ if $p_1\geq\sqrt{2}/2$, $q_1\leq\sqrt{3}/3$, $p_2\geq\sqrt{1-\pi/4}$, $q_2\leq\sqrt{6}/6$, $p_3=0$, $q_3\geq\sqrt{6}/6$, $p_4\leq\sqrt{\pi-2}/2$ and $q_4\geq\sqrt{3}/3$.

It is not difficult to verify that $H_p(a,b)$ is strictly decreasing and $C_p(a,b)$ is strictly increasing with respect to $p\in[0,1]$ for fixed $a,b>0$ with $a\neq b$.

The first aim of this paper is to prove that $p_1=\sqrt{2}/2$, $q_1=\sqrt{3}/3$, $p_2=\sqrt{1-\pi/4}$, $q_2=\sqrt{6}/6$, $p_3=\sqrt{[\log(1+\sqrt{2})+\sqrt{2}-2)]/2}$, $q_3=\sqrt{6}/6$, $p_4=\sqrt{\pi-2}/2$ and $q_4=\sqrt{3}/3$ are the best possible parameters in $[0,1]$ such that the double inequalities (1.7) and (1.8) hold for all $a,b>0$ with $a\neq b$.

The second purpose of the article is to present the best possible parameters $\alpha_1=\alpha_1(p)$, $\beta_1=\beta_1(p)$, $\alpha_2=\alpha_2(q)$, $\beta_2=\beta_2(q)$, $\alpha_3=\alpha_3(r)$, $\beta_3=\beta_3(r)$, $\alpha_4=\alpha_4(s)$ and $\beta_4=\beta_4(s)$ such that the double inequalities

$$\alpha_1 A(a,b) + (1-\alpha_1)H_p(a,b) < N_{AG}(a,b) < \beta_1 A(a,b) + (1-\beta_1)H_p(a,b),$$
$$\alpha_2 A(a,b) + (1-\alpha_2)H_q(a,b) < N_{GA}(a,b) < \beta_2 A(a,b) + (1-\beta_2)H_q(a,b),$$
$$\alpha_3 C_r(a,b) + (1-\alpha_3)A(a,b) < N_{QA}(a,b) < \beta_3 C_r(a,b) + (1-\beta_3)A(a,b),$$
$$\alpha_4 C_s(a,b) + (1-\alpha_4)A(a,b) < N_{AQ}(a,b) < \beta_4 C_s(a,b) + (1-\beta_4)A(a,b)$$

hold for all $p\in[\sqrt{2}/2,1]$, $q\in[\sqrt{1-\pi/4},1]$, $r\in[\sqrt{6}/6,1]$, $s\in[\sqrt{3}/3,1]$ and $a,b>0$ with $a\neq b$.

2. Lemmas

In order to prove our main results we need several lemmas, which we will present in this section.

Lemma 2.1 ([8]). *Let $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ be two real sequences with $b_n>0$ and $\lim_{n\to\infty} a_n/b_n = s$. Then the power series $\sum_{n=0}^\infty a_n t^n$ is convergent for all $t\in\mathbb{R}$ and*

$$\lim_{t\to\infty}\frac{\sum_{n=0}^\infty a_n t^n}{\sum_{n=0}^\infty b_n t^n} = s$$

if the power series $\sum_{n=0}^\infty b_n t^n$ is convergent for all $t\in\mathbb{R}$.

Lemma 2.2 ([1]). *Let $-\infty<a<b<+\infty$ and $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions $[f(x)-f(a)]/[g(x)-g(a)]$ and $[f(x)-f(b)]/[g(x)-g(b)]$. If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.3 ([2]). *Let $A(t)=\sum_{k=0}^{\infty}a_k t^k$ and $B(t)=\sum_{k=0}^{\infty}b_k t^k$ be two real power series converging on $(-r,r)$ $(r>0)$ with $b_k>0$ for all k. If the non-constant sequence $\{a_k/b_k\}$ is increasing (decreasing) for all k, then the function $A(t)/B(t)$ is strictly increasing (decreasing) on $(0,r)$.*

Lemma 2.4. *The function*

$$f(x)=\frac{\sinh(2x)\cosh(x)-2x\cosh(x)}{\sinh(3x)-3\sinh(x)}$$

is strictly increasing form $(0,\infty)$ onto $(1/3,1/2)$.

Proof. Let

$$a_n=\frac{\frac{1}{2}\left(3^{2n+3}+1\right)-2(2n+3)}{(2n+3)!},\quad b_n=\frac{3^{2n+3}-3}{(2n+3)!}.$$

Then simple computations lead to

$$\begin{aligned} f(x)&=\frac{\frac{1}{2}[\sinh(3x)+\sinh(x)]-2x\cosh(x)}{\sinh(3x)-3\sinh(x)}\\ &=\frac{\frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{3^{2n+1}}{(2n+1)!}x^{2n+1}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}\right]-2x\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}}{\sum_{n=0}^{\infty}\frac{3^{2n+1}}{(2n+1)!}x^{2n+1}-3\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}} \qquad (2.1)\\ &=\frac{\sum_{n=0}^{\infty}a_n x^{2n}}{\sum_{n=0}^{\infty}b_n x^{2n}}, \end{aligned}$$

$$\frac{a_0}{b_0}=\frac{1}{3},\quad \lim_{n\to\infty}\frac{a_n}{b_n}=\frac{1}{2}, \qquad (2.2)$$

$$\frac{a_{n+1}}{b_{n+1}}-\frac{a_n}{b_n}=\frac{4\left[(7+8n)3^{2n+2}+1\right]}{3\left(3^{2n+2}-1\right)\left(3^{2n+4}-1\right)}>0,\quad b_n>0 \qquad (2.3)$$

for all $n\ge 0$.

It follows from (2.1)–(2.3) together with Lemmas 2.1 and 2.3 that the function $f(x)$ is strictly increasing on $(0,\infty)$ and

$$\lim_{x\to 0^+}f(x)=\frac{a_0}{b_0}=\frac{1}{3},\quad \lim_{x\to\infty}f(x)=\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{1}{2}. \qquad (2.4)$$

Therefore, Lemma 2.4 follows from (2.4) and the monotonicity of f on the interval $(0,\infty)$. □

Lemma 2.5. *The function*

$$g(x)=\frac{4\sin(x)-\sin(2x)-2x}{3\sin(x)-\sin(3x)}$$

is strictly increasing form $(0,\pi/2)$ onto $(1/6,1-\pi/4)$.

Proof. Let $g_1(x)=4\sin(x)-\sin(2x)-2x$ and $g_2(x)=3\sin(x)-\sin(3x)$. Then

$$g(x)=\frac{g_1(x)}{g_2(x)},\quad g_1(0^+)=g_2(0^+)=0, \qquad (2.5)$$

$$\frac{g_1'(x)}{g_2'(x)}=\frac{1}{3[1+\cos(x)]}. \qquad (2.6)$$

From (2.6) we clearly see that the function $g_1'(x)/g_2'(x)$ is strictly increasing on $(0,\pi/2)$. Then Lemma 2.2 and (2.5) lead to the conclusion that $g(x)$ is strictly increasing on $(0,\pi/2)$. Note that

$$g(0^+)=\lim_{x\to 0^+}\frac{g_1'(x)}{g_2'(x)}=\frac{1}{6},\quad g\left(\frac{\pi}{2}\right)=1-\frac{\pi}{4}. \qquad (2.7)$$

Therefore, Lemma 2.5 follows from (2.7) and the monotonicity of $g(x)$ on the interval $(0,\pi/2)$. □

Lemma 2.6. *The function*

$$h(x) = \frac{\sinh(2x) - 4\sinh(x) + 2x}{\sinh(3x) - 3\sinh(x)}$$

is strictly decreasing form $(0, \log(1+\sqrt{2}))$ *onto* $((\log(1+\sqrt{2}) + \sqrt{2} - 2)/2, 1/6)$.

Proof. Let

$$c_n = \frac{2^{2n+3} - 4}{(2n+3)!}, \quad d_n = \frac{3^{2n+3} - 3}{(2n+3)!}.$$

Then simple computation lead to

$$h(x) = \frac{\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+1} - 4\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} + 2x}{\sum_{n=0}^{\infty} \frac{3^{2n+1}}{(2n+1)!} x^{2n+1} - 3\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}} = \frac{\sum_{n=0}^{\infty} c_n x^{2n}}{\sum_{n=0}^{\infty} d_n x^{2n}}, \tag{2.8}$$

$$\frac{c_{n+1}}{d_{n+1}} - \frac{c_n}{d_n} = -\frac{8\left(5 \times 2^{2n} - 4\right) 3^{2n+3} + 9 \times 2^{2n+3}}{9\left(3^{2n+2} - 1\right)\left(3^{2n+4} - 1\right)} < 0, \quad d_n > 0 \tag{2.9}$$

for all $n \geq 0$.

It follows from Lemma 2.3 and (2.8) together with (2.9) that $h(x)$ is strictly decreasing on $(0, \infty)$. Note that

$$h(0^+) = \frac{c_0}{d_0} = \frac{1}{6}, \quad h(\log(1+\sqrt{2})) = \frac{\log(1+\sqrt{2}) + \sqrt{2} - 2}{2}. \tag{2.10}$$

Therefore, Lemma 2.6 follows from (2.10) and the monotonicity of $h(x)$ on the interval $(0, \infty)$. □

Lemma 2.7. *The function*

$$k(x) = \frac{2x\cos(x) - \sin(2x)\cos(x)}{3\sin(x) - \sin(3x)}$$

is strictly decreasing form $(0, \pi/4)$ *onto* $((\pi - 2)/4, 1/3)$.

Proof. Let $k_1(x) = 2x\cos(x) - \sin(2x)\cos(x)$ and $k_2(x) = 3\sin(x) - \sin(3x)$. Then

$$k(x) = \frac{k_1(x)}{k_2(x)}, \quad k_1(0) = k_2(0) = 0, \tag{2.11}$$

$$\frac{k_1'(x)}{k_2'(x)} = \frac{1}{2} - \frac{x}{3\sin(2x)}, \tag{2.12}$$

$$k(0^+) = \lim_{x \to 0^+} \frac{k_1'(x)}{k_2'(x)} = \frac{1}{3}, \quad k\left(\frac{\pi}{4}\right) = \frac{\pi - 2}{4}. \tag{2.13}$$

It is well known that the function $x/\sin(2x)$ is strictly increasing on $(0, \pi/4)$, then (2.12) leads to the conclusion that the function $k_1'(x)/k_2'(x)$ is strictly decreasing on $(0, \pi/4)$.

Therefore, Lemma 2.7 follows easily from Lemma 2.2, (2.11) , (2.13) and the monotonicity of $k_1'(x)/k_2'(x)$ on the interval $(0, \pi/4)$. □

3. Main Results

Theorem 3.1. *Let* $p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \in [0, 1]$. *Then the double inequalities*

$$H_{p_1(a,b)} < N_{AG}(a,b) < H_{q_1}(a,b), \quad H_{p_2(a,b)} < N_{GA}(a,b) < H_{q_2}(a,b),$$

$$C_{p_3(a,b)} < N_{QA}(a,b) < C_{q_3}(a,b), \quad C_{p_4(a,b)} < N_{AQ}(a,b) < C_{q_4}(a,b)$$

hold for all $a, b > 0$ *with* $a \neq b$ *if and only if* $p_1 \geq \sqrt{2}/2$, $q_1 \leq \sqrt{3}/3$, $p_2 \geq \sqrt{1 - \pi/4}$, $q_2 \leq \sqrt{6}/6$, $p_3 \leq \sqrt{[\log(1+\sqrt{2}) + \sqrt{2} - 2]/2}$, $q_3 \geq \sqrt{6}/6$, $p_4 \leq \sqrt{\pi - 2}/2$ *and* $q_4 \geq \sqrt{3}/3$.

Proof. Since the Neuman means $N_{AG}(a,b)$, $N_{GA}(a,b)$, $N_{QA}(a,b)$ and $N_{AQ}(a,b)$, and the one-parameter means $H_\lambda(a,b)$ and $C_\mu(a,b)$ are symmetric and homogeneous of degree 1 with respect to a and b, without loss of generality, we assume that $a > b > 0$.

Let $\lambda, \mu \in [0,1]$, $v = (a-b)/(a+b) \in (0,1)$, $x = \tanh^{-1}(v) \in (0,\infty)$, $y = \arcsin(v) \in (0,\pi/2)$, $z = \sinh^{-1}(v) \in (0, \log(1+\sqrt{2}))$ and $w = \arctan(v) \in (0,\pi/4)$. Then it follows from (1.2)–(1.6) that

$$H_\lambda(a,b) = A(a,b)\left(1-\lambda^2 v^2\right), \quad C_\mu(a,b) = A(a,b)\left(1+\mu^2 v^2\right), \tag{3.1}$$

$$\begin{aligned} H_\lambda(a,b) - N_{AG}(a,b) &= A(a,b)v^2\left[\frac{v-\left(1-v^2\right)\tanh^{-1}(v)}{2v^3} - \lambda^2\right] \\ &= A(a,b)v^2\left[\frac{\sinh(2x)\cosh(x) - 2x\cosh(x)}{\sinh(3x) - 3\sinh(x)} - \lambda^2\right] = A(a,b)v^2\left[f(x)-\lambda^2\right], \end{aligned} \tag{3.2}$$

$$\begin{aligned} H_\lambda(a,b) - N_{GA}(a,b) &= A(a,b)v^2\left[\frac{2v - v\sqrt{1-v^2} - \arcsin(v)}{2v^3} - \lambda^2\right] \\ &= A(a,b)v^2\left[\frac{4\sin(y) - \sin(2y) - 2y}{3\sin(y) - \sin(3y)} - \lambda^2\right] = A(a,b)v^2\left[g(y)-\lambda^2\right], \end{aligned} \tag{3.3}$$

$$\begin{aligned} N_{QA}(a,b) - C_\mu(a,b) &= A(a,b)v^2\left[\frac{v\sqrt{1+v^2} + \sinh^{-1}(v) - 2v}{2v^3} - \mu^2\right] \\ &= A(a,b)v^2\left[\frac{\sinh(2z) - 4\sinh(z) + 2z}{\sinh(3z) - 3\sinh(z)} - \mu^2\right] = A(a,b)v^2\left[h(z)-\mu^2\right], \end{aligned} \tag{3.4}$$

$$\begin{aligned} N_{AQ}(a,b) - C_\mu(a,b) &= A(a,b)v^2\left[\frac{\left(1+v^2\right)\arctan(v) - v}{2v^3} - \mu^2\right] \\ &= A(a,b)v^2\left[\frac{2w\cos(w) - \sin(2w)\cos(w)}{3\sin(w) - \sin(3w)} - \mu^2\right] = A(a,b)v^2\left[k(w)-\mu^2\right], \end{aligned} \tag{3.5}$$

where the functions $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ and $k(\cdot)$ are respectively defined as in Lemmas 2.4, 2.5, 2.6 and 2.7.

Therefore, Theorem 3.1 follows easily from Lemmas 2.4–2.7 and (3.2)–(3.5). □

Theorem 3.2. *Let $p \in [\sqrt{2}/2, 1]$, $q \in [\sqrt{1-\pi/4}, 1]$, $r \in [\sqrt{6}/6, 1]$ and $s \in [\sqrt{3}/3, 1]$. Then the double inequalities*

$$\alpha_1 A(a,b) + (1-\alpha_1)H_p(a,b) < N_{AG}(a,b) < \beta_1 A(a,b) + (1-\beta_1)H_p(a,b), \tag{3.6}$$

$$\alpha_2 A(a,b) + (1-\alpha_2)H_q(a,b) < N_{GA}(a,b) < \beta_2 A(a,b) + (1-\beta_2)H_q(a,b), \tag{3.7}$$

$$\alpha_3 C_r(a,b) + (1-\alpha_3)A(a,b) < N_{QA}(a,b) < \beta_3 C_r(a,b) + (1-\beta_3)A(a,b), \tag{3.8}$$

$$\alpha_4 C_s(a,b) + (1-\alpha_4)A(a,b) < N_{AQ}(a,b) < \beta_4 C_s(a,b) + (1-\beta_4)A(a,b) \tag{3.9}$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha_1 \le 1 - 1/(2p^2)$, $\beta_1 \ge 1 - 1/(3p^2)$, $\alpha_2 \le 1 - (4-\pi)/(4q^2)$, $\beta_2 \ge 1 - 1/(6q^2)$, $\alpha_3 \le [\log(1+\sqrt{2}) + \sqrt{2} - 2]/(2r^2)$, $\beta_3 \ge 1/(6r^2)$, $\alpha_4 \le (\pi-2)/(4s^2)$ and $\beta_4 \ge 1/(3s^2)$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $v = (a-b)/(a+b) \in (0,1)$. Then from (1.2)–(1.5) and (3.1) we clearly see that inequalities (3.6)–(3.9) are respectively equivalent to the inequalities

$$(1-\beta_1)p^2 < \frac{v - (1-v^2)\tanh^{-1}(v)}{2v^3} < (1-\alpha_1)p^2, \tag{3.10}$$

$$(1-\beta_2)q^2 < \frac{2v - v\sqrt{1-v^2} - \arcsin(v)}{2v^3} < (1-\alpha_2)q^2, \tag{3.11}$$

$$\alpha_3 r^2 < \frac{v\sqrt{1+v^2} + \sinh^{-1}(v) - 2v}{2v^3} < \beta_3 r^2, \tag{3.12}$$

$$\alpha_4 s^2 < \frac{\left(1+v^2\right)\arctan(v) - v}{2v^3} < \beta_4 s^2. \tag{3.13}$$

Let $x = \tanh^{-1}(v) \in (0, \infty)$, $y = \arcsin(v) \in (0, \pi/2)$, $z = \sinh^{-1}(v) \in (0, \log(1+\sqrt{2}))$ and $w = \arctan(v) \in (0, \pi/4)$. Then simple computations lead to

$$\frac{v - (1-v^2)\tanh^{-1}(v)}{2v^3} = \frac{\sinh(2x)\cosh(x) - 2x\cosh(x)}{\sinh(3x) - 3\sinh(x)}, \tag{3.14}$$

$$\frac{2v - v\sqrt{1-v^2} - \arcsin(v)}{2v^3} = \frac{4\sin(y) - \sin(2y) - 2y}{3\sin(y) - \sin(3y)}, \tag{3.15}$$

$$\frac{v\sqrt{1+v^2} + \sinh^{-1}(v) - 2v}{2v^3} = \frac{\sinh(2z) - 4\sinh(z) + 2z}{\sinh(3z) - 3\sinh(z)}, \tag{3.16}$$

$$\frac{\left(1+v^2\right)\arctan(v) - v}{2v^3} = \frac{2w\cos(w) - \sin(2w)\cos(w)}{3\sin(w) - \sin(3w)}. \tag{3.17}$$

Therefore, inequality (3.6) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1 - 1/(2p^2)$ and $\beta_1 \geq 1 - 1/(3p^2)$ follows from (3.10) and (3.14) together with Lemma 2.4, inequality (3.7) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 1 - (4-\pi)/(4q^2)$ and $\beta_2 \geq 1 - 1/(6q^2)$ follows from (3.11) and (3.15) together with Lemma 2.5, inequality (3.8) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq [\log(1+\sqrt{2}) + \sqrt{2} - 2]/(2r^2)$ and $\beta_3 \geq 1/(6r^2)$ follows from (3.12) and (3.16) together with Lemma 2.6 and inequality (3.9) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq (\pi - 2)/(4s^2)$ and $\beta_4 \geq 1/(3s^2)$ follows from (3.13) and (3.17) together with Lemma 2.7. □

From (1.2) we clearly see that

$$N_{AG}(a,b) = \frac{1}{2}\left[A(a,b) + \frac{G^2(a,b)}{L(a,b)}\right], \quad N_{GA}(a,b) = \frac{1}{2}\left[G(a,b) + \frac{A^2(a,b)}{P(a,b)}\right], \tag{3.18}$$

$$N_{AQ}(a,b) = \frac{1}{2}\left[A(a,b) + \frac{Q^2(a,b)}{T(a,b)}\right], \quad N_{QA}(a,b) = \frac{1}{2}\left[Q(a,b) + \frac{A^2(a,b)}{M(a,b)}\right]. \tag{3.19}$$

Theorem 3.2, (3.18) and (3.19) lead to Theorem 3.3 immediately.

Theorem 3.3. *Let $p \in [\sqrt{2}/2, 1]$, $q \in [\sqrt{1-\pi/4}, 1]$, $r \in [\sqrt{6}/6, 1]$ and $s \in [\sqrt{3}/3, 1]$. Then the double inequalities*

$$\frac{G^2(a,b)}{(2\beta_1 - 1)A(a,b) + 2(1-\beta_1)H_p(a,b)} < L(a,b) < \frac{G^2(a,b)}{(2\alpha_1 - 1)A(a,b) + 2(1-\alpha_1)H_p(a,b)},$$

$$\frac{A^2(a,b)}{2\beta_2 A(a,b) + 2(1-\beta_2)H_q(a,b) - G(a,b)} < P(a,b) < \frac{A^2(a,b)}{2\alpha_2 A(a,b) + 2(1-\alpha_2)H_q(a,b) - G(a,b)},$$

$$\frac{A^2(a,b)}{2\beta_3 C_r(a,b) + 2(1-\beta_3)A(a,b) - Q(a,b)} < M(a,b) < \frac{A^2(a,b)}{2\alpha_3 C_r(a,b) + 2(1-\alpha_3)A(a,b) - Q(a,b)},$$

$$\frac{Q^2(a,b)}{2\beta_4 C_s(a,b) + (1-2\beta_4)A(a,b)} < T(a,b) < \frac{Q^2(a,b)}{2\alpha_4 C_s(a,b) + (1-2\alpha_4)A(a,b)}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1 - 1/(2p^2)$, $\beta_1 \geq 1 - 1/(3p^2)$, $\alpha_2 \leq 1 - (4-\pi)/(4q^2)$, $\beta_2 \geq 1 - 1/(6q^2)$, $\alpha_3 \leq [\log(1+\sqrt{2}) + \sqrt{2} - 2]/(2r^2)$, $\beta_3 \geq 1/(6r^2)$, $\alpha_4 \leq (\pi-2)/(4s^2)$ and $\beta_4 \geq 1/(3s^2)$.

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