



# Unified implicit common fixed point theorems under non-negative complex valued functions satisfying the identity of indiscernible

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## Abstract

In this paper, we consider a non-negative complex valued function satisfying the identity of indiscernible and utilize the same to prove some common fixed point theorems for two pairs of non-vacuously weakly compatible mappings satisfying an implicit relation having rational terms as its co-ordinates. Some illustrative examples are also given which demonstrate the validity of the hypotheses of our results. In process, a host of previously known results in the context of complex as well as real valued metric spaces are generalized and improved. ©2016 All rights reserved.

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## 1. Introduction and Preliminaries

Azam et al. [3] introduced the concept of complex valued metric spaces and obtained sufficient conditions for the existence of a unique common fixed point for a pair of mappings satisfying a suitable rational

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contractive type condition. Thereafter, Rouzkard and Imdad [18] established some common fixed point theorems satisfying certain rational expressions in complex valued metric spaces to generalize the results proved in [3] with the note that complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many such results of analysis involving quotients cannot be generalized to cone metric spaces as the definition of a metric spaces banks on the underlying Banach space which is not a division Ring. However in complex valued metric spaces, one can study improvements of a host results of analysis involving divisions.

Let  $C$  be the set of complex numbers and  $z_1, z_2 \in C$ . Azam et al. [3] defined a partial ordering  $\lesssim$  on  $C$  (for all  $z_1, z_2 \in C$ ) as follows:

$z_1 \lesssim z_2$  if and only if  $Re(z_1) \leq Re(z_2)$  and  $Im(z_1) \leq Im(z_2)$ . It follows that  $z_1 \lesssim z_2$ , if one of the following conditions are satisfied:

- (C1)  $Re(z_1) = Re(z_2)$  and  $Im(z_1) = Im(z_2)$ ;
- (C2)  $Re(z_1) < Re(z_2)$  and  $Im(z_1) = Im(z_2)$ ;
- (C3)  $Re(z_1) = Re(z_2)$  and  $Im(z_1) < Im(z_2)$ ;
- (C4)  $Re(z_1) < Re(z_2)$  and  $Im(z_1) < Im(z_2)$ .

In particular, we write  $z_1 \not\lesssim z_2$  if  $z_1 \neq z_2$  and one of (C2), (C3) and (C4) is satisfied while  $z_1 \prec z_2$  if only (C4) is satisfied. Note that

$$\begin{aligned} 0 \lesssim z_1 \not\lesssim z_2 &\Rightarrow |z_1| < |z_2|, \\ z_1 \lesssim z_2, z_2 \prec z_3 &\Rightarrow z_1 \prec z_3. \end{aligned}$$

**Definition 1.1** ([3]). A mapping  $d : X \times X \rightarrow C$  is called a complex valued metric on a non-empty set  $X$  if the following conditions are satisfied:

- (CM1)  $0 \lesssim d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0 \iff x = y$ ;
- (CM2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (CM3)  $d(x, y) \lesssim d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a complex valued metric on  $X$  and  $(X, d)$  is called a complex valued metric space.

After the notion of complex valued metric spaces, K. P. R. Rao [17] considered the concept of complex valued b-metric spaces.

**Definition 1.2.** Let  $X$  be a nonempty set and  $s \geq 1$ , a given real number. A function  $d : X \times X \rightarrow C$  is called a complex valued b-metric (cvbm) if it satisfies the following conditions:

- (cvbm-1)  $0 \lesssim d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0 \iff x = y$ ;
- (cvbm-2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (cvbm-3)  $d(x, y) \lesssim s[d(x, z) + d(z, y)]$  for all  $x, y, z \in X$ .

The pair  $(X, d)$  is called a complex valued b-metric space.

We relax last two conditions of Definitions 1.1 and 1.2 and define a class of functions satisfying merely the first condition of foregoing definitions which are in fact the class of non-negative complex valued functions satisfying the identity of indiscernible.

**Definition 1.3.** Let  $X$  be a non-empty set. A complex-valued function  $m_c : X \times X \rightarrow C$  is said to be a non-negative function satisfying the identity of indiscernible if

$$0 \lesssim m_c(x, y) \text{ for all } x, y \in X \text{ and } m_c(x, y) = 0 \text{ if and only if } x = y.$$

The following functions meet such requirements.

**Example 1.4.** For  $X = R$ , define a function  $m_c : X \times X \rightarrow C$  as

$$m_c(x, y) = \begin{cases} i(e^{(x-y)} - 1) & \text{if } x \geq y; \\ ie^{y-x} & \text{if } x < y. \end{cases}$$

Then  $m_c(x, y)$  is a complex valued non-negative function satisfying the identity of indiscernible.

**Example 1.5.** If  $X \subset C^+$ , containing all those elements of  $C^+$  which are comparable in view of partial ordering ' $\preceq$ ' defined earlier, where  $C^+ = \{a + ib \in C : a, b \in [0, \infty)\}$ .

Define  $m_c : X \times X \rightarrow C$  as

$$m_c(x, y) = \begin{cases} x - y & \text{if } x \succeq y; \\ y - x + i & \text{if } x \prec y. \end{cases}$$

Then  $m_c(x, y)$  is a complex valued non-negative function satisfying the identity of indiscernible.

A topology  $\tau(m_c)$  on  $X$  is given by  $U \in \tau(m_c)$  if and only if for each  $x \in U$ ,  $B(x, r) \subset U$  for some  $r \succ 0$ ,  $r \in C$ , where  $B(x, r) = \{y \in X : m_c(x, y) \prec r\}$ .

*Remark 1.6.* In view of Definition 1.3, complex valued metric and complex valued b-metric are the examples of the non-negative complex valued functions satisfying the identity of indiscernible.

Utilizing the above concept of Azam et al. [3], Verma et al. [20] defined max function for complex numbers as follows:

**Definition 1.7** ([20]). The max function for complex numbers with partial order relation  $\preceq$  is defined as

1.  $\max\{z_1, z_2\} = z_2 \Rightarrow z_1 \preceq z_2$ ;
2.  $z_1 \preceq \max\{z_2, z_3\} \Rightarrow z_1 \preceq z_2$  or  $z_1 \preceq z_3$ .

On the similar lines min function can also be defined as

1.  $\min\{z_1, z_2\} = z_1 \Rightarrow z_1 \preceq z_2$ ;
2.  $\min\{z_1, z_2\} \preceq z_3 \Rightarrow z_1 \preceq z_3$  or  $z_2 \preceq z_3$ .

**Definition 1.8** ([11]). Let  $X$  be a non-empty set and let  $A$  and  $S$  be two self mappings of  $X$ . A point  $x$  in  $X$  is called a coincidence point of  $A$  and  $S$  if and only if  $Ax = Sx$ . Also, we shall call  $w$  to be a point of coincidence of  $A$  and  $S$  if  $w = Ax = Sx$ .

The following definition can be restated in context of nonnegative complex-valued function  $m_c : X \times X \rightarrow C$ , satisfying the identity of indiscernible.

**Definition 1.9.** Let  $A$  and  $S$  be a pair of self mappings defined on  $X$ . Then the pair  $(A, S)$  is said to be non-vacuously weakly compatible, if :

1.  $C(A, S) \neq \phi$ ;
2. The pair of mappings  $A$  and  $S$  commute on the set of coincidence points  
i.e.  $ASu = SAu$  for every  $u \in C(A, S)$  where  $C(A, S)$  is the set of coincidence points of the mappings  $A$  and  $S$ .

In this paper, some common fixed point theorems for two pairs of non-vacuously weakly compatible mappings satisfying an implicit relation under a nonnegative complex-valued function  $m_c : X \times X \rightarrow C$  satisfying the identity of indiscernible (i.e. Definition 1.3) are proved.

## 2. An Implicit Relation

A simple and natural way to prove unified metrical fixed point theorems is to employ implicit relations instead of using an explicit contractive condition. Popa [13, 14] initiated this idea of unification which had produced thus far a consistent literature on fixed point, common fixed point and coincidence point for both single valued and multi-valued mappings in various ambient spaces. Motivated by Popa [15, 16], Ali and Imdad [2] and Imdad et al. [8, 9, 10], we consider an implicit function under minimal requirement and utilize the same to prove some common fixed point theorems.

**Definition 2.1.** Let  $\mathcal{F}_6$  be set of all functions  $F(t_1, t_2, t_3, t_4, t_5, t_6) : C^6 \rightarrow C$  satisfying the condition

$$F(t, 0, t, t, 0, t) \succ 0, \quad \forall t \succ 0. \quad (2.1)$$

**Example 2.2.**  $F(t_1, t_2, \dots, t_6) = t_1 - at_3 - b\{t_2 + t_5\} - c\{t_4 + t_6\}$ , where  $a, b, c \geq 0$ ,  $a + 2c < 1$ .

**Example 2.3.**  $F(t_1, t_2, \dots, t_6) = t_1 - at_3 - b \max\{t_2, t_4, t_6\} - c \max\{t_3, t_5\}$ , where  $a, b, c \geq 0$ ,  $a + b + c < 1$ .

**Example 2.4.**  $F(t_1, t_2, \dots, t_6) = t_1 - k \max\{t_2, t_3, \dots, t_6\}$ ,  $k \in (0, 1)$ .

**Example 2.5.**  $F(t_1, t_2, \dots, t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$ , where  $a, b, c, d, e \geq 0$ ,  $b + c + e < 1$ .

**Example 2.6.**  $F(t_1, t_2, \dots, t_6) = t_1 - k \min\{t_2 + t_3, t_3 + t_4, t_5 + t_6\}$ ,  $k \in (0, 1)$ .

**Example 2.7.**  $F(t_1, t_2, \dots, t_6) = t_1 - k \min\{t_3, \max\{t_2, t_4, t_6\}, t_5\}$ ,  $k \in (0, \infty)$ .

**Example 2.8.**  $F(t_1, t_2, \dots, t_6) = t_1 - k \max\{t_2, t_3, t_5, \frac{t_4+t_6}{2}\}$ ,  $k \in (0, 1)$ .

**Example 2.9.**  $F(t_1, t_2, \dots, t_6) = t_1 - \lambda t_3 - \frac{\mu(t_2.t_5)}{1+t_3} - \frac{\gamma(t_4.t_5)}{1+t_3}$  where  $\lambda, \mu, \gamma \geq 0$ ,  $\lambda + \mu + \gamma < 1$ .

**Example 2.10.**

$$F(t_1, t_2, \dots, t_6) = \begin{cases} t_1 - \lambda t_3 - \frac{\mu(t_2.t_5+t_4.t_6)}{t_2+t_5} - \frac{\gamma(t_2.t_6+t_4.t_5)}{t_4+t_6} & \text{if } t_2 + t_5 \neq 0, t_4 + t_6 \neq 0; \\ t_1 & \text{if } t_2 + t_5 = 0 \text{ or } t_4 + t_6 = 0, \end{cases}$$

where  $\lambda, \mu, \gamma \geq 0$ ,  $\lambda + \mu + \gamma < 1$ .

**Example 2.11.**

$$F(t_1, t_2, \dots, t_6) = \begin{cases} t_1 - \lambda t_3 - \frac{\mu(t_2.t_5)}{t_3+t_4+t_6} & \text{if } t_3 + t_4 + t_6 \neq 0; \\ t_1 & \text{if } t_3 + t_4 + t_6 = 0, \end{cases}$$

where  $\lambda, \mu \geq 0$ ,  $\lambda + \mu < 1$ .

**Example 2.12.**  $F(t_1, t_2, \dots, t_6) = t_1 - At_3 - \frac{B(t_2.t_5)}{1+t_3} - \frac{C(t_4.t_6)}{1+t_3} - \frac{D(t_2.t_6)}{1+t_3} - \frac{E(t_4.t_5)}{1+t_3}$ , where  $A, B, C, D, E \geq 0$ ,  $A + B + C + D + E < 1$ .

**Example 2.13.**  $F(t_1, t_2, \dots, t_6) = t_1 - k \max\{t_3, \frac{t_2+t_5}{2}, \frac{t_4+t_6}{2}\}$ ,  $k \in (0, 1)$ .

**Example 2.14.**  $F(t_1, t_2, \dots, t_6) = t_1 - k \max\{t_3, \sqrt{t_2.t_5}, \sqrt{t_4.t_6}\}$ ,  $k \in (0, 1)$ .

**Example 2.15.**  $F(t_1, t_2, \dots, t_6) = t_1 - \alpha \max\{t_2, t_3, t_5\} - (1 - \alpha)\{at_4 + bt_6\}$ , where  $\alpha \in (0, 1)$  and  $a + b < 1$ .

**Example 2.16.**  $F(t_1, t_2, \dots, t_6) = t_1 - k \max\left\{\frac{2t_2+t_5}{2}, \frac{2t_2+t_4}{2}, \frac{2t_2+t_6}{2}\right\}$ ,  $k \in (0, 1)$ .

**Example 2.17.**  $F(t_1, t_2, \dots, t_6) = t_1 - \alpha \frac{t_2.t_6+t_4.t_5}{1+t_4+t_6}$ ,  $\alpha \in (0, 1)$ .

Further, for some  $t \in C$ , if  $Re(t) \geq Im(t)$ , then following examples show that  $F \in \mathcal{F}_6$ .

**Example 2.18.**  $F(t_1, t_2, \dots, t_6) = t_1^2 - at_1(t_2 + t_3 + t_5) - b(t_4.t_6)$ , where  $a, b \geq 0$ ,  $a + b < 1$ .

**Example 2.19.**  $F(t_1, t_2, \dots, t_6) = t_1^2 - k \max\{t_2.t_3, t_3.t_5, t_2.t_5, t_4.t_6\}$ ,  $k \in (0, 1)$ .

**Example 2.20.**  $F(t_1, t_2, \dots, t_6) = t_1^2 - at_3^2 - b \frac{t_2.t_5}{1+t_4+t_6}$ , where  $a, b \geq 0$ ,  $a < 1$ .

### 3. Results without symmetry

Following theorem is proved under implicit relation  $F \in \mathcal{F}_6$ , having rational terms as its coordinates.

**Theorem 3.1.** Let  $X$  be a non-empty set and  $m_c : X \times X \rightarrow C$  a non-negative complex valued function satisfying the identity of indiscernible. If  $A, B, S$  and  $T$  are four self mappings of  $X$  such that

1.

$$F\left(\frac{m_c(Ax, By)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}\right) \lesssim 0, \tag{3.1}$$

for all  $x, y \in X$  with  $Ax \neq By$  and  $F \in \mathcal{F}_6$ .

2. The pairs  $(A, S)$  and  $(B, T)$  are non-vacuously weakly compatible,

then  $A, B, S$  and  $T$  have a unique common fixed point.

*Proof.* Since the pairs  $(A, S)$  and  $(B, T)$  are non-vacuously weakly compatible, there exist  $x, y \in X$  such that

$$Ax = Sx \quad \text{and} \quad ASx = SAx \tag{3.2}$$

and

$$By = Ty \quad \text{and} \quad BTy = TBy. \tag{3.3}$$

First of all, we prove that  $Ax = By$ . On the contrary suppose  $Ax \neq By$ , then using Inequality (3.1), we have

$$F\left(\frac{m_c(Ax, By)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}\right) \lesssim 0.$$

From (3.2) and (3.3), we have

$$F(m_c(Ax, By), 0, m_c(Ax, By), m_c(Ax, By), 0, m_c(Ax, By)) \lesssim 0,$$

which is contradiction as  $F \in \mathcal{F}_6$  yielding thereby  $Ax = By, Ax = Sx = By = Ty$  and

$$A^2x = AAx = ASx = SAx. \tag{3.4}$$

Now, we prove that  $Ax = A^2x$ . If  $A^2x \neq Ax = By$ . On using Inequality (3.1), we have

$$F\left(\frac{m_c(A^2x, By)}{1 + m_c(A^2x, SAx)}, \frac{m_c(A^2x, SAx)}{1 + m_c(By, Ty)}, \frac{m_c(SAx, Ty)}{1 + m_c(A^2x, SAx)}, \frac{m_c(A^2x, Ty)}{1 + m_c(By, Ty)}, \frac{m_c(By, Ty)}{1 + m_c(A^2x, SAx)}, \frac{m_c(SAx, By)}{1 + m_c(By, Ty)}\right) \lesssim 0$$

$$\Rightarrow F(m_c(A^2x, Ax), 0, m_c(A^2x, Ax), m_c(A^2x, Ax), 0, m_c(A^2x, Ax)) \lesssim 0,$$

which contradicts (2.1). Therefore  $A^2x = Ax = By$  or  $A(Ax) = Ax$  which amounts to say that  $Ax$  is a fixed point of  $A$ .

Also  $A^2x = SAx = Ax$ , this implies that  $Ax$  is a fixed point of  $S$ . Proceeding on earlier lines, we prove that  $B^2y = By$ . Therefore  $Ax = A^2x = By = B^2y = BBY = BAx$  i.e.  $BAx = Ax$ . This implies that  $Ax$  is fixed point of  $B$ .

On the other hand,  $Ax = A^2x = By = B^2y = BBY = BTy = TBy = Ax$  i.e.  $TAx = Ax$ , therefore,  $Ax$  is a fixed point of  $T$ . Hence  $Ax$  is a common fixed point of  $A, B, S$  and  $T$ . To prove the uniqueness of common fixed point, suppose that  $z$  and  $w$  are two distinct fixed points of  $A, B, S$  and  $T$ , then using inequality (3.1), we have

$$F\left(\frac{m_c(Az, Bw)}{1 + m_c(Az, Sz)}, \frac{m_c(Az, Sz)}{1 + m_c(Bw, Tw)}, \frac{m_c(Sz, Tw)}{1 + m_c(Az, Sz)}, \frac{m_c(Az, Tw)}{1 + m_c(Bw, Tw)}, \frac{m_c(Bw, Tw)}{1 + m_c(Az, Sz)}, \frac{m_c(Sz, Bw)}{1 + m_c(Bw, Tw)}\right) \lesssim 0$$

or

$$F\left(\frac{m_c(z, w)}{1 + m_c(z, z)}, \frac{m_c(z, z)}{1 + m_c(w, w)}, \frac{m_c(z, w)}{1 + m_c(z, z)}, \frac{m_c(z, w)}{1 + m_c(w, w)}, \frac{m_c(w, w)}{1 + m_c(z, z)}, \frac{m_c(z, w)}{1 + m_c(w, w)}\right) \lesssim 0$$

or

$$F(m_c(z, w), 0, m_c(z, w), m_c(z, w), 0, m_c(z, w)) \lesssim 0,$$

which contradicts (2.1). Therefore  $z = w$ . Hence  $z = Ax$  is the unique common fixed point of  $A, B, S$  and  $T$ . This completes the proof. □

Following example demonstrates the Theorem 3.1.

**Example 3.2.** If  $X = [0, 1]$ . Define  $m_c : X \times X \rightarrow C$  as

$$m_c(x, y) = \begin{cases} i(x - y) & \text{if } x \geq y; \\ i(y - x + 1) & \text{if } x < y. \end{cases}$$

Then  $m_c(x, y)$  is a non-negative complex valued function satisfying the identity of indiscernible.

Define mappings  $A, B, S$  and  $T$  by

$$Ax = \frac{x^2}{4}, \quad Bx = \frac{x^2}{9}, \quad Sx = \frac{x}{2} \quad \text{and} \quad Tx = \frac{x}{3}.$$

Clearly pairs  $(A, S)$  and  $(B, T)$  are non-vacuously weakly compatible pairs for coincidence point  $x = 0$  in  $X$ .

In order to verify inequality (3.1), we utilize subsequent function  $F$  such that  $F \in \mathcal{F}_6$ .

$$F(t_1, t_2, \dots, t_6) = t_1 - at_3 - b \max\{t_2, t_4, t_6\} - c \max\{t_3, t_5\}, \tag{3.5}$$

where  $a, b, c \geq 0$  such that  $a + b + c < 1$ , so that

$$\begin{aligned} &F\left(\frac{m_c(Ax, By)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}, \right. \\ &\quad \left. \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}\right) \\ &= \frac{m_c(Ax, By)}{1 + m_c(Ax, Sx)} - a \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)} - b \max\left\{\frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \right. \\ &\quad \left. \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}\right\} - c \max\left\{\frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}\right\}, \quad \forall x, y \in X. \end{aligned}$$

By routine calculation, it is easy to verify that  $A, B, S$  and  $T$  satisfy (3.1) with function  $F$ , defined by (3.5) for  $x, y \in [0, 1]$  with  $Ax \neq By$  and  $a = \frac{1}{4}, b = \frac{1}{4}$  and  $c = \frac{9}{20}$  (clearly  $a + b + c < 1$ ). For instance, we discuss the following cases for randomly chosen combinations of values of  $x, y \in [0, 1]$

**Case-I:** If  $x = 0$  and  $y = 1$ , we have

$$\begin{aligned} &F\left(\frac{m_c(Ax, By)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}, \right. \\ &\quad \left. \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}\right) \\ &= 1.11i - (0.1193 + 1.1305i) = -0.1193 - 0.09075i \lesssim 0 \end{aligned}$$

so that inequality (3.1) is satisfied.

**Case-II:** If  $x = 1$  and  $y = \frac{1}{2}$ , we have

$$\begin{aligned} &F\left(\frac{m_c(Ax, By)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}, \right. \\ &\quad \left. \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}\right) \\ &= 0.1085 + 0.0867i - (0.4676 + 0.388805i) = -0.3591 - 0.302105i \lesssim 0, \end{aligned}$$

as desired.

**Case-III:** For,  $x = 1$  and  $y = 1$ , we have

$$F\left(\frac{m_c(Ax, By)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}\right) = 0.06781 + 0.054i - (0.4731 + 0.36235i) = -0.40529 - 0.30835i \lesssim 0,$$

as required.

**Case-IV:** Finally, for  $x = \frac{1}{2}$  and  $y = \frac{1}{2}$ , we have

$$F\left(\frac{m_c(Ax, By)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}\right) = -0.1965 - 0.336i \lesssim 0,$$

as required.

Thus Inequality (3.1) of Theorem 3.1 is satisfied for all  $x, y \in [0, 1]$  with  $a = \frac{1}{4}$ ,  $b = \frac{1}{4}$  and  $c = \frac{9}{20}$  so that all the conditions of Theorem 3.1 are satisfied. Notice that  $x = 0$  remains fixed under  $A, B, S, T$  and is indeed unique.

Observe that Theorem 3.1 remains true for complex valued metric spaces as well as complex valued b- metric spaces, resulting the following corollary.

**Corollary 3.3.** *Let  $(X, d)$  be a complex valued metric space (or complex valued b- metric space) and  $A, B, S$  and  $T$  be self mappings of  $X$  such that Inequality (3.1) holds for all  $x, y \in X$  with  $Ax \neq By$  and  $F \in \mathcal{F}_6$ . If  $(A, S)$  and  $(B, T)$  are non-vacuously weakly compatible pairs, then  $A, B, S$  and  $T$  have a unique common fixed point.*

By setting  $A = B$  and  $S = T$  in Theorem 3.1, we deduce the following corollary involving a pair of mappings.

**Corollary 3.4.** *Let  $X$  be a non-empty set and  $m_c : X \times X \rightarrow C$  a non-negative complex valued function satisfying the identity of indiscernible. Suppose that  $A$  and  $S$  are two self mappings of  $X$  such that*

$$F\left(\frac{m_c(Ax, Ay)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)}, \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)}, \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)}, \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)}\right) \lesssim 0,$$

for all  $x, y \in X$  with  $Ax \neq Ay$  and  $F \in \mathcal{F}_6$ .

If pair  $(A, S)$  is non-vacuously weakly compatible, then  $A$  and  $S$  have a unique common fixed point.

Invoking Examples 2.2 - 2.17 of implicit functions, one can have the following corollary covering several known as well as unknown theorems in one go.

**Corollary 3.5.** *Let  $X$  be a non-empty set and  $m_c : X \times X \rightarrow C$  a non-negative complex valued function satisfying the identity of indiscernible. If  $A, B, S$  and  $T$  are four self mappings of  $X$  such that any one of the following inequality is satisfied (for all  $x, y \in X$  with  $Ax \neq By$ ).*

**I:**

$$m_c(Ax, By) \lesssim a\left(\frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}\right) + b\left\{\frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)} + \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}\right\} + c\left\{\frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)} + \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}\right\},$$

where  $a, b, c \geq 0$ ,  $a + 2c < 1$ .

**II:**

$$m_c(Ax, By) \lesssim a \left( \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)} \right) + b \max \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)} \right. \\ \left. \frac{m_c(Sx, By)}{1 + m_c(By, Ty)} \right\} + c \max \left\{ \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)} \right\},$$

where  $a, b, c \geq 0, a + b + c < 1$ .

**III:**

$$m_c(Ax, By) \lesssim k \max \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \right. \\ \left. \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Sx, By)}{1 + m_c(By, Ty)} \right\}, \quad \text{where } k \in (0, 1).$$

**IV:**

$$m_c(Ax, By) \lesssim a \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)} + b \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)} + c \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)} \\ + d \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)} + e \frac{m_c(Sx, By)}{1 + m_c(By, Ty)},$$

where  $a, b, c, d, e \geq 0, b + c + e < 1$ .

**V:**

$$m_c(Ax, By) \lesssim k \min \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)} + \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)} + \right. \\ \left. \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)} + \frac{m_c(Sx, By)}{1 + m_c(By, Ty)} \right\}, \quad \text{where } k \in (0, 1).$$

**VI:**

$$m_c(Ax, By) \lesssim k \min \left\{ \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \max \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)}, \right. \right. \\ \left. \left. \frac{m_c(Sx, By)}{1 + m_c(By, Ty)} \right\}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)} \right\}, \quad \text{where } k \in (0, \infty).$$

**VII:**

$$m_c(Ax, By) \lesssim k \max \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)}, \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}, \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}, \right. \\ \left. \frac{1}{2} \left( \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)} + \frac{m_c(Sx, By)}{1 + m_c(By, Ty)} \right) \right\}, \quad \text{where } k \in (0, 1).$$

**VIII:**

$$m_c(Ax, By) \lesssim \lambda \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)} + \mu \frac{\frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)} \cdot \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}}{1 + \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}} + \gamma \frac{\frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)} \cdot \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}}{1 + \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)}},$$

where  $\lambda, \mu, \gamma \geq 0, \lambda + \mu + \gamma < 1$ .

**IX:**

$$m_c(Ax, By) \begin{cases} \lesssim \lambda \frac{m_c(Sx, Ty)}{1 + m_c(Ax, Sx)} + \mu \frac{\frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)} \cdot \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)} + \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)} \cdot \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}}{\frac{m_c(Ax, Sx)}{1 + m_c(By, Ty)} + \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)}} + \\ \gamma \frac{\frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)} \cdot \frac{m_c(By, Ty)}{1 + m_c(Ax, Sx)} + \frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)} \cdot \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}}{\frac{m_c(Ax, Ty)}{1 + m_c(By, Ty)} + \frac{m_c(Sx, By)}{1 + m_c(By, Ty)}} \text{ if } D_1 \neq 0, D_2 \neq 0; \\ = 0 \text{ if } D_1 = 0 \text{ or } D_2 = 0, \end{cases}$$



where  $D_1 = \frac{m_c(Ax, Sx)}{1+m_c(By, Ty)} + \frac{m_c(By, Ty)}{1+m_c(Ax, Sx)}$  and  $D_2 = \frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} + \frac{m_c(Sx, By)}{1+m_c(By, Ty)}$ ,  $\lambda, \mu, \gamma \geq 0$ ,  $\lambda + \mu + \gamma < 1$ .

**X:**

$$m_c(Ax, By) \begin{cases} \lesssim \lambda \frac{m_c(Sx, Ty)}{1+m_c(Ax, Sx)} + \mu \frac{\frac{m_c(Ax, Sx)}{1+m_c(By, Ty)} \cdot \frac{m_c(By, Ty)}{1+m_c(Ax, Sx)}}{\frac{m_c(Ax, Sx)}{1+m_c(By, Ty)} + \frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} + \frac{m_c(Sx, By)}{1+m_c(By, Ty)}} & \text{if } D_1 \neq 0; \\ = 0 & \text{if } D_1 = 0, \end{cases}$$

where  $\lambda, \mu \geq 0$ ,  $\lambda + \mu < 1$  and  $D_1 = \frac{m_c(Ax, Sx)}{1+m_c(By, Ty)} + \frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} + \frac{m_c(Sx, By)}{1+m_c(By, Ty)}$ .

**XI:**

$$m_c(Ax, By) \lesssim A \frac{m_c(Sx, Ty)}{1+m_c(Ax, Sx)} + B \frac{\frac{m_c(Ax, Sx)}{1+m_c(By, Ty)} \cdot \frac{m_c(By, Ty)}{1+m_c(Ax, Sx)}}{1 + \frac{m_c(Sx, Ty)}{1+m_c(Ax, Sx)}} + C \frac{\frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} \cdot \frac{m_c(Sx, By)}{1+m_c(By, Ty)}}{1 + \frac{m_c(Sx, Ty)}{1+m_c(Ax, Sx)}} \\ + D \frac{\frac{m_c(Ax, Sx)}{1+m_c(By, Ty)} \cdot \frac{m_c(Sx, By)}{1+m_c(By, Ty)}}{1 + \frac{m_c(Sx, Ty)}{1+m_c(Ax, Sx)}} + E \frac{\frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} \cdot \frac{m_c(By, Ty)}{1+m_c(Ax, Sx)}}{1 + \frac{m_c(Sx, Ty)}{1+m_c(Ax, Sx)}},$$

where  $A, B, C, D, E \geq 0$ ,  $A + B + C + D + E < 1$ .

**XII:**

$$m_c(Ax, By) \lesssim k \max \left\{ \frac{m_c(Sx, Ty)}{1+m_c(Ax, Sx)}, \frac{1}{2} \left( \frac{m_c(Ax, Sx)}{1+m_c(By, Ty)} + \frac{m_c(By, Ty)}{1+m_c(Ax, Sx)} \right), \right. \\ \left. \frac{1}{2} \left( \frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} + \frac{m_c(Sx, By)}{1+m_c(By, Ty)} \right) \right\}, \quad \text{where } k \in (0, 1).$$

**XIII:**

$$m_c(Ax, By) \lesssim k \max \left\{ \frac{m_c(Sx, Ty)}{1+m_c(Ax, Sx)}, \sqrt{\frac{m_c(Ax, Sx)}{1+m_c(By, Ty)} \cdot \frac{m_c(By, Ty)}{1+m_c(Ax, Sx)}}, \right. \\ \left. \sqrt{\frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} \cdot \frac{m_c(Sx, By)}{1+m_c(By, Ty)}} \right\}, \quad \text{where } k \in (0, 1).$$

**XIV:**

$$m_c(Ax, By) \lesssim \alpha \max \left\{ \frac{m_c(Ax, Sx)}{1+m_c(By, Ty)}, \frac{m_c(Sx, Ty)}{1+m_c(Ax, Sx)}, \frac{m_c(By, Ty)}{1+m_c(Ax, Sx)} \right\} \\ (1 - \alpha) \left\{ a \frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} + b \frac{m_c(Sx, By)}{1+m_c(By, Ty)} \right\},$$

where  $\alpha \in (0, 1)$  and  $a + b < 1$  for  $a, b \geq 0$ .

**XV:**

$$m_c(Ax, By) \lesssim k \max \left\{ \frac{1}{2} \left( \frac{2m_c(Ax, Sx)}{1+m_c(By, Ty)} + \frac{m_c(By, Ty)}{1+m_c(Ax, Sx)} \right), \frac{1}{2} \left( \frac{2m_c(Ax, Sx)}{1+m_c(By, Ty)} + \right. \right. \\ \left. \left. \frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} \right), \frac{2m_c(Ax, Sx)}{1+m_c(By, Ty)} + \frac{m_c(Sx, By)}{1+m_c(By, Ty)} \right\},$$

where  $k \in (0, 1)$ .

**XVI:**

$$m_c(Ax, By) \lesssim \alpha \frac{\frac{m_c(Ax, Sx)}{1+m_c(By, Ty)} \cdot \frac{m_c(Sx, By)}{1+m_c(By, Ty)} + \frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} \cdot \frac{m_c(By, Ty)}{1+m_c(Ax, Sx)}}{1 + \frac{m_c(Ax, Ty)}{1+m_c(By, Ty)} + \frac{m_c(Sx, By)}{1+m_c(By, Ty)}}, \quad \text{where } \alpha \in (0, 1).$$

If  $(A, S)$  and  $(B, T)$  are non-vacuously weakly compatible pairs, then  $A, B, S$  and  $T$  have a unique common fixed point.

*Proof.* The proof of this corollary is obvious in view of Examples 2.2 - 2.17. □

Next, by setting  $A = B$  and  $S = T$  in Corollary 3.5 (I-XVI), the following corollary is obtained.

**Corollary 3.6.** *Let  $X$  be a non-empty set and  $m_c : X \times X \rightarrow C$  a non-negative complex valued function satisfying the identity of indiscernible and  $A$  and  $S$  be self mappings of  $X$  satisfying any one of the following inequalities for all  $x, y \in X$ ;*

**I:**

$$m_c(Ax, Ay) \lesssim a \left( \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)} \right) + b \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)} + \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)} \right\} + c \left\{ \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)} + \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)} \right\}, \quad \text{where } a, b, c \geq 0, a + 2c < 1.$$

**II:**

$$m_c(Ax, Ay) \lesssim a \left( \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)} \right) + b \max \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)}, \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)}, \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)} \right\} + c \max \left\{ \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)} \right\},$$

where  $a, b, c \geq 0, a + b + c < 1$ .

**III:**

$$m_c(Ax, Ay) \lesssim k \max \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)}, \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)}, \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)}, \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)} \right\}, \quad \text{where } k \in (0, 1).$$

**IV:**

$$m_c(Ax, Ay) \lesssim a \frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)} + b \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)} + c \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)} + d \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)} + e \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)},$$

where  $a, b, c, d, e \geq 0, b + c + e < 1$ .

**V:**

$$m_c(Ax, Ay) \lesssim k \min \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)} + \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)}, \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)}, \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)} + \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)} \right\},$$

where  $k \in (0, 1)$ .

**VI:**

$$m_c(Ax, Ay) \lesssim k \min \left\{ \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)}, \max \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)}, \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)}, \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)} \right\}, \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)} \right\}, \quad \text{where } k \in (0, \infty).$$

**VII:**

$$m_c(Ax, Ay) \lesssim k \max \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)}, \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)}, \frac{1}{2} \left( \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)} + \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)} \right) \right\}, \quad \text{where } k \in (0, 1).$$

**VIII:**

$$m_c(Ax, Ay) \lesssim \lambda \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)} + \mu \frac{\frac{m_c(Ax, Sx)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Ay, Sy)}{1+m_c(Ax, Sx)}}{1 + \frac{m_c(Sx, Sy)}{1+m_c(Ax, Sx)}} + \gamma \frac{\frac{m_c(Ax, Sy)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Ay, Sy)}{1+m_c(Ax, Sx)}}{1 + \frac{m_c(Sx, Sy)}{1+m_c(Ax, Sx)}},$$

where  $\lambda, \mu, \gamma \geq 0, \lambda + \mu + \gamma < 1$ .

**IX:**

$$m_c(x, y) \begin{cases} \lesssim \lambda \frac{m_c(Sx, Sy)}{1+m_c(Ax, Sx)} + \mu \frac{\frac{m_c(Ax, Sx)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Ay, Sy)}{1+m_c(Ax, Sx)} + \frac{m_c(Ax, Sy)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Sx, Ay)}{1+m_c(Ay, Sy)}}{\frac{m_c(Ax, Sx)}{1+m_c(Ay, Sy)} + \frac{m_c(Ay, Sy)}{1+m_c(Ax, Sx)}} + \\ \gamma \frac{\frac{m_c(Ax, Sy)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Ay, Sy)}{1+m_c(Ax, Sx)} + \frac{m_c(Ax, Sy)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Sx, Ay)}{1+m_c(Ay, Sy)}}{\frac{m_c(Ax, Sy)}{1+m_c(Ay, Sy)} + \frac{m_c(Sx, Ay)}{1+m_c(Ay, Sy)}} \text{ if } D_1 \neq 0, D_2 \neq 0; \\ = 0 \text{ if } D_1 = 0 \text{ or } D_2 = 0, \end{cases}$$

where  $D_1 = \frac{m_c(Ax, Sx)}{1+m_c(Ay, Sy)} + \frac{m_c(Ay, Sy)}{1+m_c(Ax, Sx)}$  and  $D_2 = \frac{m_c(Ax, Sy)}{1+m_c(Ay, Sy)} + \frac{m_c(Sx, Ay)}{1+m_c(Ay, Sy)}$ ,  $\lambda, \mu, \gamma \geq 0, \lambda + \mu + \gamma < 1$ .

**X:**

$$m_c(x, y) \begin{cases} \lesssim \lambda \frac{m_c(Sx, Sy)}{1+m_c(Ax, Sx)} + \mu \frac{\frac{m_c(Ax, Sx)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Ay, Sy)}{1+m_c(Ax, Sx)}}{\frac{m_c(Ax, Sx)}{1+m_c(Ay, Sy)} + \frac{m_c(Ay, Sy)}{1+m_c(Ax, Sx)} + \frac{m_c(Sx, Ay)}{1+m_c(Ay, Sy)}} \text{ if } D_1 \neq 0; \\ = 0 \text{ if } D_1 = 0, \end{cases}$$

where  $\lambda, \mu \geq 0, \lambda + \mu < 1$  and  $D_1 = \frac{m_c(Ax, Sx)}{1+m_c(Ay, Sy)} + \frac{m_c(Ax, Sy)}{1+m_c(Ay, Sy)} + \frac{m_c(Sx, Ay)}{1+m_c(Ay, Sy)}$ .

**XI:**

$$m_c(Ax, Ay) \lesssim A \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)} + B \frac{\frac{m_c(Ax, Sx)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Ay, Sy)}{1+m_c(Ax, Sx)}}{1 + \frac{m_c(Sx, Sy)}{1+m_c(Ax, Sx)}} + C \frac{\frac{m_c(Ax, Sy)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Sx, Ay)}{1+m_c(Ay, Sy)}}{1 + \frac{m_c(Sx, Sy)}{1+m_c(Ax, Sx)}} + D \frac{\frac{m_c(Ax, Sx)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Sx, Ay)}{1+m_c(Ay, Sy)}}{1 + \frac{m_c(Sx, Sy)}{1+m_c(Ax, Sx)}} + E \frac{\frac{m_c(Ax, Sy)}{1+m_c(Ay, Sy)} \cdot \frac{m_c(Ay, Sy)}{1+m_c(Ax, Sx)}}{1 + \frac{m_c(Sx, Sy)}{1+m_c(Ax, Sx)}},$$

where  $A, B, C, D, E \geq 0, A + B + C + D + E < 1$ .

**XII:**

$$m_c(Ax, Ay) \lesssim k \max \left\{ \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)}, \frac{1}{2} \left( \frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)} + \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)} \right), \frac{1}{2} \left( \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)} + \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)} \right) \right\}, \text{ where } k \in (0, 1).$$

**XIII:**

$$m_c(Ax, Ay) \lesssim k \max \left\{ \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)}, \sqrt{\frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)} \cdot \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)}}, \sqrt{\frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)} \cdot \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)}} \right\}, \text{ where } k \in (0, 1).$$

**XIV:**

$$m_c(Ax, Ay) \lesssim \alpha \max \left\{ \frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)}, \frac{m_c(Sx, Sy)}{1 + m_c(Ax, Sx)}, \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)} \right\} (1 - \alpha) \left\{ a \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)} + b \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)} \right\},$$

where  $\alpha \in (0, 1)$  and  $a + b < 1$  for  $a, b \geq 0$ .

**XV:**

$$m_c(Ax, Ay) \lesssim k \max \left\{ \frac{1}{2} \left( \frac{2m_c(Ax, Sx)}{1 + m_c(Ay, Sy)} + \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)} \right), \frac{1}{2} \left( \frac{2m_c(Ax, Sx)}{1 + m_c(Ay, Sy)} + \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)} \right), \frac{2m_c(Ax, Sx)}{1 + m_c(Ay, Sy)} + \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)} \right\},$$

where  $k \in (0, 1)$ .

**XVI:**

$$m_c(Ax, Ay) \lesssim \alpha \frac{\frac{m_c(Ax, Sx)}{1 + m_c(Ay, Sy)} \cdot \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)} + \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)} \cdot \frac{m_c(Ay, Sy)}{1 + m_c(Ax, Sx)}}{1 + \frac{m_c(Ax, Sy)}{1 + m_c(Ay, Sy)} + \frac{m_c(Sx, Ay)}{1 + m_c(Ay, Sy)}}, \quad \text{where } \alpha \in (0, 1).$$

If  $(A, S)$  is non-vacuously weakly compatible pair, then  $A$  and  $S$  have a unique common fixed point.

*Remark 3.7.* Corollaries corresponding to contraction conditions (I-XVI), mentioned in Corollary 3.5 and Corollary 3.6 are new results even in complex as well as real valued metric spaces.

## 4. Results under minimal symmetry

Our next result is proved for two pairs of non-vacuously weakly compatible mappings satisfying implicit relation defined in Definition 2.1.

**Theorem 4.1.** Let  $X$  be a non-empty set and  $m_c : X \times X \rightarrow C$  a non-negative complex valued function satisfying the identity of indiscernible. If  $A, B, S$  and  $T$  are four self mappings of  $X$  such that

1. for all  $x, y \in X$  with  $Ax \neq By$  and  $F \in \mathcal{F}_6$ ,

$$F(m_c(Ax, By), m_c(Sx, Ax), m_c(Sx, Ty), m_c(Ty, Ax), m_c(Ty, By), m_c(Sx, By)) \lesssim 0; \quad (4.1)$$

2.  $m_c(u, v) = m_c(v, u)$ , whenever  $u \in X$  and  $v \in X$  are points of coincidence of  $(A, S)$  and  $(B, T)$  respectively;
3. the pairs  $(A, S)$  and  $(B, T)$  are non-vacuously weakly compatible,

then  $A, B, S$  and  $T$  have a unique common fixed point.

*Proof.* Since  $(A, S)$  and  $(B, T)$  are two non-vacuously weakly compatible pairs of mappings then there exist  $x, y \in X$  such that

$$Ax = Sx = u(\text{say}) \quad \text{and} \quad ASx = SAx \quad (4.2)$$

and

$$By = Ty = v(\text{say}) \quad \text{and} \quad BTy = TBy. \quad (4.3)$$

Clearly  $u$  and  $v$  are points of coincidence of pairs  $(A, S)$  and  $(B, T)$  respectively.

First of all, we prove that  $Ax = By$ . Otherwise, if  $Ax \neq By$ , then using (4.1), (4.2) and (4.3), we have

$$F(m_c(Ax, By), 0, m_c(Ax, By), m_c(By, Ax), 0, m_c(Ax, By)) \lesssim 0. \quad (4.4)$$

As  $u$  and  $v$  are points of coincidence of the pairs  $(A, S)$  and  $(B, T)$  respectively, therefore in view of condition (2),

$$m_c(u, v) = m_c(v, u) \quad \text{or} \quad m_c(Ax, By) = m_c(By, Ax).$$

Now, from (4.4), we have

$$F(m_c(Ax, By), 0, m_c(Ax, By), m_c(Ax, By), 0, m_c(Ax, By)) \lesssim 0,$$

which contradicts the requirement of  $\mathcal{F}_6$  as  $F \in \mathcal{F}_6$ . Therefore  $Ax = By$  which implies that  $Ax = Sx = By = Ty$ , so that

$$A^2x = AAx = ASx = SAx. \quad (4.5)$$

Next, we prove that  $Ax$  is a fixed point of  $A$ . For this, it is sufficient to show that  $Ax = A^2x$ . On the contrary, suppose  $A^2x \neq Ax = By$ . Then by inequality (4.1), we have

$$\begin{aligned} F(m_c(AAx, By), m_c(SAx, AAx), m_c(SAx, Ty), m_c(Ty, AAx), m_c(Ty, By), m_c(SAx, By)) \lesssim 0. \\ \Rightarrow F(m_c(A^2x, Ax), 0, m_c(A^2x, Ax), m_c(Ty, A^2x), 0, m_c(A^2x, Ax), ) \lesssim 0. \end{aligned} \quad (4.6)$$

On utilizing (4.5), one can get  $A(Ax) = S(Ax) = w$  (say).

Clearly  $w = Ax$  and  $v$  are points of coincidence of the pairs  $(A, S)$  and  $(B, T)$  respectively. Then by condition (2), one can obtain

$$m_c(v, w) = m_c(w, v) \text{ i.e. } m_c(Ty, A^2x) = m_c(A^2x, Ty) = m_c(A^2x, Ax).$$

From (4.6), we have

$$F(m_c(A^2x, Ax), 0, m_c(A^2x, Ax), m_c(A^2x, Ax), 0, m_c(A^2x, Ax)) \lesssim 0,$$

which is a contradiction as  $F \in \mathcal{F}_6$ . Hence  $A^2x = Ax$  or  $A(Ax) = Ax$ . This implies that  $Ax$  is a fixed point of  $A$ . Also  $A^2x = SAx = Ax$ . This implies that  $Ax$  is a fixed point of  $S$ . Proceeding as above, we can prove that  $B^2y = By$ , therefore  $Ax = A^2x = By = B^2y = BBy = BAx$ , this leads to  $BAx = Ax$  that is  $Ax$  is a fixed point of  $B$ .

On the other hand  $Ax = A^2x = By = B^2y = BBy = BTy = TBy = TAx$  i.e.  $TAx = Ax$ . Hence  $Ax$  is a fixed point of  $T$ . Therefore  $Ax$  is a common fixed point of  $A, B, S$  and  $T$ . Finally uniqueness of fixed point is established. To prove the uniqueness of common fixed point, let  $w$  be another common fixed point of  $A, B, S$  and  $T$ , such that  $w \neq Ax = z$ . Then using inequality (4.1), we have

$$\begin{aligned} F(m_c(Az, Bw), m_c(Sz, Az), m_c(Sz, Tw), m_c(Tw, Az), m_c(Tw, Bw), m_c(Sz, Bw)) \lesssim 0 \\ F(m_c(z, w), m_c(z, z), m_c(z, w), m_c(w, z), m_c(w, w), m_c(z, w)) \lesssim 0. \end{aligned} \quad (4.7)$$

If  $Bw = Tw = k$  (say) and  $Az = Sz = h$  (say), then  $h$  and  $k$  are the points of coincidences of pairs  $(A, S)$  and  $(B, T)$  respectively. Using condition (2), one gets

$$m_c(h, k) = m_c(k, h) \text{ or } m_c(Az, Tw) = m_c(Tw, Az) \text{ or } m_c(z, w) = m_c(w, z).$$

Making use of preceding observations in (4.7), one can obtain

$$F(m_c(z, w), 0, m_c(z, w), m_c(z, w), 0, m_c(z, w)) \lesssim 0,$$

which is a contradiction as  $F \in \mathcal{F}_6$ , this concludes that  $z = w$ . Hence  $z = Ax$  is a unique common fixed point of  $A, B, S$  and  $T$ . This completes the proof.  $\square$

Following example demonstrates the Theorem 4.1.

**Example 4.2.** Let  $X = [0, 1]$ . Define  $m_c : X \times X \rightarrow C$  by

$$m_c(x, y) = \begin{cases} i(e^{(x-y)} - 1) & \text{if } x \geq y; \\ ie^{y-x} & \text{if } x < y. \end{cases}$$

Then  $m_c(x, y)$  is a non-negative complex valued function and satisfies the identity of indiscernible.

Define the mappings  $A, B, S$  and  $T$  by

$$Ax = \frac{1+x}{2}, \quad Bx = \frac{1+2x}{3} \quad Sx = \frac{1+3x}{4} \quad \text{and} \quad Tx = \frac{1+4x}{5}.$$

Clearly pairs  $(A, S)$  and  $(B, T)$  are non-vacuously weakly compatible pairs of mappings with point  $x = 1$  as a coincidence point and also point of coincidence as

$$A(1) = S(1) = 1 \text{ and } B(1) = T(1) = 1.$$

Hence  $m_c(u, v) = m_c(v, u)$  (for point of coincidence  $u = 1$  of pair  $(A, S)$  and point of coincidence  $v = 1$  of  $(B, T)$ ).

Now, define  $F(t_1, t_2, \dots, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$ , where  $k \in (0, 1)$ , clearly  $F \in \mathcal{F}_6$ .

So that for all  $x, y \in X$ ,

$$\begin{aligned} F(m_c(Ax, By), m_c(Sx, Ax), m_c(Sx, Ty), m_c(Ty, Ax), m_c(Ty, By), m_c(Sx, By)) \\ = m_c(Ax, By) - k \max\{m_c(Sx, Ax), m_c(Sx, Ty), m_c(Ty, Ax), \\ m_c(Ty, By), m_c(Sx, By)\}. \end{aligned}$$

In order to verify the Inequality (4.1) with aforementioned expression of  $F \in \mathcal{F}_6$ , the following cases are discussed for randomly chosen values of  $x, y \in [0, 1]$  such that  $Ax \neq By$ .

**Case I-** If  $x = 0$  and  $y = 1$ , then

$$\begin{aligned} F(m_c(Ax, By), m_c(Sx, Ax), m_c(Sx, Ty), m_c(Ty, Ax), m_c(Ty, By), m_c(Sx, By)) \\ = 1.6487i - k(2.1170i) \lesssim 0, \end{aligned}$$

for  $k \in [0.78, 1)$ .

**Case II-** If  $x = 1$  and  $y = \frac{1}{2}$ , we have

$$\begin{aligned} F(m_c(Ax, By), m_c(Sx, Ax), m_c(Sx, Ty), m_c(Ty, Ax), m_c(Ty, By), m_c(Sx, By)) \\ = 0.39i - k(1.491i) \lesssim 0, \end{aligned}$$

for  $k \in [0.27, 1)$ .

**Case III-** If  $x = 0$  and  $y = 0$ , we get

$$\begin{aligned} F(m_c(Ax, By), m_c(Sx, Ax), m_c(Sx, Ty), m_c(Ty, Ax), m_c(Ty, By), m_c(Sx, By)) \\ = 0.1813i - k(1.3498i) \lesssim 0, \end{aligned}$$

for  $k \in [0.14, 1)$ .

**Case IV-** If  $x = \frac{1}{2}$  and  $y = 1$ , we have

$$\begin{aligned} F(m_c(Ax, By), m_c(Sx, Ax), m_c(Sx, Ty), m_c(Ty, Ax), m_c(Ty, By), m_c(Sx, By)) \\ = 1.284i - k(1.455i) \lesssim 0, \end{aligned}$$

for  $k \in [0.89, 1)$ .

So that Inequality (4.1) of Theorem 4.1 is satisfied for all  $x, y \in [0, 1]$  and all the conditions of Theorem 4.1 are satisfied. Notice that  $x = 1$  is a fixed point of  $A, B, S, T$ , which is indeed unique.

Aforesaid theorem is also valid for complex valued metric spaces and complex valued b-metric spaces, even on relaxing the condition (2), the following corollary is obtained immediately.

**Corollary 4.3.** *Let  $(X, d)$  be a complex valued metric space (or Complex valued b-metric space) and  $A, B, S$  and  $T$  be self mappings of  $X$  such that Inequality (4.1) holds for all  $x, y \in X$  with  $F \in \mathcal{F}_6$ . If  $(A, S)$  and  $(B, T)$  are non-vacuously weakly compatible pairs, then  $A, B, S$  and  $T$  have a unique common fixed point.*

Restricting Theorem 4.1 to single pair of mappings, we deduce the following corollary.

**Corollary 4.4.** *Let  $X$  be a non-empty set and  $m_c : X \times X \rightarrow C$  a non-negative complex valued function satisfying the identity of indiscernible. If  $A$  and  $S$  are self mappings of  $X$  such that*

$$F(m_c(Ax, Ay), m_c(Sx, Ax), m_c(Sx, Sy), m_c(Sy, Ax), m_c(Sy, Ay), m_c(Sx, Ay)) \lesssim 0$$

for all  $x, y \in X$  with  $Ax \neq Ay$  and  $F \in \mathcal{F}_6$ .

If pair  $(A, S)$  is a pair of non-vacuously weakly compatible mappings such that  $m_c(u, v) = m_c(v, u)$ , wherein  $u$  and  $v$  are points of coincidence of  $(A, S)$  then  $A$  and  $S$  have a unique common fixed point.

Invoking Examples 2.2 - 2.17, one can derive the following unified corollary:

**Corollary 4.5.** *Let  $X$  be a non-empty set and  $m_c : X \times X \rightarrow C$  a non-negative complex valued function satisfying the identity of indiscernible. If  $A, B, S$  and  $T$  are four self mappings of  $X$  such that any one of the following conditions is satisfied (for all  $x, y \in X$ ):*

**I:**

$$m_c(Ax, By) \lesssim a.m_c(Sx, Ty) + b\{m_c(Sx, Ax) + m_c(Ty, By)\} + c\{m_c(Ty, Ax) + m_c(Sx, By)\},$$

where  $a, b, c \geq 0$ ,  $a + 2c < 1$ .

**II:**

$$m_c(Ax, By) \lesssim a.m_c(Sx, Ty) + b \max\{m_c(Sx, Ax), m_c(Ty, Ax), m_c(Sx, By)\} + c \max\{m_c(Sx, Ax), m_c(Ty, By)\},$$

where  $a, b, c \geq 0$ ,  $a + b + c < 1$ .

**III:**

$$m_c(Ax, By) \lesssim k \max\{m_c(Sx, Ax), m_c(Sx, Ty), m_c(Ty, Ax), m_c(Ty, By), m_c(Sx, By)\},$$

where  $k \in (0, 1)$ .

**IV:**

$$m_c(Ax, By) \lesssim a.m_c(Sx, Ax) + b.m_c(Sx, Ty) + c.m_c(Ty, Ax) + d.m_c(Ty, By) + e.m_c(Sx, By),$$

where  $a, b, c, d, e \geq 0$ ,  $b + c + e < 1$ .

**V:**

$$m_c(Ax, By) \lesssim k \min\{m_c(Sx, Ax) + m_c(Sx, Ty), m_c(Sx, Ty) + m_c(Ty, Ax), m_c(Ty, By) + m_c(Sx, By)\}, \quad \text{where } k \in (0, 1).$$

**VI:**

$$m_c(Ax, By) \lesssim k \min\{m_c(Sx, Ty), \max\{m_c(Sx, Ax), m_c(Ty, Ax), m_c(Sx, By)\}, m_c(Ty, By)\}, \quad \text{where } k \in (0, \infty).$$

**VII:**

$$m_c(Ax, By) \lesssim k \max\left\{m_c(Sx, Ax), m_c(Sx, Ty), m_c(Ty, By), \frac{m_c(Ty, Ax) + m_c(Sx, By)}{2}\right\},$$

where  $k \in (0, 1)$ .

**VIII:**

$$m_c(Ax, By) \lesssim \lambda.m_c(Sx, Ty) + \mu \frac{m_c(Sx, Ax).m_c(Ty, By)}{1 + m_c(Sx, Ty)} + \gamma \frac{m_c(Ty, Ax).m_c(Sx, By)}{1 + m_c(Sx, Ty)},$$

where  $\lambda, \mu, \gamma \geq 0$ ,  $\lambda + \mu + \gamma < 1$ .

**IX:**

$$m_c(Ax, By) \begin{cases} \lesssim \lambda m_c(Sx, Ty) + \mu \frac{m_c(Sx, Ax).m_c(Ty, By) + m_c(Ty, Ax).m_c(Sx, By)}{m_c(Sx, Ax) + m_c(Ty, By)} + \\ \gamma \frac{m_c(Ty, Ax).m_c(Ty, By) + m_c(Sx, Ax).m_c(Sx, By)}{m_c(Ty, Ax) + m_c(Sx, By)} \text{ if } D_1 \neq 0, D_2 \neq 0; \\ = 0 \text{ if } D_1 = 0 \text{ or } D_2 = 0, \end{cases}$$

where  $D_1 = m_c(Sx, Ax) + m_c(Ty, By)$  and  $D_2 = m_c(Ty, Ax) + m_c(Sx, By)$ ,  $\lambda, \mu, \gamma \geq 0$ ,  $\lambda + \mu + \gamma < 1$ .

**X:**

$$m_c(Ax, By) \begin{cases} \lesssim \lambda.m_c(Sx, Ty) + \mu \frac{m_c(Sx, Ax).m_c(Ty, By)}{m_c(Sx, Ty)+m_c(Ty, Ax)+m_c(Sx, By)} & \text{if } D_1 \neq 0; \\ = 0 & \text{if } D_1 = 0, \end{cases}$$

where  $\lambda, \mu \geq 0$ ,  $\lambda + \mu < 1$  and  $D_1 = m_c(Sx, Ty) + m_c(Ty, Ax) + m_c(Sx, By)$ .

**XI:**

$$m_c(Ax, By) \lesssim A.m_c(Sx, Ty) + B. \frac{m_c(Sx, Ax).m_c(Ty, By)}{1 + m_c(Sx, Ty)} + C. \frac{m_c(Ty, Ax).m_c(Sx, By)}{1 + m_c(Sx, Ty)} + D. \frac{m_c(Sx, Ax).m_c(Sx, By)}{1 + m_c(Sx, Ty)} + E. \frac{m_c(Ty, Ax).m_c(Ty, By)}{1 + m_c(Sx, Ty)},$$

where  $A, B, C, D, E \geq 0$ ,  $A + B + C + D + E < 1$ .

**XII:**

$$m_c(Ax, By) \lesssim k \max \left\{ m_c(Sx, Ty), \frac{m_c(Sx, Ax) + m_c(Ty, By)}{2}, \frac{m_c(Ty, Ax) + m_c(Sx, By)}{2} \right\},$$

where  $k \in (0, 1)$ .

**XIII:**

$$m_c(Ax, By) \lesssim k \max \{ m_c(Sx, Ty), \sqrt{m_c(Sx, Ax).m_c(Ty, By)}, \sqrt{m_c(Ty, Ax).m_c(Sx, By)} \},$$

where  $k \in (0, 1)$ .

**XIV:**

$$m_c(Ax, By) \lesssim \alpha \max \{ m_c(Sx, Ax), m_c(Sx, Ty), m_c(Ty, By) \} + (1 - \alpha) \{ a.m_c(Ty, Ax) + b.m_c(Sx, By) \},$$

where  $\alpha \in (0, 1)$  and  $a + b < 1$  for  $a, b \geq 0$ .

**XV:**

$$m_c(Ax, By) \lesssim k \max \left\{ \frac{2m_c(Sx, Ax) + m_c(Ty, By)}{2}, \frac{2m_c(Sx, Ax) + m_c(Ty, Ax)}{2}, \frac{2m_c(Sx, Ax) + m_c(Sx, By)}{2} \right\},$$

where  $k \in (0, 1)$ .

**XVI:**

$$m_c(Ax, By) \lesssim \alpha \frac{m_c(Sx, Ax).m_c(Sx, By) + m_c(Ty, Ax).m_c(Ty, By)}{1 + m_c(Ty, Ax) + m_c(Sx, By)},$$

where  $\alpha \in (0, 1)$ .

If  $(A, S)$  and  $(B, T)$  are pairs of non-vacuously weakly compatible mappings such that  $m_c(u, v) = m_c(v, u)$ , wherein  $u$  and  $v$  are points of coincidence of  $(A, S)$  and  $(B, T)$  respectively. Then  $A, S, B$  and  $T$  have a unique common fixed point.

*Proof.* This corollary follows immediately in view of Examples 2.2 - 2.17. □

*Remark 4.6.* A multitude of corollaries corresponding to contraction conditions (I-XVI) are new in the present specific setting. In fact, contraction condition (I), condition (XII) and condition (XVI) of Theorem 4.1 are respectively Theorem 2.1 of Sandeep Bhatt et al. [5], Theorem 2.2 of K.P.R. Sastry [19] and Theorem 9 of Sumit chandok et al. [6]. Consequently, results proved in [5], [19] and in [6] can also be extended to our setting.



Restricting Corollary 4.5 to a pair of mappings  $(A, S)$  (I-XVI) under the additional condition  $m_c(u, v) = m_c(v, u)$ , wherein  $u$  and  $v$  are points of coincidence of  $(A, S)$ , we deduce the following corollary:

**Corollary 4.7.** *Let  $X$  be a non-empty set and  $m_c : X \times X \rightarrow C$  a non-negative complex valued function satisfying the identity of indiscernible. If  $A$  and  $S$  are four self mappings of  $X$  such that any one of the following conditions is satisfied (for all  $x, y \in X$ ):*

**I:**

$$m_c(Ax, Ay) \lesssim a.m_c(Sx, Sy) + b\{m_c(Sx, Ax) + m_c(Sy, Ay)\} + c\{m_c(Sy, Ax) + m_c(Sx, Ay)\},$$

where  $a, b, c \geq 0$ ,  $a + 2c < 1$ .

**II:**

$$m_c(Ax, Ay) \lesssim a.m_c(Sx, Sy) + b \max\{m_c(Sx, Ax), m_c(Sy, Ax), m_c(Sx, Ay)\} + c \max\{m_c(Sx, Ax), m_c(Sy, Ay)\},$$

where  $a, b, c \geq 0$ ,  $a + b + c < 1$ .

**III:**

$$m_c(Ax, Ay) \lesssim k \max\{m_c(Sx, Ax), m_c(Sx, Sy), m_c(Sy, Ax), m_c(Sy, Ay), m_c(Sx, Ay)\},$$

where  $k \in (0, 1)$ .

**IV:**

$$m_c(Ax, Ay) \lesssim a.m_c(Sx, Ax) + b.m_c(Sx, Sy) + c.m_c(Sy, Ax) + d.m_c(Sy, Ay) + e.m_c(Sx, Ay),$$

where  $a, b, c, d, e \geq 0$ ,  $b + c + e < 1$ .

**V:**

$$m_c(Ax, Ay) \lesssim k \min\{m_c(Sx, Ax) + m_c(Sx, Sy), m_c(Sx, Sy) + m_c(Sy, Ax), m_c(Sy, Ay) + m_c(Sx, Ay)\},$$

where  $k \in (0, 1)$ .

**VI:**

$$m_c(Ax, Ay) \lesssim k \min\{m_c(Sx, Sy), \max\{m_c(Sx, Ax), m_c(Sy, Ax), m_c(Sx, Ay)\}, m_c(Sy, Ay)\},$$

where  $k \in (0, \infty)$ .

**VII:**

$$m_c(Ax, Ay) \lesssim k \max\left\{m_c(Sx, Ax), m_c(Sx, Sy), m_c(Sy, Ay), \frac{(m_c(Sy, Ax) + m_c(Sx, Ay))}{2}\right\},$$

where  $k \in (0, 1)$ .

**VIII:**

$$m_c(Ax, Ay) \lesssim \lambda.m_c(Sx, Sy) + \mu \frac{m_c(Sx, Ax).m_c(Sy, Ay)}{1 + m_c(Sx, Sy)} + \gamma \frac{m_c(Sy, Ax).m_c(Sx, Ay)}{1 + m_c(Sx, Sy)},$$

where  $\lambda, \mu, \gamma \geq 0$ ,  $\lambda + \mu + \gamma < 1$ .

**IX:**

$$m_c(Ax, Ay) \begin{cases} \lesssim \lambda.m_c(Sx, Sy) + \mu \frac{m_c(Sx, Ax).m_c(Sy, Ay) + m_c(Sy, Ax).m_c(Sx, Ay)}{m_c(Sx, Ax) + m_c(Sy, Ay)} + \\ \gamma \frac{m_c(Sy, Ax).m_c(Sy, Ay) + m_c(Sx, Ax).m_c(Sx, Ay)}{m_c(Sy, Ax) + m_c(Sx, Ay)} & \text{if } D_1 \neq 0, D_2 \neq 0; \\ = 0 & \text{if } D_1 = 0 \text{ or } D_2 = 0, \end{cases}$$

where  $D_1 = m_c(Sx, Ax) + m_c(Sy, Ay)$  and  $D_2 = m_c(Sy, Ax) + m_c(Sx, Ay)$ ,  $\lambda, \mu, \gamma \geq 0$ ,  $\lambda + \mu + \gamma < 1$ .

**X:**

$$m_c(Ax, Ay) \begin{cases} \lesssim \lambda.m_c(Sx, Sy) + \mu \frac{m_c(Sx,Ax).m_c(Sy,Ay)}{m_c(Sx,Sy)+m_c(Sy,Ax)+m_c(Sx,Ay)} & \text{if } D_1 \neq 0; \\ = 0 & \text{if } D_1 = 0, \end{cases}$$

where  $\lambda, \mu \geq 0$ ,  $\lambda + \mu < 1$  and  $D_1 = m_c(Sx, Sy) + m_c(Sy, Ax) + m_c(Sx, Ay)$ .

**XI:**

$$m_c(Ax, Ay) \lesssim Am_c(Sx, Sy) + B. \frac{m_c(Sx, Ax).m_c(Sy, Ay)}{1 + m_c(Sx, Sy)} + C. \frac{m_c(Sy, Ax).m_c(Sx, Ay)}{1 + m_c(Sx, Sy)} \\ + D. \frac{m_c(Sx, Ax).m_c(Sx, Ay)}{1 + m_c(Sx, Sy)} + E \frac{m_c(Sy, Ax).m_c(Sy, Ay)}{1 + m_c(Sx, Sy)},$$

where  $A, B, C, D, E \geq 0$ ,  $A + B + C + D + E < 1$ .

**XII:**

$$m_c(Ax, Ay) \lesssim k \max \left\{ m_c(Sx, Sy), \frac{m_c(Sx, Ax) + m_c(Sy, Ay)}{2}, \frac{m_c(Sy, Ax) + m_c(Sx, Ay)}{2} \right\},$$

where  $k \in (0, 1)$ .

**XIII:**

$$m_c(Ax, Ay) \lesssim k \max \{ m_c(Sx, Sy), \sqrt{m_c(Sx, Ax).m_c(Sy, Ay)}, \sqrt{m_c(Sy, Ax).m_c(Sx, Ay)} \},$$

where  $k \in (0, 1)$ .

**XIV:**

$$m_c(Ax, Ay) \lesssim \alpha \max \{ m_c(Sx, Ax), m_c(Sx, Sy), m_c(Sy, Ay) \} \\ + (1 - \alpha) \{ a.m_c(Sy, Ax) + b.m_c(Sx, Ay) \},$$

where  $\alpha \in (0, 1)$  and  $a + b < 1$  for  $a, b \geq 0$ .

**XV:**

$$m_c(Ax, Ay) \lesssim k \max \left\{ \frac{2m_c(Sx, Ax) + m_c(Sy, Ay)}{2}, \frac{2m_c(Sx, Ax) + m_c(Sy, Ax)}{2}, \frac{2m_c(Sx, Ax) + m_c(Sx, Ay)}{2} \right\},$$

where  $k \in (0, 1)$ .

**XVI:**

$$m_c(Ax, Ay) \lesssim \alpha \frac{m_c(Sx, Ax).m_c(Sx, Ay) + m_c(Sy, Ax).m_c(Sy, Ay)}{1 + m_c(Sy, Ax) + m_c(Sx, Ay)},$$

where  $\alpha \in (0, 1)$ .

If pair  $(A, S)$  is pair of non-vacuously weakly compatible mappings such that  $m_c(u, v) = m_c(v, u)$ , wherein  $u$  and  $v$  are points of coincidence of  $(A, S)$ , then  $A$  and  $S$  have a unique common fixed point.

*Proof.* This corollary is immediate in view of Examples 2.2 - 2.17. □

*Remark 4.8.* Inequality (I) of Corollary 4.7 represents the Theorem 1 of Sanjib Datta et al. [7]. While in Corollary 4.7 (II), (III), some similar results to Theorems 2.2 and 2.3 of A.Bhatt et al. [4] are obtained in the context of real valued minimal condition metric spaces.

With similar notation  $m_c$ , setting  $S = T = I$  in Corollary 4.5 (I-XVI), following corollary is obtained in the framework of complex valued metric spaces.

**Corollary 4.9.** Let  $X$  be a non-empty set and  $m_c : X \times X \rightarrow C$  be a complex valued metric on  $X$ . If  $A$  and  $B$  are two self mappings of  $X$  such that any one of the following conditions is satisfied (for all  $x, y \in X$ ):

**I:**

$$m_c(Ax, By) \lesssim am_c(x, y) + b\{m_c(x, Ax) + m_c(y, By)\} + c\{m_c(y, Ax) + m_c(x, By)\},$$

where  $a, b, c \geq 0$ ,  $a + 2c < 1$ .

**II:**

$$m_c(Ax, By) \lesssim am_c(x, y) + b \max\{m_c(x, Ax), m_c(y, Ax), m_c(x, By)\} \\ + c \max\{m_c(x, Ax), m_c(y, By)\},$$

where  $a, b, c \geq 0$ ,  $a + b + c < 1$ .

**III:**

$$m_c(Ax, By) \lesssim k \max\{m_c(x, Ax), m_c(x, y), m_c(y, Ax), m_c(y, By), m_c(x, By)\},$$

where  $k \in (0, 1)$ .

**IV:**

$$m_c(Ax, By) \lesssim am_c(x, Ax) + b.m_c(x, y) + c.m_c(y, Ax) + d.m_c(y, By) + e.m_c(x, By),$$

where  $a, b, c, d, e \geq 0$ ,  $b + c + e < 1$ .

**V:**

$$m_c(Ax, By) \lesssim k \min\{m_c(x, Ax) + m_c(x, y), m_c(x, y) + m_c(y, Ax), \\ m_c(y, By) + m_c(x, By)\},$$

where  $k \in (0, 1)$ .

**VI:**

$$m_c(Ax, By) \lesssim k \min\{m_c(x, y), \max\{m_c(x, Ax), m_c(y, Ax), m_c(x, By)\}, m_c(y, By)\},$$

where  $k \in (0, \infty)$ .

**VII:**

$$m_c(Ax, By) \lesssim k \max\left\{m_c(x, Ax), m_c(x, y), m_c(y, By), \frac{(m_c(y, Ax) + m_c(x, By))}{2}\right\},$$

where  $k \in (0, 1)$ .

**VIII:**

$$m_c(Ax, By) \lesssim \lambda m_c(x, y) + \mu \frac{m_c(x, Ax).m_c(y, By)}{1 + m_c(x, y)} + \gamma \frac{m_c(y, Ax).m_c(x, By)}{1 + m_c(x, y)},$$

where  $\lambda, \mu, \gamma \geq 0$ ,  $\lambda + \mu + \gamma < 1$ .

**IX:**

$$m_c(Ax, By) \begin{cases} \lesssim \lambda.m_c(x, y) + \mu \frac{m_c(x, Ax).m_c(y, By) + m_c(y, Ax).m_c(x, By)}{m_c(x, Ax) + m_c(y, By)} + \\ \gamma \frac{m_c(y, Ax).m_c(x, By) + m_c(x, Ax).m_c(y, By)}{m_c(y, Ax) + m_c(x, By)} \text{ if } D_1 \neq 0, D_2 \neq 0; \\ = 0 \text{ if } D_1 = 0 \text{ or } D_2 = 0, \end{cases}$$

where  $D_1 = m_c(x, Ax) + m_c(y, By)$  and  $D_2 = m_c(y, Ax) + m_c(x, By)$ ,  $\lambda, \mu, \gamma \geq 0$ ,  $\lambda + \mu + \gamma < 1$ .

**X:**

$$m_c(Ax, By) \begin{cases} \lesssim \lambda m_c(x, y) + \mu \frac{m_c(x, Ax).m_c(y, By)}{m_c(x, y) + m_c(y, Ax) + m_c(x, By)} \text{ if } D_1 \neq 0; \\ = 0 \text{ if } D_1 = 0, \end{cases}$$

where  $\lambda, \mu \geq 0$ ,  $\lambda + \mu < 1$  and  $D_1 = m_c(x, y) + m_c(y, Ax) + m_c(x, By)$ .

**XI:**

$$m_c(Ax, By) \lesssim A.m_c(x, y) + B. \frac{m_c(x, Ax).m_c(y, By)}{1 + m_c(x, y)} + C. \frac{m_c(y, Ax).m_c(x, By)}{1 + m_c(x, y)} \\ + D. \frac{m_c(x, Ax).m_c(x, By)}{1 + m_c(x, y)} + E. \frac{m_c(y, Ax).m_c(y, By)}{1 + m_c(x, y)},$$

where  $A, B, C, D, E \geq 0$ ,  $A + B + C + D + E < 1$ .

**XII:**

$$m_c(Ax, By) \lesssim k \max \left\{ m_c(x, y), \frac{m_c(x, Ax) + m_c(y, By)}{2}, \frac{m_c(y, Ax) + m_c(x, By)}{2} \right\},$$

where  $k \in (0, 1)$ .

**XIII:**

$$m_c(Ax, By) \lesssim k \max \{ m_c(x, y), \sqrt{m_c(x, Ax).m_c(y, By)}, \sqrt{m_c(y, Ax).m_c(x, By)} \},$$

where  $k \in (0, 1)$ .

**XIV:**

$$m_c(Ax, By) \lesssim \alpha \max \{ m_c(x, Ax), m_c(x, y), m_c(y, By) \} + (1 - \alpha) \{ a.m_c(y, Ax) + b.m_c(x, By) \},$$

where  $\alpha \in (0, 1)$  and  $a + b < 1$  for  $a, b \geq 0$ .

**XV:**

$$m_c(Ax, By) \lesssim k \max \left\{ \frac{2m_c(x, Ax) + m_c(y, By)}{2}, \frac{2m_c(x, Ax) + m_c(y, Ax)}{2}, \frac{2m_c(x, Ax) + m_c(x, By)}{2} \right\},$$

where  $k \in (0, 1)$ .

**XVI:**

$$m_c(Ax, By) \lesssim \alpha \frac{m_c(x, Ax).m_c(x, By) + m_c(y, Ax).m_c(y, By)}{1 + m_c(y, Ax) + m_c(x, By)},$$

where  $\alpha \in (0, 1)$ .

Then  $A$  and  $B$  have a unique common fixed point.

*Proof.* This corollary follows immediately in view of Examples 2.2 - 2.17. □

*Remark 4.10.* Some contraction conditions (e.g. VIII, IX, X, XI) in the above corollary are well-known and generalize certain relevant results from [1, 3, 12, 18]. In fact

1. Inequality VIII represents the Theorem 2.1 and inequality IX represents the Theorem 2.11 of Rouzkard and Imdad[18]. Also if we set  $\gamma = 0$  in inequality VIII, we obtain Theorem 4 of Azam et al.[3].
2. Inequality X represents the Theorem 3.1 of Nashine et al.[12].
3. Inequality XI is the contraction condition in Theorem 6 of Jamshed Ahmed et al.[1].

While other inequalities in Corollary 4.9 are new in context of complex valued metric spaces.

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