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Existence of positive solution for a fractional order nonlinear differential system involving a changing sign perturbation

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Abstract

In this paper, we study a class of singular fractional order differential system with a changing-sign perturbation which arises from fluid dynamics, biological models, electrical networks with uncertain physical parameters and parametrical variations in time. Under suitable growth condition, the singular changing-sign system is transformed to an approximately singular fractional order differential system with positive nonlinear term, then the existence of positive solution is established by using the known fixed point theorem.

Keywords: Singular phenomena, changing-sign perturbation, positive solution, Riemann-Stieltjes integral conditions.
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1. Introduction

The study of many problems in fluid dynamics, biological models, electrical networks, chemical kinetics often leads to mathematical models in the form of nonlinear fractional order differential equations. Generally,

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some perturbations and uncertainties usually exist in these real world differential models due to some uncertain physical parameters and parametrical variations in time. These perturbations and uncertainties can be introduced in the underlying mathematical model [3, 5, 7, 9, 10, 15, 16, 18].

In this paper, we consider a class of singular fractional order differential system with a changing-sign perturbation

$$\begin{cases} -D_{0+}^{\alpha}u(t)=f(t,v(t))+q(t), \quad -D_{0+}^{\beta}v(t)=g(t,u(t)),\ 0<t<1, \\ u(0)=0,\ u(1)=\int_0^1 u(s)dA(s),\ v(0)=0,\ v(1)=\int_0^1 v(s)dB(s), \end{cases} \tag{1.1}$$

where $1<\alpha,\beta\leq 2$, D_{0+}^{α} and D_{0+}^{β} are the standard Riemann-Liouville derivatives, A and B are functions of bounded variation, $\int_0^1 u(s)dA(s)$ and $\int_0^1 v(s)dB(s)$ denote the Riemann-Stieltjes integrals of u,v with respect to A and B, $f,g\in C((0,1)\times[0,+\infty),[0,+\infty))$, f and g may be singular at $t=0$ and $t=1$, $q:(0,1)\to(-\infty,+\infty)$ is Lebesgue integrable and q can have infinitely many singularities in [0, 1].

In recent years, a handful of papers have appeared to study differential equation with some changing-sign perturbation, but most of them treated with the changing-sign perturbation problems with the form $f(t,u)+M\geq 0$ for some $M>0$ [1, 11], especially, containing fractional derivatives, for detail, see [8, 14, 17, 19, 20, 21, 22, 23, 24]. In this paper, we handle the singular fractional order differential system with the form $f(t,u)+q(t)\to-\infty$ at some singular point. By finding the unique solution of the linear nonlocal boundary value problem, we transform the changing-sign singular fractional order differential system to an approximately positive differential system, and then the existence of positive solution is established by using the known fixed point theorem. Here we also point out the boundary condition of system (1.1) is nonlocal which involves the Riemann-Stieltjes integral, i.e. it can cover the multi-point BCs and also integral BCs in a single framework.

2. Basic Definitions and Preliminaries

In this paper, we restrict our attention to the Riemann-Liouville fractional derivatives and some properties of the Riemann-Liouville fractional integral and derivative operators, see[6, 12, 13].

Lemma 2.1 ([2]). *Given $h(t)\in L^1(0,1)$, the problem*

$$\begin{cases} -D_{0+}^{\alpha}u(t)=h(t), \quad 0<t<1, \\ u(0)=0, \qquad u(1)=0, \end{cases} \tag{2.1}$$

has the unique solution

$$u(t)=\int_0^1 G_A(t,s)h(s)ds, \tag{2.2}$$

where $G_A(t,s)$ is the Green function of (2.1) and is given by

$$G_A(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} [t(1-s)]^{\alpha-1}, & 0\leq t\leq s\leq 1, \\ [t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}, & 0\leq s\leq t\leq 1. \end{cases} \tag{2.3}$$

Lemma 2.2 ([14]). *If $h(t)\in L^1(0,1)$, the Green function of the following problem*

$$\begin{cases} -D_{0+}^{\alpha}u(t)=h(t),\ 0<t<1, \\ u(0)=0, u(1)=\int_0^1 u(s)dA(s), \end{cases} \tag{2.4}$$

is given by

$$H_A(t,s)=\frac{t^{\alpha-1}}{1-\mathcal{A}}\mathcal{G}_A(s)+G_A(t,s), \qquad \mathcal{A}=\int_0^1 t^{\alpha-1}dA(t), \tag{2.5}$$

where

$$\mathcal{G}_A(s) = \int_0^1 G_A(t,s)dA(t).$$

Lemma 2.3 ([14])**.** *Let $0 \le \mathcal{A} < 1$, $\mathcal{G}_A \ge 0$ for $s \in [0,1]$, then the Green function defined by* (2.5) *satisfies:*

(1) $H_A(t,s) > 0$ *for all* $(t,s) \in (0,1)\times(0,1)$,

(2) *There exist two constants c_A, d_A such that*

$$c_A t^{\alpha-1}\mathcal{G}_A(s) \le H_A(t,s) \le d_A t^{\alpha-1} \le d_A, \quad t,s \in [0,1]. \tag{2.6}$$

Here we also address that for the equation

$$\begin{cases} -D_{0+}^{\beta} v(t) = h(t), \ 0 < t < 1, \\ v(0) = 0, \ v(1) = \displaystyle\int_0^1 v(s)dB(s), \end{cases} \tag{2.7}$$

we can get results similar to (2.3) and (2.5) as well as (2.6), and the results will be denoted by subscript B.

For clarity in presentation, we also list below some assumptions to be used later in the paper.

(H1) A, B are functions of bounded variation such that $\mathcal{G}_A(s) \ge 0$, $\mathcal{G}_B(s) \ge 0$ for $s \in [0,1]$ and $0 \le \mathcal{A}, \mathcal{B} < 1$;

(H2) $f, g : (0,1)\times[0,+\infty) \to [0,+\infty)$ are continuous, and are nondecreasing on second variable, and there exist two continuous functions $p : (0,1) \to [0,+\infty)$ and $h : [0,+\infty) \to [0,+\infty)$ such that

$$f(t,u) \le p(t)h(u).$$

(H3) There exists a constant $\lambda > 0$ such that for any $(t,v) \in (0,1)\times[0,+\infty)$

$$g(t,cv) \ge c^{\lambda} g(t,v), \forall\ 0 < c \le 1,$$

with $0 < \int_0^1 g(t,1)dt < +\infty$.

(H4) $q : (0,1) \to (-\infty,+\infty)$ is Lebesgue integrable such that $0 < \int_0^1 q_-(t)dt < c_A^{-1}d_A^2$ and

$$\int_0^1 [p(s) + q_+(s)]\, ds \le d_A^{-1}\left(\max_{0\le\tau\le M} h(\tau) + 1\right)^{-1},$$

where

$$q_+(t) = \max\{q(t),0\},\ q_-(t) = \min\{q(t),0\},\ M = d_B\int_0^1 g(\tau,1)d\tau.$$

(H5)

$$\lim_{u\to+\infty}\frac{f(t,u)}{u} = +\infty, \quad \lim_{u\to+\infty}\frac{g(t,u)}{u} = +\infty$$

for t uniformly on any close subinterval of (0, 1).

Remark 2.4. Let $\Omega \subset (0,1)$ be a zero measure set and $q : (0,1)\setminus\Omega \to (-\infty,+\infty)$ be continuous and integrable, then q can have infinitely many singularities.

Remark 2.5. If q satisfies **(H4)** and $\lim_{t\to t_0} q(t) = -\infty$ for some point $t_0 \in (0,1)$, then the system can be changing-sign.

Remark 2.6. For any $(t,v) \in (0,1)\times[0,+\infty)$ and $\widetilde{c} \ge 1$, by **(H3)**, it is easy to get

$$g(t,\widetilde{c}v) \le \widetilde{c}^{\lambda} g(t,v).$$

Throughout this paper, we will work in the space $E = C[0,1]$, which is a Banach space if it is endowed with the norm $\| u \| = \max_{t\in[0,1]} | u(t) |$ for any $u \in E$. Let

$$P = \left\{ u \in C[0,1] : u(t) \geq \frac{c_A}{d_A} t^{\alpha-1} ||u||, t \in [0,1] \right\}.$$

Obviously P is a normal cone in the Banach space E. For $u \in E$, let us define a function $[\cdot]^*$ by

$$[u(t)]^* = \begin{cases} u(t), & u(t) \geq 0, \\ 0, & u(t) < 0. \end{cases} \tag{2.8}$$

By Lemma 2.3, the following equation

$$\begin{cases} -D_{0+}^{\alpha} u(t) = q_-(t), \ 0 < t < 1, \\ u(0) = 0, u(1) = \displaystyle\int_0^1 u(s) dA(s), \end{cases} \tag{2.9}$$

has unique solution which satisfies

$$w(t) = \int_0^1 H_A(t,s) q_-(s) ds \leq d_A \int_0^1 q_-(s) ds. \tag{2.10}$$

In what follows, we consider the following approximately singular fractional order differential system

$$\begin{cases} -D_{0+}^{\alpha} u(t) = f(t, v(t)) + q_+(t), \quad -D_{0+}^{\beta} v(t) = g(t, [u(t) - w(t)]^*), \ 0 < t < 1, \\ u(0) = 0, \ u(1) = \displaystyle\int_0^1 u(s) dA(s), \ v(0) = 0, \ v(1) = \displaystyle\int_0^1 v(s) dB(s). \end{cases} \tag{2.11}$$

Clearly, by Lemma 2.3, we know that the system (2.11) is equivalent to the following integral system

$$\begin{cases} u(t) = \displaystyle\int_0^1 H_A(t,s)[f(s, v(s)) + q_+(s)] ds, \\ v(t) = \displaystyle\int_0^1 H_B(t,s) g(s, [u(s) - w(s)]^*) ds. \end{cases} \tag{2.12}$$

Obviously, the above nonlinear integral system (2.12) can be transformed to the following equivalent nonlinear integral equation

$$u(t) = \int_0^1 H_A(t,s) \left[f\left(s, \int_0^1 H_B(s,\tau) g(\tau, [u(\tau) - w(\tau)]^*) d\tau \right) + q_+(s) \right] ds. \tag{2.13}$$

Now let us define a nonlinear integral operator T: $E \to E$ by

$$(Tu)(t) = \int_0^1 H_A(t,s) \left[f\left(s, \int_0^1 H_B(s,\tau) g(\tau, [u(\tau) - w(\tau)]^*) d\tau \right) + q_+(s) \right] ds, t \in [0,1], \tag{2.14}$$

we have that if $u(t)$ is a fixed point of T in E, then system (2.11) has one solution (u, v) with

$$\begin{cases} u = u(t), \\ v = \displaystyle\int_0^1 H_B(t,s) g(s, [u(s) - w(s)]^*) ds. \end{cases}$$

Lemma 2.7. *If (u, v) with $u(t) \geq w(t)$ for any $t \in [0,1]$ is a positive solution of approximately singular fractional order differential system (2.11), then $(u - w, v)$ is a positive solution of singular fractional order differential system with a negative perturbation (1.1), where*

$$v(t) = \int_0^1 H_B(t,s)g(s, u(s) - w(s))ds.$$

Proof. In fact, if (u, v) is a positive solution system (2.11) and satisfies $u(t) \geq w(t)$ for any $t \in [0, 1]$, then, by (2.11) and the definition of $[u(t)]^*$, we have

$$\begin{cases} -D_{0+}^{\alpha}u(t) = f(t, v(t)) + q_+(t), \quad -D_{0+}^{\beta}v(t) = g(t, u(t) - w(t)), \\ u(0) = 0, \ u(1) = \int_0^1 u(s)dA(s), \ v(0) = 0, \ v(1) = \int_0^1 v(s)dB(s). \end{cases} \tag{2.15}$$

Let $u_1 = u - w$, then we have

$$\begin{aligned} &D_{0+}^{\alpha}u_1(t) = D_{0+}^{\alpha}u(t) - D_{0+}^{\alpha}w(t) = D_{0+}^{\alpha}u(t) + q_-(t), \\ &u_1(0) = 0, \ u_1(1) = \int_0^1 u_1(s)dA(s), \end{aligned}$$

which implies that

$$\begin{cases} -D_{0+}^{\alpha}u_1(t) + q_-(t) = f(t, v(t)) + q_+(t), \quad -D_{0+}^{\beta}v(t) = g(t, u_1(t)), \\ u_1(0) = 0, \ u_1(1) = \int_0^1 u_1(s)dA(s), \ v(0) = 0, \ v(1) = \int_0^1 v(s)dB(s). \end{cases} \tag{2.16}$$

Notice that $q(t) = q_+(t) - q_-(t)$, we have $(u_1, v) = (u - w, v)$ is a positive solution of singular fractional order differential system with a negative perturbation (1.1). □

Lemma 2.8. *$T : P \to P$ is a completely continuous operator.*

Proof. For any fixed $u \in P$, we can find a constant $L_1 > 0$ such that $||u|| \leq L_1$. Take $\hat{L} = \max\{L_1, 1\}$, then

$$0 \leq [u(s) - w(s)]^* \leq u(s) \leq ||u|| \leq L_1 \leq \hat{L}. \tag{2.17}$$

It follows from (2.17) and **(H2)-(H3)** that

$$g(\tau, [u(\tau) - w(\tau)]^*) \leq g(\tau, \hat{L}) \leq \hat{L}^{\lambda} g(\tau, 1).$$

And then

$$\int_0^1 H_B(s,\tau)g(\tau, [u(\tau) - w(\tau)]^*)d\tau \leq d_B \int_0^1 \hat{L}^{\lambda} g(\tau, 1)d\tau = d_B \hat{L}^{\lambda} \int_0^1 g(\tau, 1)d\tau := \hat{M}.$$

Consequently, for any $t \in [0, 1]$, we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 H_A(t,s) \left[f\left(s, \int_0^1 H_B(s,\tau)g(\tau, [u(\tau) - w(\tau)]^*)d\tau\right) + q_+(s)\right] ds \\ &\leq d_A \int_0^1 \left[p(s)h\left(\int_0^1 H_B(s,\tau)g(\tau, [u(\tau) - w(\tau)]^*)d\tau\right) + q_+(s)\right] ds \\ &\leq d_A \int_0^1 \left[p(s)h\left(d_B \hat{L}^{\lambda} \int_0^1 g(\tau, 1)d\tau\right) + q_+(s)\right] ds \\ &\leq d_A \left(\max_{0 \leq \tau \leq \hat{M}} h(\tau) + 1\right) \int_0^1 [p(s) + q_+(s)]\, ds < +\infty, \end{aligned}$$

which implies that the operator $T : P \to E$ is bounded.

Next for any $u \in P$, by Lemma 2.3, we have

$$\begin{aligned} ||Tu|| &= \max_{0\le t\le 1}\left\{\int_0^1 H_A(t,s)\left[f\left(s,\int_0^1 H_B(s,\tau)g(\tau,[u(\tau)-w(\tau)]^*)d\tau\right)+q_+(s)\right]ds\right\} \\ &\le d_A\int_0^1\left[f\left(s,\int_0^1 H_B(s,\tau)g(\tau,[u(\tau)-w(\tau)]^*)d\tau\right)+q_+(s)\right]ds. \end{aligned} \tag{2.18}$$

On the other hand, by Lemma 2.3, we also have

$$(Tu)(t) \ge c_A t^{\alpha-1}\int_0^1\left[f\left(s,\int_0^1 H_B(s,\tau)g(\tau,[u(\tau)-w(\tau)]^*)d\tau\right)+q_+(s)\right]ds. \tag{2.19}$$

So, (2.18) and (2.19) yield

$$(Tu)(t) \ge \frac{c_A}{d_A}||Tu||t^{\alpha-1}, t\in[0,1],$$

which yields that $T(P) \subset P$.

At the end, using standard arguments, according to the Ascoli-Arzela Theorem, one can show that $T : P \to P$ is a completely continuous operator. □

3. Main results

Theorem 3.1. *Suppose **(H1)** – **(H5)** hold. Then the singular fractional order differential system with a changing-sign perturbation* (1.1) *has at least one positive solution.*

Proof. Take $\Omega_1 = \{u \in P : ||u|| \le 1\}$, then for any $u \in \partial\Omega_1$, we have

$$0 \le [u(s)-w(s)]^* \le u(s) \le ||u|| \le 1. \tag{3.1}$$

Thus for any $u \in \partial\Omega_1$, it follows from (3.1) and **(H2)-(H3)** that

$$\begin{aligned} ||Tu|| &= \max_{0\le t\le 1}\left\{\int_0^1 H_A(t,s)\left[f\left(s,\int_0^1 H_B(s,\tau)g(\tau,[u(\tau)-w(\tau)]^*)d\tau\right)+q_+(s)\right]ds\right\} \\ &\le d_A\int_0^1\left[p(s)h\left(\int_0^1 H_B(s,\tau)g(\tau,[u(\tau)-w(\tau)]^*)d\tau\right)+q_+(s)\right]ds \\ &\le d_A\int_0^1\left[p(s)h\left(d_B\int_0^1 g(\tau,1)d\tau\right)+q_+(s)\right]ds \\ &\le d_A\left(\max_{0\le\tau\le M}h(\tau)+1\right)\int_0^1[p(s)+q_+(s)]\,ds \\ &\le 1 = ||u||, \end{aligned}$$

where $M = d_B\int_0^1 g(\tau,1)d\tau$. Therefore,

$$||Tu|| \le ||u||, \quad u\in\partial\Omega_1. \tag{3.2}$$

On the other hand, choose constants a, b and L such that

$$[a,b]\subset(0,1),\ \frac{c_A}{2d_A}a^{\alpha-1}L\int_a^b\frac{\mathcal{G}_A(s)}{1-\mathcal{A}}ds > 1.$$

By **(H5)**, there exists a constant enough large

$$K^* > \max\left\{2c_A\int_0^1 q_-(s)ds, 1\right\}$$

such that

$$f(t,x) \geq Lx, \ t \in [a,b], \ x \geq K^*. \tag{3.3}$$

Furthermore, from **(H5)**, there exists a constant $K > K^*$ such that

$$g(t,y) \geq \left(a^{\alpha-1} c_A \int_a^b \mathcal{G}_A(\tau)d\tau\right)^{-1} y, \ t \in [a,b], \ y \geq K. \tag{3.4}$$

Now take $R \geq \max\left\{\frac{2Kd_A}{c_A a^{\alpha-1}}, K\right\}$. Obviously,

$$R \geq K > K^* > \max\left\{2c_A \int_0^1 q_-(s)ds, 1\right\}.$$

Let $\Omega_2 = \{u \in P : ||u|| \leq R\}$. Then for any $u \in \partial\Omega_2$ and for any $t \in [a,b]$, we have

$$\begin{aligned} u(t) - w(t) &\geq u(t) - d_A t^{\alpha-1} \int_0^1 q_-(s)ds \geq u(t) - c_A \int_0^1 q_-(s)ds \frac{u(t)}{||u||} \\ &= u(t) - c_A \int_0^1 q_-(s)ds \frac{u(t)}{R} \geq \frac{1}{2}u(t) \\ &\geq \frac{c_A}{2d_A} t^{\alpha-1} R \geq \frac{c_A}{2d_A} a^{\alpha-1} R \geq K > 0. \end{aligned} \tag{3.5}$$

And then, it follows from (3.5) that, for any $s \in [a,b]$,

$$\begin{aligned} \int_a^b H_B(s,\tau) g(\tau, [u(\tau) - w(\tau)]^*) d\tau &= \int_a^b H_B(s,\tau) g(\tau, u(\tau) - w(\tau)) d\tau \\ &\geq \left(a^{\alpha-1} c_A \int_a^b \mathcal{G}_A(\tau)d\tau\right)^{-1} \int_a^b c_A s^{\alpha-1} \mathcal{G}_A(\tau)(u(\tau) - w(\tau))d\tau \\ &\geq \frac{c_A}{2d_A} a^{\alpha-1} R \geq K > K^* > 0. \end{aligned} \tag{3.6}$$

So for any $u \in \partial\Omega_2$ and $t \in [0,1]$, by (3.4) and (3.6), we have

$$\begin{aligned} ||Tu|| &\geq \int_0^1 H_A(1,s) \left[f\left(s, \int_0^1 H_B(s,\tau) g(\tau, [u(\tau) - w(\tau)]^*) d\tau\right) + q_+(s)\right] ds \\ &\geq \int_0^1 \frac{\mathcal{G}_A(s)}{1 - \mathcal{A}} f\left(s, \int_0^1 H_B(s,\tau) g(\tau, [u(\tau) - w(\tau)]^*) d\tau\right) ds \\ &\geq \int_a^b \frac{\mathcal{G}_A(s)}{1 - \mathcal{A}} f\left(s, \int_a^b H_B(s,\tau) g(\tau, [u(\tau) - w(\tau)]^*) d\tau\right) ds \\ &\geq L \int_a^b \frac{\mathcal{G}_A(s)}{1 - \mathcal{A}} \int_a^b H_B(s,\tau) g(\tau, [u(\tau) - w(\tau)]^*) d\tau ds \\ &\geq \frac{c_A}{2d_A} a^{\alpha-1} RL \int_a^b \frac{\mathcal{G}_A(s)}{1 - \mathcal{A}} ds \geq R = ||u||. \end{aligned}$$

Thus, we have $||Tu|| \geq ||u||, u \in \partial\Omega_2$. Thus T has a fixed point u_0 such that $1 \leq ||u_0|| \leq R$ from [4].

In what follows, we prove $u_0(t) \geq w(t)$ for any $t \in [0,1]$. In fact, for any $t \in [0,1]$, by **(H4)**, we have

$$\begin{aligned} u_0(t) - w(t) &\geq u_0(t) - d_A t^{\alpha-1} \int_0^1 q_-(s)ds \geq \frac{c_A}{d_A} t^{\alpha-1} - d_A t^{\alpha-1} \int_0^1 q_-(s)ds \\ &\geq \left(\frac{c_A}{d_A} - d_A \int_0^1 q_-(s)ds\right) t^{\alpha-1} \geq 0. \end{aligned} \tag{3.7}$$

By Lemma 2.7 and (3.7), the singular fractional order differential system with a changing-sign perturbation (1.1) has at least one positive solution. The proof of Theorem 3.1 is completed. □

4. An example

To demonstrate the application of our results, we give a simple example. Firstly, we take $\alpha = \frac{3}{2},\ \beta = \frac{4}{3}$, and

$$A(t) = \begin{cases} 0, & t \in \left[0, \dfrac{1}{3}\right), \\ \dfrac{1}{2}, & t \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right), \\ 1, & t \in \left[\dfrac{2}{3}, 1\right], \end{cases} \qquad B(t) = \begin{cases} 0, & t \in \left[0, \dfrac{1}{2}\right), \\ 2, & t \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right), \\ 1, & t \in \left[\dfrac{3}{4}, 1\right]. \end{cases}$$

By Lemma 2.3, there exist some positive constants $c_A,\ d_A, c_B,\ d_B$ such that

$$c_A t^{\frac{1}{2}} \mathcal{G}_A(s) \le H_A(t,s) \le d_A t^{\frac{1}{2}} \le d_A, \quad t, s \in [0,1], \tag{4.1}$$

and

$$c_B t^{\frac{1}{3}} \mathcal{G}_B(s) \le H_B(t,s) \le d_B t^{\frac{1}{3}} \le d_B, \quad t, s \in [0,1]. \tag{4.2}$$

Consider the following singular fractional order differential system with a changing-sign perturbation

$$\begin{cases} -D_{0+}^{\frac{3}{2}} u(t) = d_A^{-1} \left(\dfrac{4}{9} d_B^2 + 1\right)^{-1} t v^2 - \dfrac{c_A^{-1} d_A^2}{4\sqrt{t}},\ -D_{0+}^{\frac{4}{3}} v(t) = \dfrac{u^{\frac{3}{2}}}{2\sqrt[4]{t}}, \\ u(0) = 0, u(1) = \displaystyle\int_0^1 u(s) dA(s),\ v(0) = 0, v(1) = \displaystyle\int_0^1 v(s) dB(s). \end{cases} \tag{4.3}$$

Analysis. Let

$$f(t,x) = d_A^{-1} \left(\frac{4}{9} d_B^2 + 1\right)^{-1} t x^2,\ g(t,y) = \frac{y^{\frac{3}{2}}}{2\sqrt[4]{t}},$$
$$p(t) = d_A^{-1} \left(\frac{4}{9} d_B^2 + 1\right)^{-1} t,\ h(x) = x^2,\ q(t) = -\frac{c_A^{-1} d_A^2}{4\sqrt{t}},$$

then we have

$$f(t,x) \le p(t) h(x), \quad \int_0^1 g(t,1) dt = \int_0^1 \frac{1}{2\sqrt[4]{t}} dt = \frac{2}{3}.$$

Clearly, $f, g : (0,1) \times [0,+\infty) \to [0,+\infty)$ are continuous, and are nondecreasing on second variable, and for any $0 < c \le 1$,

$$g(t, cy) = \frac{c^{\frac{3}{2}} y^{\frac{3}{2}}}{2\sqrt[4]{t}} \ge \frac{c^2 y^{\frac{3}{2}}}{2\sqrt[4]{t}} = c^2 g(t,y).$$

Thus **(H2)** and **(H3)** hold.

On the other hand, clearly,

$$q_-(t) = \frac{c_A^{-1} d_A^2}{4\sqrt{t}},\ q_+(t) = 0,\ M = d_B \int_0^1 g(\tau, 1) d\tau = \frac{2}{3} d_B,$$

so

$$d_A^{-1} \left(\max_{0 \le \tau \le M} h(\tau) + 1\right)^{-1} = d_A^{-1} \left(\frac{4}{9} d_B^2 + 1\right)^{-1}.$$

Thus, we have

$$\int_0^1 q_-(t) dt = \int_0^1 \frac{c_A^{-1} d_A^2}{4\sqrt{t}} dt = \frac{1}{2} c_A^{-1} d_A^2 < c_A^{-1} d_A^2,$$
$$\int_0^1 (p(s) + q_+(s)) ds = \int_0^1 d_A^{-1} \left(\frac{4}{9} d_B^2 + 1\right)^{-1} s ds = \frac{1}{2} d_A^{-1} \left(\frac{4}{9} d_B^2 + 1\right)^{-1} < d_A^{-1} \left(\max_{0 \le \tau \le M} h(\tau) + 1\right)^{-1}.$$

So **(H4)** hold. Moreover, **(H5)** is also easy to be verified.

In addition, we have

$$0 \le \mathcal{A} = \int_0^1 t^{\frac{1}{2}} dA(t) = \frac{1}{2}\left(\frac{2}{3}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{2}} \approx 0.6969 < 1,$$

$$0 \le \mathcal{B} = \int_0^1 t^{\frac{1}{3}} dB(t) = 2\left(\frac{1}{2}\right)^{\frac{1}{3}} - \left(\frac{3}{4}\right)^{\frac{1}{3}} \approx 0.6788 < 1.$$

Clearly, $\mathcal{G}_A(s), \mathcal{G}_B(s) \ge 0$ for $s \in [0,1]$ also hold, which implies that **(H1)** is satisfied.

Hence all conditions of Theorem 3.1 are satisfied, and consequently from Theorem 3.1, the singular fractional order differential system with a changing-sign perturbation (4.3) has at least one positive solution.

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