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A coincident point principle for two weakly compatible mappings in partial S-metric spaces

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Abstract

We show the existence of common fixed point and a coincident point for two weakly compatible self-mappings defined on a complete partial S-metric space X, where the contraction in the assumption of the main result has three control functions, α, ψ, ϕ.

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1. Introduction

There are many results on the existence of a fixed point for self mappings on various metric spaces. For example, see [1, 2, 3, 4, 5, 6, 8, 11, 13, 15, 17, 18, 19, 20, 21]. However, many researchers prove the existence and uniqueness of a coincident point and common fixed point for two self-mappings on different types of metric spaces. In particular, the S-metric space which was introduced by Sedghi in [16]. The S-metric space is a space with three dimensions. In our paper, we work in partial S-metric space which was introduced in [12] as a generalization of S-metric spaces. Also, most of these results, under different contraction principles, use control functions.

Definition 1.1 ([12])**.** Let X be a nonempty set. A partial S-metric space on X is a function $S_p : X^3 \to [0, \infty)$ that satisfies the following conditions, for all $x, y, z, t \in X$:

(i) $S_p(x, y, z) \geq 0$,

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(ii) $x = y$ if and only if $S_p(x,x,x) = S_p(y,y,y) = S_p(x,x,y)$,

(iii) $S_p(x,y,z) \leq S_p(x,x,t) + S_p(y,y,t) + S_p(z,z,t) - S_p(t,t,t)$,

(iv) $S_p(x,x,x) \leq S_p(x,y,z)$,

(v) $S_p(x,x,y) = S_p(y,y,x)$.

The pair (X, S_p) is called a partial S-metric space.

Next, we recall some basic definitions for the convenience of readers.

Definition 1.2 ([12])**.** A sequence $\{x_n\}_{n=0}^{\infty}$ of elements in X is called p-Cauchy if $\lim_{n,m} S_p(x_n, x_n, x_m)$ exists and is finite. A partial S-metric space (X, S_p) is called complete if for each p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ there exists a $z \in X$ such that

$$S_p(z,z,z) = \lim_n S_p(z,z,x_n) = \lim_{n,m} S_p(x_n,x_n,x_m).$$

A sequence $\{x_n\}_n$ in a partial S-metric space (X, S_p) is called 0-Cauchy if

$$\lim_{n,m} S_p(x_n,x_n,x_m) = 0.$$

We say that (X, S_p) is 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $S_p(x,x,x) = 0$.

Definition 1.3 ([7])**.** A function $\psi : [0,\infty)^2 \to [0,\infty)$ is said to be a generalized altering distance function of two variables if:

1. ψ is continuous,
2. ψ is monotone increasing in both variables,
3. $\psi(x,y) = 0$ only if $x = y = 0$.

The class of all such functions is denoted by Ω. We define $\alpha(x) = \psi(x,x)$ for $x \in [0,\infty)$. Clearly, $\alpha(x) = 0$ if and only if $x = 0$.

Definition 1.4 ([14])**.** Let X be a nonempty set, n a positive integer and $F : X \to X$ a mapping. $X = \bigcup_{i=1}^{n} A_i$ is said to be a cyclic representation of X with respect to F if:

1. $A_i, i = 1,2,\ldots,n$ are nonempty sets,
2. $F(A_1) \subset A_2, F(A_2) \subset A_3, \ldots, F(A_{n-1}) \subset A_n, F(A_n) \subset A_1$.

Definition 1.5 ([10])**.** Let X be a nonempty set, n a positive integer and $f, g : X \to X$ two mappings. $X = \bigcup_{i=1}^{n} A_i$ is said to be a cyclic representation of X with respect to f and g if:

1. $A_i, i = 1,2,\ldots,n$ are nonempty sets,
2. $g(A_1) \subset f(A_2), g(A_2) \subset f(A_3), \ldots, g(A_{n-1}) \subset f(A_n), g(A_n) \subset f(A_1)$.

Definition 1.6 ([9])**.** Let f and g be two self-maps on X. If $fw = gw = z$, for some $w \in X$, then w is called a coincidence point of f and g, and z is called a point of coincidence of f and g. If $w = z$, then z is called a common fixed point of f and g.

Definition 1.7 ([9])**.** Consider two self-maps f and g defined on a nonempty set X. If $fgx = gfx$, for all $x \in X$, then f and g are said to be commuting maps. If they commute only at their coincidence points, then they are said to be weakly compatible, that is, if $fgx = gfx$, whenever, $fx = gx$.

2. Main result

In this section, we prove our main result with very useful corollary.

Theorem 2.1. *Let (X, S_p) be a 0-complete partial S-metric space and $A_1, \ldots, A_{n_0}$ a nonempty subset of X, where $X = \bigcup_{i=1}^{n_0} A_i$. Let $f, g : X \to X$ be two self-mappings such that $X = \bigcup_{i=1}^{n_0} A_i$ is a cyclic representation of X with respect to f and g. For any $x \in A_i$ and $y \in A_{i+1}$ we have*

$$\alpha(S_p(gx, gx, gy)) \leq \psi(S_p(fx, fx, fy), S_p(fx, fx, gx)) - \phi(S_p(fx, fx, fy), S_p(fx, fx, gx)),$$

where $A_{n+1} = A_1$, $\psi, \phi \in \Omega$ and $\alpha(x) = \psi(x, x)$ for $x \in [0, \infty)$. Suppose that $f(A_i)$, for all i are closed subsets of X. If f_n is one-to-one, then there exists a $z \in \bigcap_{i=1}^{n_0} A_i$ such that $fz = gz$. In particular, if f and g are weakly compatible, then they have a unique common fixed point.

Proof. Let $x_1 \in A_1$; then by the cyclic representation of X, we can find an element $x_2 \in A_2$ such that $gx_1 = fx_2$. Also, for x_2 we can find an $x_3 \in A_3$ such that $gx_2 = fx_3$. Continuing this process, we construct the sequence $\{x_n\}$ defined by $gx_n = fx_{n+1}$ for all natural numbers n.

First, assume there exists a natural number k such that $fx_k = fx_{k+1}$, hence $fx_{n+1} = gx_n$. It follows that x_k is a coincidence point of f and g.

Now, suppose that $fx_{n+1} \neq fx_n$ for all n. Then by the definition of X, there exists an $i_m \in \{1, 2, \ldots, n\}$ such that $x_n \in A_{i_m+1}$ and $x_{n-1} \in A_{i_m}$. Thus, we have

$$\begin{aligned} \alpha(S_p(gx_n, gx_n, gx_{n-1})) &\leq \psi(S_p(fx_n, fx_n, fx_{n-1}), S_p(fx_n, fx_n, gx_n)) \\ &\quad - \phi(S_p(fx_n, fx_n, fx_{n-1}), S_p(fx_n, fx_n, gx_n)), \end{aligned}$$

$$\begin{aligned} \alpha(S_p(fx_{n+1}, fx_{n+1}, fx_n)) &\leq \psi(S_p(fx_n, fx_n, fx_{n-1}), S_p(fx_n, fx_n, fx_{n+1})) \\ &\quad - \phi(S_p(fx_n, fx_n, fx_{n-1}), S_p(fx_n, fx_n, fx_{n+1})) \\ &\leq \psi(S_p(fx_n, fx_n, fx_{n-1}), S_p(fx_n, fx_n, fx_{n+1})). \end{aligned} \tag{2.1}$$

Assuming $\alpha(x) = \psi(x, x)$, we deduce that

$$\psi(S_p(fx_{n+1}, fx_{n+1}, fx_n), S_p(fx_{n+1}, fx_{n+1}, fx_n)) \leq \psi(S_p(fx_n, fx_n, fx_{n-1}), S_p(fx_{n+1}, fx_{n+1}, fx_n)).$$

If $S_p(fx_n, fx_n, fx_{n-1}) < S_p(fx_{n+1}, fx_{n+1}, fx_n)$, then

$$\begin{aligned} \alpha(S_p(fx_{n+1}, fx_{n+1}, fx_n)) &\leq \psi(S_p(fx_n, fx_n, fx_{n-1}), S_p(fx_{n+1}, fx_{n+1}, fx_n)) \\ &< \psi(S_p(fx_{n+1}, fx_{n+1}, fx_n), S_p(fx_{n+1}, fx_{n+1}, fx_n)) \\ &= \alpha(S_p(fx_{n+1}, fx_{n+1}, fx_n)), \end{aligned}$$

which leads to a contradiction, because we know that α is monotone increasing and $S_p(fx_{n+1}, fx_{n+1}, fx_n) \neq 0$ and hence $\alpha(S_p(fx_{n+1}, fx_{n+1}, fx_n)) \neq 0$. Therefore, $S_p(fx_{n+1}, fx_{n+1}, fx_n) \leq S_p(fx_n, fx_n, fx_{n-1})$. Thus $\{S_p(fx_{n+1}, fx_{n+1}, fx_n)\}_{n \geq 1}$ is a decreasing sequence of nonnegative real numbers, so there exists an $r \geq 0$, such that

$$S_p(fx_{n+1}, fx_{n+1}, fx_n) \to r \text{ as } n \to \infty.$$

Taking the limit as $n \to \infty$ in inequality (2.1), we obtain:

$$\alpha(r) \leq \psi(r, r) - \phi(r, r) < \psi(r, r) = \alpha(r).$$

Hence, $\alpha(r) = 0$ which implies that $r = 0$. Therefore, $\lim_{n\to\infty} S_p(fx_n, fx_n, fx_{n+1}) = 0$.

To show that $\{fx_n\}_{n\geq 1}$ is a 0-Cauchy sequence, assume that $\{fx_n\}_{n\geq 1}$ is not. Hence, there would exists an $\varepsilon > 0$ for which we can find subsequences $\{fx_{n_k}\}$ and $\{fx_{m_k}\}$ of $\{fx_n\}$ with $n_k > m_k > k$ such that

$$S_p(fx_{m_k}, fx_{m_k}, fx_{n_k}) \geq \varepsilon.$$

Choose n_k and m_k to be the smallest integers satisfying the above inequality. Thus,

$$S_p(fx_{m_k}, fx_{m_k}, fx_{n_k}) \geq \varepsilon.$$

Notice that

$$\begin{aligned} S_p(fx_{m_k}, fx_{m_k}, fx_{n_k}) &\leq S_p(fx_{m_k}, fx_{m_k}, fx_{n_k-1}) + 2S_p(fx_{n_k}, fx_{n_k}, fx_{n_k-1}) \\ &< \varepsilon + 2S_p(fx_{n_k-1}, fx_{n_k-1}, fx_{n_k}). \end{aligned}$$

Thus

$$\varepsilon \leq S_p(fx_{m_k}, fx_{m_k}, fx_{n_k}) < \varepsilon,$$

which leads to a contradiction.

Thus, $\{fx_n\}$ is a 0-Cauchy sequence. Since (X, S_p) is 0-complete, there exists a $z \in X$ such that $\lim_{n\to\infty} S_p(fx_n, fx_n, z) = 0$. Hence,

$$S_p(z, z, z) = \lim_{n\to\infty} S_p(fx_n, fx_n, z) = \lim_{n\to\infty} S_p(fx_n, fx_n, fx_m) = 0.$$

Therefore, $fx_n \to z$ as $n \to \infty$ in the partial S-metric space (X, S_p). Since all of $f(A_i)$ are closed in X, so $z \in f(A_i)$ for all i.

Thus, $z \in \bigcap_{i=1}^n f(A_i)$ and there exists a $z_i \in A_i$ such that $fz_i = z$. Also, we know that f is a one-to-one map, so we have $fz_1 = fz_2 = \cdots = fz_n = z$ which implies that $z_1 = z_2 = \cdots = z_n = z'$. Therefore, $fz' = z$ for $z' \in \bigcap_{i=1}^n A_i$ and $\lim_{n\to\infty} fx_n = z = fz'$.

Now, fix $i \in \{1, \ldots, n\}$ such that $z \in A_i$ and $gz \in A_{i+1}$. Take a subsequence $\{fx_{n_k}\}$ of $\{fx_n\}$ with$fx_{n_k} \in f(A_{i-1})$ where $x_{n_k} \in A_{i-1}$ and also converge to z. Thus,

$$S_p(z, z, z) = \lim_{n\to\infty} S_p(fx_n, fx_n, z) = \lim_{n\to\infty} S_p(fx_{n_k}, fx_{n_k}, z) = 0,$$

$$\begin{aligned} \alpha(S_p(gz', gz', gx_{n_k})) &= \alpha(S_p(gz', gz', fx_{n_{k+1}})) \\ &\leq \psi(S_p(fz', fz', fx_{n_k}), S_p(fz', fz', gz')) \\ &\quad - \phi(S_p(fz', fz', fx_{n_k}), S_p(fz', fz', gz')). \end{aligned}$$

Taking the limit as $n \to \infty$ and using the properties of ψ and ϕ, we have

$$\begin{aligned} \psi(S_p(gz', gz', fz'), S_p(gz', gz', fz')) &= \alpha(S_p(gz', gz', fz')) \\ &\leq \psi(S_p(fz', fz', fz'), S_p(fz', fz', gz')) \\ &\quad - \phi(S_p(fz', fz', fz'), S_p(fz', fz', gz')) \\ &\leq \psi(S_p(fz', fz', fz'), S_p(fz', fz', gz')). \end{aligned}$$

Since ψ is monotone increasing, we get

$$S_p(gz', gz', fz') \leq S_p(fz', fz', fz').$$

But, by the property of partial S-metric spaces, we have

$$S_p(fz', fz', fz') \leq S_p(gz', gz', fz').$$

Thus

$$S_p(fz', fz', fz') = S_p(gz', gz', fz').$$

If $S_p(fz', fz', fz') \neq 0$, then $S_p(fz', fz', fz') > 0$ and

$$\begin{aligned}\psi(S_p(fz',fz',fz'),S_p(fz',fz',fz')) &= \alpha(S_p(fz',fz',fz'))\\ &\le \psi(S_p(fz',fz',fz'),S_p(fz',fz',fz'))\\ &\quad - \phi(S_p(fz',fz',fz'),S_p(fz',fz',gz'))\\ &\le \psi(S_p(fz',fz',fz'),S_p(fz',fz',fz')).\end{aligned}$$

Given the fact that $\psi,\phi\in\Omega$, this leads to a contradiction. Thus

$$S_p(fz',fz',fz')=S_p(gz',gz',fz')=S_p(gz',gz',gz')=0$$

and $gz'=fz'=z$.

Since f and g are weakly compatible, we have $ggz'=gfz'=ffz'$, that is $fz=gz$.

Now, we show that $fz=z$. Since $gz'\in X$, we have $gz'\in A_i$ for some $i\in\{1,\dots,n\}$. We know that $z'\in\bigcap_{i=1}^n A_i$, so we have $z'\in A_{i-1}$ and

$$\begin{aligned}\alpha(S_p(gz',gz',ggz')) &\le \psi(S_p(fz',fz',fgz'),S_p(gz',gz',gz'))\\ &\quad - \phi(S_p(fz',fz',fgz'),S_p(fz',fz',gz'))\\ &\le \psi(S_p(fz',fz',fgz'),S_p(fz',fz',gz')).\end{aligned}$$

Since $fz'=gz'$, we deduce

$$\begin{aligned}\alpha(S_p(gz',gz',ggz')) &\le \psi(S_p(gz',gz',ggz'),S_p(gz',gz',gz'))\\ &\quad - \phi(S_p(gz',gz',ggz'),S_p(gz',gz',gz'))\\ &\le \psi(S_p(gz',gz',ggz'),S_p(gz',gz',gz'))\\ &\le \psi(S_p(gz',gz',ggz'),S_p(gz',gz',ggz')).\end{aligned}$$

Since $\psi\in\Omega$ and $S_p(gz',gz',gz')\le S_p(gz',gz',ggz')$, we have

$$S_p(gz',gz',ggz')=0,$$

and hence, $gz=gz'=ggz'=gz=fz$. Thus $fz=gz=z$.

Now, assume that there exists another common fixed point $z^*\in X$ of f and g. Hence,

$$\begin{aligned}\alpha(S_p(z,z,z^*)) &= \alpha(S_p(gz,gz,gz^*))\\ &\le \psi(S_p(fz,fz,fz^*),S_p(fz,fz,gz))\\ &\quad - \phi(S_p(fz,fz,fz^*),S_p(fz,fz,gz))\\ &\le \psi(S_p(fz,fz,fz^*),S_p(fz,fz,gz))\\ &= \psi(S_p(z,z,z^*),S_p(z,z,z)).\end{aligned}$$

Since $\psi\in\Omega$ and $S_p(z,z,z)\le S_p(z,z,z^*)$, therefore $S_p(z,z,z^*)=0$ and hence $z=z^*$, as desired. □

Now, we state the following immediate corollary.

Corollary 2.2. *Let (X,S_p) be a complete partial S-metric space, n a positive integer, and $A_1,\cdots,A_n$ nonempty closed subsets of X such that $X=\bigcup_1^n A_i$ is a cyclic representation of X with respect to the self-mapping g on X. Assume that there exist $\psi,\phi\in\Omega$ such that*

$$\alpha(S_p(gx,gx,gy))\le\psi(S_p(x,x,y),S_p(x,x,gx))-\phi(S_p(x,x,y),S_p(x,x,gx))$$

is satisfied for any $x\in A_i$ and $y\in A_{i+1}$ for $i\in\{1,\cdots,n\}$, where $A_{n+1}=A_1$, and for $x\in[0,\infty)$, $\alpha(x)=\psi(x,x)$. Then g has a unique fixed point $z\in\bigcap_1^n A_i$.

Proof. Just take $fx=x$ in Theorem 2.1. □

Example 2.3. Let $X = [0, 1]$ and define the function $S_p : X \times X \times X \to R^+$ by $S_p(x, y, z) = \max(x, y, z)$. Then (X, S_p) is a complete partial S-metric space. Let $f, g : X \to X$ be such that $fx = \frac{x}{4}$ and $gx = \frac{x^2}{16}$ for all $x \in X$. Let $\psi, \phi \in \Omega$ be defined by $\psi(x, y) = x + y$ and $\phi(x, y) = \max(x, y)$ for all $x, y, z \in [0, \infty)$. Let $A_i = [0, 1]$ for $i = 1, 2, \ldots, n$.

Note that all the conditions of Theorem 2.1 are satisfied and we obtain $0 \in \bigcap_{i=1}^{n} A_i$ as coincident and common fixed point of f and g.

3. Conclusion

In closing, the authors invite the readers to try to prove our main result, weakening or eliminating the assumption that f and g are weakly compatible. Also, it is possible to prove a similar result if we change the contraction to

$$\alpha(x, y)S_p(gx, gx, gy) \leq \psi(S_p(fx, fx, fy), S_p(fx, fx, gx)),$$

where ψ as defined in our main theorem and $\alpha : X \times X \to (0, \infty)$.

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