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The Borwein-Preiss variational principle for nonconvex countable systems of equilibrium problems

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Abstract

The aim of the present paper, by using the Borwein-Preiss variational principle, we prove existence results for countable systems of equilibrium problems. We establish some sufficient conditions which can guarantee two existence theorems for countable systems of equilibrium problems on closed subsets of complete metric spaces and on weakly compact subsets of real Banach spaces, respectively.

Keywords: Borwein-Preiss variational principle, bifunction, complete metric space, equilibrium problems, gauge-type function, nonconvex.
2010 MSC: 47H10, 54H25.

1. Introduction

The Ekeland variational principle (for short, EVP) was discovered by Ekeland in 1974 (see in [9] and [10]). The principle has received a great deal of attention and found many applications in different fields in analysis. It is a potent and flexible tool in analysis and in optimization problems for lower semicontinuous function on a complete metric space. Moreover, this Ekeland variational principle is equivalent to the Caristi's fixed point theorem, the Danež drop property, and the Petal flower theorem (see in [11], [14], and [17]). In 1987, for the application to differentiability problems of convex functions, Borwein and Preiss [7] revised this principle in the Banach spaces. The generalization of the Borwein-Preiss variational principle

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on a complete metric space appeared in 2005 (see in [8]).

One of the most important problems in nonlinear analysis is the so-called equilibrium problem considered by Takahashi [19] (see also in [6]). Let A be a nonempty set and $\phi : A \times A \to \mathbb{R}$ a bifunction. The problem consists of finding an element $a^* \in A$ such that

$$\phi(a^*, a) \geq 0, \quad \forall\, a \in A. \tag{EP}$$

Furthermore, equilibrium problems have been extensively studied (e.g. [5], [12], [15] and the references therein). The reason of studies of equilibrium problems is that it was applied among its particular cases, variational inequalities (monotone or otherwise), Nash equilibrium problems, optimization problems, and saddlepoint (minimax) problems (see [12] for a survey). Recently, Ansari, Schaible, and Yao [3] introduced a system of equilibrium problems and established existence results of the problems (see also in [13], [16] and [18]).

In 2005, Bianchi, Kassay and Pini [5] studied existence results of equilibrium problems and a system of equilibrium problems via the Ekeland variational principle (see also in [2] and [4]). Very recently, Alleche and Rădulescu [1] studied the Ekeland variational principle for equilibrium problems and a system of equilibrium problems under real Banach spaces. They also proved a result to guarantee for an existence of solutions for countable systems of equilibrium problems in the non weakly compact case which is a generalization of ([2], Theorem 15).

In this paper, by using the Borwein-Preiss variational principle, we prove existence results for countable systems of equilibrium problems. We establish some sufficient conditions which can guarantee for existence theorems of countable systems of equilibrium problems on closed subsets of complete metric spaces and on weakly compact subsets of real Banach spaces, respectively.

2. Preliminaries

The purpose of this section, we will introduce the significant definitions, theorems and corollary for using in the following section.

Definition 2.1 ([8]). Let (X, d) be a metric space. We say that a continuous function $\rho : X \times X \to [0, \infty]$ is a gauge-type function on a complete metric space (X, d) provided that

(i) $\rho(x, x) = 0$ for all $x \in X$,

(ii) for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $y, z \in X$ we have $\rho(y, z) \leq \delta$ implies that $d(y, z) < \epsilon$.

The following theorem, it is well-known that the Borwein-Preiss variational principle on a complete metric space.

Theorem 2.2 ([8]). *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function bounded from below. Suppose that ρ is a gauge-type function and $\{\delta_i\}_{i=0}^{\infty}$ is a sequence of positive numbers, and suppose that $\epsilon > 0$ and $z \in X$ satisfy*

$$f(z) \;\leq\; \inf_{x \in X} f(x) + \epsilon.$$

Then, there exist y and a sequence $\{x_i\} \subset X$ such that

($\bar{a}$.) $\rho(z, y) \;\leq\; \frac{\epsilon}{\delta_0}$, $\rho(x_i, y) \;\leq\; \frac{\epsilon}{2^i \delta_0}$,

($\bar{b}$.) $f(y) + \sum_{i=0}^{\infty} \delta_i \rho(y, x_i) \;\leq\; f(z)$, *and*

($\bar{c}$.) $f(x) + \sum_{i=0}^{\infty} \delta_i \rho(x, x_i) > f(y) + \sum_{i=0}^{\infty} \delta_i \rho(y, x_i)$, *for all* $x \in X \setminus \{y\}$.

Corollary 2.3. *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function bounded from below. Suppose that ρ is a gauge-type function and $\{\delta_i\}_{i=0}^{\infty}$ is a sequence of positive numbers. Then for any $\epsilon > 0$, there exist y and a sequence $\{x_i\} \subset X$ satisfying the following two conditions:*

(1) $f(y) + \sum_{i=0}^{\infty} \delta_i \rho(y, x_i) \leq \inf_{x \in X} f(x) + \epsilon$;

(2) $f(w) + \sum_{i=0}^{\infty} \delta_i \rho(w, x_i) \geq f(y) + \sum_{i=0}^{\infty} \delta_i \rho(y, x_i)$ *for all* $w \in X$.

Proof. It is obvious from Theorem 2.2. □

Definition 2.4 ([1])**.** The function f is said to be sequentially upper (resp. sequentially lower) semicontinuous on a subset A of X if it is sequentially upper (resp. sequentially lower) semicontinuous at every point of A.

Definition 2.5 ([1])**.** Let X be a Banach space (more generally it may be a Hausdorff topological space), $x \in X$ and $f : X \to \mathbb{R}$ a function. the function f is said to be

(1) sequentially upper semicontinuous at x if for every sequence $\{x_n\}_n \to x \in X$, we have
$$f(x) \geq \lim_{n \to +\infty} \sup f(x_n),$$
where $\limsup_{n \to +\infty} f(x_n) = \inf_n \sup_{k \geq n} f(x_k)$;

(2) sequentially lower semicontinuous at x if $-f$ is sequentially upper semicontinuous at x.

Hausdorff topological space is called sequentially compact if every sequence has a converging subsequence. A subset A of a Hausdorff topological space is called sequentially compact if it is sequentially compact as a topological subspace.

3. Main theorem

Let I be a countable index set. Assume that A_i is a closed subset of a complete metric space (X_i, d_i) and ρ_i is a gauge-type function on (X_i, d_i), for every $i \in I$. The system of equilibrium problems is a problem of finding $x^* = \{x_i^*\}_{i \in I}$ such that

$$\phi_i(x^*, y_i) \geq 0, \quad \forall\, i \in I \text{ and } y_i \in A_i, \qquad \text{(SEP)}$$

where $\phi_i : A \times A_i \to \mathbb{R}$, $A = \prod_{i \in I} A_i$ with A_i some given sets. Without loss of generality, we may assume that d_i and ρ_i are bounded by 1 for all $i \in I$. An element of the set $A^i = \prod_{j \in I} A_j$ with $j \neq i$ will be represented by x^i. Therefore, $\bar{x} \in A$ can be written as $\bar{x} = (x^i, x_i) \in A^i \times A_i$. The space $X = \prod_{i \in I} X_i$ will be endowed by the product topology. The distance d and gauge-type function ρ on X defined by

$$d(x, y) = \sum_{i \in I} \frac{1}{2^i} d_i(x_i, y_i) \ \ , \forall\, x = \{x_i\}_{i \in I},\, y = \{y_i\}_{i \in I} \in X,$$

and

$$\rho(x, y) = \sum_{i \in I} \frac{1}{2^i} \rho_i(x_i, y_i) \ \ , \forall\, x = \{x_i\}_{i \in I},\, y = \{y_i\}_{i \in I} \in X$$

respectively. Therefore, the space (X, d) is a complete metric space.

The following theorem is the Borwein-Preiss variational principle for nonconvex countable systems of equilibrium problems defined on complete metric spaces.

Theorem 3.1. *Let A_i be a nonempty closed subset of a complete metric space (X_i, d_i) and $\phi_i : A \times A_i \to \mathbb{R}$, for every $i \in I$. Assume that, for every $i \in I$, the following conditions hold:*

(1) $\phi_i(\bar{a}, a_i) = 0$, for every $\bar{a} = (a^i, a_i) \in A$;

(2) $\phi_i(\bar{a}, b_i) \leq \phi_i(\bar{a}, c_i) + \phi_i(\bar{c}, b_i)$ for every $b_i, c_i \in A_i$, and $\bar{a}, \bar{c} \in A$ such that $\bar{c} = (c^i, c_i)$;

(3) ϕ_i is lower bounded and lower semicontinuous in its second variable.

Suppose that $\{\delta^n\}_{n=0}^{\infty}$ is a sequence of positive numbers and ρ_i is a gauge-type function and suppose that $\epsilon > 0$ and $\bar{z} = \{z_i\}_{i\in I} \in A$, there exists $m_i \in A_i$ with

$$\phi_i(\bar{z}, m_i) \leq \inf_{x_i \in A_i} \phi_i(\bar{z}, x_i) + \epsilon,$$

for all $i \in I$. Then, for every $\bar{z} = \{z_i\}_{i\in I} \in A$, there exist $\bar{y} = \{y_i\}_{i\in I} \in A$ and a sequence $\{\bar{x}_n\}_{n=0}^{\infty} = \{(x_i^n)_{i\in I}\}_{n=0}^{\infty} \in A$ such that, for all $i \in I$,

(i) $\phi_i(\bar{z}, y_i) + \sum_{n=0}^{\infty} \delta^n \rho_i(y_i, \bar{x}_n) \leq 0$;

(ii) $\phi_i(\bar{y}, x_i) + \sum_{n=0}^{\infty} \delta^n (\rho_i(x_i, \bar{x}_n) - \rho_i(y_i, \bar{x}_n)) > 0, \ \forall x_i \in A_i$ and $x_i \neq y_i$.

Proof. We define sequences $\{x_i^n\}$ and F_i^n for all $i \in I$ inductively. Let $x^0 := \bar{z} \in A$ and for all $i \in I$

$$F_i^0 := \{x_i \in A_i | \phi_i(x^0, x_i) + \delta^0 \rho_i(x_i, x_i^0) \leq 0\}. \tag{3.1}$$

From the condition (1), we obtain that $x_i^0 \in F_i^0$. Hence, by the condition (3), F_i^0 is nonempty and closed set. For all $i \in I$ and $x_i \in F_i^0$, we note that

$$\begin{aligned}\delta^0 \rho_i(x_i, x_i^0) &\leq -\phi_i(x^0, x_i) \leq \phi_i(x^0, x_i^0) - \phi_i(x^0, x_i) \\ &\leq \phi_i(x^0, x_i^0) - \inf_{x_i \in A_i} \phi_i(x^0, x_i) \leq \epsilon.\end{aligned}$$

For all $i \in I$, we choose $x_i^1 \in F_i^0$ such that

$$\begin{aligned}\phi_i(x^0, x_i^1) + \delta^0 \rho_i(x_i^1, x_i^0) &\leq \inf_{x_i \in F_i^0} \Big[\phi_i(x^0, x_i) + \delta^0 \rho_i(x_i, x_i^0)\Big] + \frac{\delta^1 \epsilon}{2\delta^0} \\ &\leq \phi_i(x^0, x_i^1) + \inf_{x_i \in F_i^0} \Big[\phi_i(x^1, x_i) + \delta^0 \rho_i(x_i, x_i^0)\Big] + \frac{\delta^1 \epsilon}{2\delta^0},\end{aligned}$$

and hence

$$\delta^0 \rho_i(x_i^1, x_i^0) \leq \inf_{x_i \in F_i^0} \Big[\phi_i(x^1, x_i) + \delta^0 \rho_i(x_i, x_i^0)\Big] + \frac{\delta^1 \epsilon}{2\delta^0}.$$

For any fixed $i \in I$, we define

$$F_i^1 := \Big\{x_i \in F_i^0 | \phi_i(x^1, x_i) + \sum_{k=0}^{1} \delta^k \rho_i(x_i, x_i^k) \leq \delta^0 \rho_i(x_i^1, x_i^0)\Big\}.$$

Similarly, for all $i \in I$, we take $x_i^2 \in F_i^1$ such that

$$\sum_{k=0}^{1} \delta^k \rho_i(x_i^2, x_i^k) \leq \inf_{x_i \in F_i^1} \Big[\phi_i(x^2, x_i) + \sum_{k=0}^{1} \delta^k \rho_i(x_i, x_i^k)\Big] + \frac{\delta^2 \epsilon}{2\delta^0}$$

and define

$$F_i^2 := \left\{ x_i \in F_i^1 | \phi_i(x^2, x_i) + \sum_{k=0}^{2} \delta^k \rho_i(x_i, x_i^k) \leq \sum_{k=0}^{1} \delta^k \rho_i(x_i^2, x_i^k) \right\}.$$

In general, for all $i \in I$ and for $j = 0, 1, \dots, n-1$, suppose that we have defined x_i^j, F_i^j satisfying

$$\sum_{k=0}^{j-1} \delta^k \rho_i(x_i^j, x_i^k) \leq \inf_{x_i \in F_i^{j-1}} \left[\phi_i(x^j, x_i) + \sum_{k=0}^{j-1} \delta^k \rho_i(x_i, x_i^k) \right] + \frac{\delta^j \epsilon}{2\delta^0}$$

and

$$F_i^j := \left\{ x_i \in F_i^{j-1} | \phi_i(x^j, x_i) + \sum_{k=0}^{j} \delta^k \rho_i(x_i, x_i^k) \leq \sum_{k=0}^{j-1} \delta^k \rho_i(x_i^j, x_i^k) \right\}, \ \forall\, i \in I.$$

For all $i \in I$, we choose $x_i^n \in F_i^{n-1}$ such that

$$\sum_{k=0}^{n-1} \delta^k \rho_i(x_i^n, x_i^k) \leq \inf_{x_i \in F_i^{n-1}} \left[\phi_i(x^n, x_i) + \sum_{k=0}^{n-1} \delta^k \rho_i(x_i, x_i^k) \right] + \frac{\delta^n \epsilon}{2\delta^0} \tag{3.2}$$

and define

$$F_i^n := \left\{ x_i \in F_i^{n-1} | \phi_i(x^n, x_i) + \sum_{k=0}^{n} \delta^k \rho_i(x_i, x_i^k) \leq \sum_{k=0}^{n-1} \delta^k \rho_i(x_i^n, x_i^k) \right\}. \tag{3.3}$$

Thus, we see that, for every $n = 1, 2, \dots,$ F_i^n is nonempty and closed set. It follows from (3.2) and (3.3) that, for all $i \in I, n = 1, 2, \dots,$ and $x_i \in F_i^n$

$$\begin{aligned}
\delta^n \rho_i(x_i, x_i^n) &\leq \sum_{k=0}^{n-1} \delta^k \rho_i(x_i^n, x_i^k) - (\phi_i(x^n, x_i) + \sum_{k=0}^{n-1} \delta^k \rho_i(x_i, x_i^k)) \\
&\leq \sum_{k=0}^{n-1} \delta^k \rho_i(x_i^n, x_i^k) - \inf_{x_i \in F_i^{n-1}} \left[\phi_i(x^n, x_i) + \sum_{k=0}^{n-1} \delta^k \rho_i(x_i, x_i^k) \right] \\
&\leq \frac{\delta^n \epsilon}{2\delta^0}.
\end{aligned}$$

Thus, we see that there exists $\delta := \frac{\epsilon}{2\delta^0}$ such that $\rho_i(x_i, x_i^n) \leq \delta$ for all $i \in I, n = 1, 2, \dots,$ and $x_i \in F_i^n$. Since ρ_i is a gauge-type function, it follows that $d_i(x_i, x_i^n) \to 0$ as $n \to \infty$ uniformly. Hence $diam(F_i^n) \to 0$ and therefore, by Cantor's intersection theorem, there exists a unique $y_i \in \cap_{n=0}^{\infty} F_i^n$ for all $i \in I$. This implies that $x_i^n \to y_i$ as $n \to \infty$. For any $i \in I$, if $x_i \neq y_i$, we can conclude that $x_i \notin \cap_{n=0}^{\infty} F_i^n$. Therefore for some j,

$$\sum_{k=0}^{j} \delta_k \rho_i(x_i, x_i^k) > \sum_{k=0}^{j-1} \delta_k \rho_i(x_i^j, x_i^k) - \phi_i(x^j, x_i). \tag{3.4}$$

This implies that

$$\begin{aligned}
\phi_i(x^j, x_i) + \sum_{k=0}^{\infty} \delta_k \rho_i(x_i, x_i^k) &\geq \phi_i(x^j, x_i) + \sum_{k=0}^{j} \delta_k \rho_i(x_i, x_i^k) \\
&> \phi_i(x^j, x_i) + \sum_{k=0}^{j-1} \delta_k \rho_i(x_i^j, x_i^k) - \phi_i(x^j, x_i) \\
&= \sum_{k=0}^{j-1} \delta_k \rho_i(x_i^j, x_i^k).
\end{aligned} \tag{3.5}$$

On the other hand, it follows from (3.1), (3.3), and $y_i \in \cap_{n=0}^{\infty} F_i^n$ that, for all $q \geq j$,

$$\begin{aligned} -\phi_i(x^0, x_i^j) &\geq \sum_{k=0}^{j-1} \delta_k \rho_i(x_i^j, x_i^k) \\ &\geq \sum_{k=0}^{q-1} \delta_k \rho_i(x_i^q, x_i^k) + \phi_i(x^j, x_i^q) \\ &\geq \phi_i(x^j, x_i^q) + \phi_i(x^q, y_i) + \sum_{k=0}^{q} \delta_k \rho_i(y_i, x_i^k) \\ &\geq \phi_i(x^j, y_i) + \sum_{k=0}^{q} \delta_k \rho_i(y_i, x_i^k). \end{aligned} \tag{3.6}$$

From (3.6), we obtain that, for any $q \geq j$ and $i \in I$,

$$0 \geq \phi_i(x^0, y_i) + \sum_{k=0}^{q} \delta_k \rho_i(y_i, x_i^k). \tag{3.7}$$

Combining (3.5) and (3.6) yields for any $q \geq j$ and $i \in I$,

$$\phi_i(x^j, x_i) - \phi_i(x^j, y_i) + \sum_{k=0}^{\infty} \delta_k \rho_i(x_i, x_i^k) > \sum_{k=0}^{q} \delta_k \rho_i(y_i, x_i^k).$$

Since $\phi_i(x^j, x_i) - \phi_i(x^j, y_i) \leq \phi_i(\bar{y}, x_i)$, we have

$$\phi_i(\bar{y}, x_i) + \sum_{k=0}^{\infty} \delta_k \rho_i(x_i, x_i^k) > \sum_{k=0}^{q} \delta_k \rho_i(y_i, x_i^k). \tag{3.8}$$

By taking limits in (3.7) and (3.8) as $q \to \infty$, we have

$$0 \geq \phi_i(\bar{z}, y_i) + \sum_{k=0}^{\infty} \delta_k \rho_i(y_i, x_i^k)$$

and

$$\phi_i(\bar{y}, x_i) + \sum_{k=0}^{\infty} \delta_k \rho_i(x_i, x_i^k) > \sum_{k=0}^{\infty} \delta_k \rho_i(y_i, x_i^k)$$

for all $i \in I$. This completes the proof. □

Example 3.2. Let A_i be a nonempty closed subset of a usual metric space $(\mathbb{R}, d)$ with the gauge-type function $\rho_i = d$, for every $i \in I$, and $A = \prod_{i \in I} A_i$. Suppose that $\phi_i : A \times A_i \to \mathbb{R}$ is a real valued function defined by

$$\phi_i(\bar{a}, b_i) = \frac{1}{a_i} - \frac{1}{b_i}, \quad \forall\ \bar{a} = \{a_i\}_{i \in I} \in A, b_i \in A_i.$$

It is easy to see that both functions (ρ_i and ϕ_i) satisfy the conditions in Theorem 3.1. Suppose that $\{\delta^n\}_{n=0}^{\infty}$ is a sequence of positive numbers and suppose that $\epsilon > 0$ and $\bar{z} = \{z_i\}_{i \in I} \in A$. Then, by Theorem 3.1, there exist $\bar{y} = \{y_i\}_{i \in I}$ with $y_i = \frac{z_i}{z_i+1} \in A_i$ and a sequence $\{x_n\}_{n=0}^{\infty} = \{(x_i^n)_{i \in I}\}_{n=0}^{\infty}$ with $x_i^n = \frac{z_i}{z_i+1} + \frac{1}{\delta^n 2^{k_n}} \in A_i$, where $\{k_n\}$ is a subsequence of a sequence of natural numbers, such that

$$\begin{aligned}\phi_i(\bar{z}, y_i) + \sum_{n=0}^{\infty} \delta^n \rho_i(y_i, \bar{x}_n) &= \frac{1}{z_i} - \frac{1}{y_i} + \sum_{n=0}^{\infty} \delta^n \rho_i(y_i, x_i^n) \\ &\leq \frac{1}{z_i} - \frac{z_i + 1}{z_i} + 1 = 0\end{aligned}$$

for all $i \in I$. Furthermore, for every $x_i \neq y_i$, we have

$$\phi_i(\bar{y}, x_i) + \sum_{n=0}^{\infty} \delta^n \rho_i(x_i, \bar{x}_n) - \sum_{n=0}^{\infty} \delta^n \rho_i(y_i, \bar{x}_n) \geq \frac{1}{y_i} - \frac{1}{x_i} + \sum_{n=0}^{\infty} \delta^n \rho_i(x_i, x_i^n) - 1 > 0.$$

Putting $I = \{1\}, X_1 = X$ and $A_1 = A$ in Theorem 3.1, we have the following result.

Corollary 3.3. *Let A be a nonempty closed subset of a complete metric space (X, d) and $\phi : A \times A \to \mathbb{R}$ be a bifunction. Assume that the following conditions hold:*

(1) $\phi(a, a) = 0$, for every $a \in A$;

(2) $\phi(a, b) \leq \phi(a, c) + \phi(c, b)$ for every $a, b, c \in A$;

(3) ϕ is lower bounded and lower semicontinuous in its second variable.

Suppose that $\{\delta^n\}_{n=0}^{\infty}$ is a sequence of positive numbers and ρ is a gauge-type function and suppose that $\epsilon > 0$ and $z \in A$. There exists $m \in A$ with

$$\phi(z, m) \leq \inf_{x \in A} \phi(z, x) + \epsilon.$$

There exist y and a sequence $\{x_n\}_{n=0}^{\infty} \subset A$ such that

(i) $\phi(z, y) + \sum_{n=0}^{\infty} \delta^n \rho(y, x_n) \leq 0$;

(ii) $\phi(y, x) + \sum_{n=0}^{\infty} \delta^n (\rho(x, x_n) - \rho(y, x_n)) > 0, \ \forall x \in A$ and $x \neq y$.

In the final of this section, we will prove existence theorems for solutions of countable systems of equilibrium problems in the weakly compact case and non weakly compact case, respectively. Here, the space X_i is replaced by a real Banach space E_i, for every $i \in I$ (Denoted by $\|\cdot\|_i$).

The following proposition can guarantee the existence of solutions to countable systems of equilibrium problems in the weakly compact case.

Proposition 3.4. *Let A_i be a nonempty weakly closed subset of a real Banach space E_i and $\phi_i : A \times A_i \to \mathbb{R}$, for every $i \in I$. Assume that the following conditions hold:*

(1) $\phi_i(\bar{a}, a_i) = 0$, for every $i \in I$ and $\bar{a} = (a^i, a_i) \in A$;

(2) $\phi_i(\bar{a}, b_i) \leq \phi_i(\bar{a}, c_i) + \phi_i(\bar{c}, b_i)$ for every $i \in I$, $b_i, c_i \in A_i$, and $\bar{a}, \bar{c} \in A$ such that $\bar{c} = (c^i, c_i)$;

(3) ϕ_i is lower bounded and lower semicontinuous in its second variable, for every $i \in I$.

(4) ϕ_i is weakly sequentially upper semicontinuous in its first variable, for every $i \in I$.

(5) A is sequentially compact subset of $E = \prod_{i \in I} E_i$ with respect to the topology σ.

Then, the system of equilibrium problems (SEP) has a solution.

Proof. For every $n \in \mathbb{N}$, we choose a sequence $\{\bar{z}_n\} = \{(z_i^n)_{i \in I}\} \subseteq A$ and $\epsilon_n = \frac{1}{n}$. By Theorem 3.1, there exists $\{\bar{x}_n\} = \{(x_i^n)_{i \in I}\} \subseteq A$ which is a solution of the system of equilibrium problems (SEP) such that

$$\phi_i(\bar{x}_n, y_i) \geq -\frac{1}{n} \|z_i^n - y_i\|_i, \forall\, y_i \in A_i.$$

Since A is a sequentially compact subset of E with respect to the topology σ, the sequence $\{\bar{x}_n\}_n$ has a converging subsequence $\{\bar{x}_{n_k}\}_k$ to some $\bar{x} = \{\bar{x}_i\}_{i\in I} \in A$ with respect to the topology σ. By the sequentially upper semicontinuity of a function ϕ_i , we see that, for every $i \in I$,

$$\begin{aligned} \phi_i(\bar{x}, y_i) &\geq \lim_{k\to\infty} \sup \phi_i(\bar{x}_i^{n_k}, y_i) \\ &\geq \lim_{k\to\infty} \sup \left(-\frac{1}{n_k} \|z_i^{n_k} - y_i\|_i \right) = 0 \ , \forall y_i \in A_i. \end{aligned}$$

Therefore $\bar{x}$ is a solution to the system of equilibrium problems (SEP). □

The ensuing theorem can guarantee the existence of solutions to countable systems of equilibrium problems in the non weakly compact case.

Theorem 3.5. *Let A_i be a nonempty weakly closed subset of a real reflexive Banach space E_i and $\phi_i : A \times A_i \to \mathbb{R}$, for every $i \in I$. Assume that the following conditions hold:*

(1) $\phi_i(\bar{a}, a_i) = 0$, for every $i \in I$ and $\bar{a} = (a^i, a_i) \in A$;

(2) $\phi_i(\bar{a}, b_i) \leq \phi_i(\bar{a}, c_i) + \phi_i(\bar{c}, b_i)$ for every $i \in I$, $b_i, c_i \in A_i$, and $\bar{a}, \bar{c} \in A$ such that $\bar{c} = (c^i, c_i)$;

(3) there exists a nonempty closed subset K_i of A_i for every $i \in I$ such that for every $\bar{x} = (x^i, x_j) \in A$ with $x_j \notin K_j$ for some $j \in I$,

$$\exists y_j \in A_j,\ \|y_j\| < \|x_j\|,\ \ \phi_j(x, y_j) \leq 0;$$

(4) ϕ_i is sequentially lower semicontinuous in its second variable on K_i for every $i \in I$;

(5) the restriction of ϕ_i on $(\prod_{i\in I} K_i) \times K_i$ is lower bounded in its second variable for every $i \in I$;

(6) the restriction of ϕ_i on $(\prod_{i\in I} K_i) \times K_i$ is weakly sequentially upper semicontinuous in its first variable, for every $i \in I$;

(7) The subset $\prod_{i\in I} K_i$ is sequentially compact subset of E with respect to the topology σ.

Then, the system of equilibrium problems (SEP) has a solution.

Proof. It follows the proof of Theorem 4.3 in [1]. □

Putting $I = \{1\}, E_1 = E$ and $A_1 = A$ in Theorem 3.5, we have the following result.

Corollary 3.6. *Let A be a nonempty weakly closed subset of a real reflexive Banach space E and $\phi : A\times A \to \mathbb{R}$ be a bifunction. Suppose that the following conditions hold:*

(1) $\phi(a, a) = 0$, for every $a \in A$;

(2) $\phi(a, b) \leq \phi(a, c) + \phi(c, b)$, for every $x, y, z \in A$;

(3) there exists a nonempty weakly compact subset K of A such that

$$\forall\, x \in\ A \setminus K, \exists y \in A,\ \|y\| < \|x\|,\ \ \phi(x, y) \leq 0;$$

(4) ϕ is sequentially lower semicontinuous in its second variable on K;

(5) the restriction of ϕ on $K \times K$ is lower bounded in its second variable;

(6) the restriction of ϕ on $K \times K$ is weakly sequentially upper semicontinuous in its first variable.

Then, the equilibrium problem (EP) has a solution.

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