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Common fixed point results for compatible-type mappings in multiplicative metric spaces

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Abstract

In this paper, we prove some common fixed point theorems for generalized contractive mappings satisfying some conditions, that is, compatible and compatible-type mappings in multiplicative metric spaces. Our results improve and generalize the corresponding results given in the literature. Moreover, we give some examples to illustrate our main results.

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1. Introduction

The study for the existence of fixed points of contractive mappings is a famous topic in metric spaces. Banach's contraction principle [4] guarantees the existence and uniqueness of fixed point of a contractive mapping in a complete metric spaces. This principle is applicable to variety of subjects such as integral equations, differential equations, image processing and many others.

In the past years, many authors generalized Banach's contraction principle in various spaces, for example, quasi-metric spaces, fuzzy metric spaces, 2-metric spaces, cone metric spaces, partial metric spaces, probabilistic metric spaces and generalized metric spaces (see, for instance, [1, 2, 3, 7, 9, 14, 15, 16, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29] and the references therein).

Especially, in 1976, Jungck [10] initially gave a common fixed point theorem for *commuting mappings* in metric spaces, which generalized Banach's contraction principle, as follows:

Theorem 1.1 (Theorem GJ)**.** *Let f be a continuous mapping of a complete metric space (X, d) into itself. Then f has a fixed point in X if and only if there exist $\alpha \in (0, 1)$ and a mapping $g : X \to X$ such that*

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(a) *f and g are commuting on X (that is, $fgx = gfx$ for all $x, y \in X$);*
(b) *$g(X) \subset f(X)$;*
(c) *$d(g(x), g(y)) \leq \alpha d(f(x), f(y))$ for all $x, y \in X$.*

Indeed, f and g have a unique common fixed point in X if the condition (c) *holds.*

Since 1976, Jungck's theorem was generalized, extended and improved in various ways by many authors. In 1982, Sessa [20] defined a generalization of commuting mappings which is called *weakly commuting mappings* in metric spaces.

In 1986, Jungck [11] introduced more generalized commuting mappings called *compatible mappings* in metric spaces which are more general than weakly commuting mappings. In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converses are not necessarily true ([11, 20]).

Further, in 1993, Jungck et al. [12] defined the concept of *compatible mappings of type* (A), which is equivalent to the concept of compatible mappings under some conditions, and proved a common fixed point theorem for compatible mappings of type (A) in a metric space. In 1995, Pathak and Khan [18] introduced more generalized compatible mappings called *compatible mappings of type* (B) and compared these mappings with compatible mappings and compatible mappings of type (A). Also, they derived some relations between these mappings and proved a common fixed point theorem for compatible mappings of type (B) in metric spaces.

On the other hand, in 2008, Bashirov et al. [5] introduced the notion of multiplicative metric spaces and studied the concept of multiplicative calculus and illustrated the usefulness of multiplicative calculus with some interesting applications. In 2011, Bashirov et al. [6] exploit the efficiency of multiplicative calculus over the Newtonian calculus. They demonstrated that the multiplicative differential equations are more suitable than the ordinary differential equations in investigating some problems in various fields.

In 2012, Ozavsar and Cevikel [17] introduced the concept of multiplicative contractive mappings in multiplicative metric spaces and prove some fixed point theorems for this type of mappings. Recently, He et al. [8] proved common fixed point theorems for four self-mappings in multiplicative metric spaces. In 2015, Kang et al. [13] introduced the concepts of compatible mappings and compatible mappings of types (A) and (B) in multiplicative metric spaces and prove some common fixed point theorems for these mappings.

In this paper, we prove some common fixed point theorems for generalized contractive mappings satisfying some conditions, that is, compatible mappings and compatible mappings of types (A) and (B) in multiplicative metric spaces. Our results improve and generalize many other results in the literature. Moreover, we give some examples to illustrate our main results.

2. Preliminaries

In this section, we give some basic and useful definitions.

Definition 2.1 ([5])**.** Let X be a nonempty set. A *multiplicative metric* is a mapping $d : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(M1) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1 \Leftrightarrow x = y$;
(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(M3) $d(x, y) \leq d(x, z) \cdot d(z, y)$ for all $x, y, z \in X$ (: multiplicative triangle inequality).

The pair (X, d) is called a *multiplicative metric space.*

Example 2.2 ([5])**.** Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. A function $d : (\mathbb{R}^+)^n \times (\mathbb{R}^+)^n \to \mathbb{R}$ is defined as follows:

$$d(x, y) = \left|\frac{x_1}{y^1}\right| \cdot \left|\frac{x_2}{y^2}\right| \cdots \left|\frac{x_n}{y^n}\right|,$$

where $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in (\mathbb{R}^+)^n$ and $|\cdot| : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as follows:

$$|a| = \begin{cases} a, & \text{if } \ a \geq 1, \\ \frac{1}{a}, & \text{if } \ a < 1. \end{cases}$$

It is clear that all the conditions of a multiplicative metric are satisfied.

Definition 2.3 ([5])**.** Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and let $x \in X$. If, for all multiplicative open ball $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$ and $\epsilon > 1$, there exists a natural number $N \in \mathbb{N}$ such that, for all $n \geq N$, $x_n \in B_\epsilon(x)$. Then the sequence $\{x_n\}$ is said to be *multiplicative convergent* to x, which is denoted by $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.4 ([17])**.** *Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and let $x \in X$. Then*

$$x_n \to x \ \ (n \to \infty) \ \ \textit{if and only if} \ \ d(x_n, x) \to 1 \ \ (n \to \infty).$$

Definition 2.5 ([17])**.** Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a *multiplicative Cauchy sequence* if, for any $\epsilon > 0$, there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Note that $\mathbb{R}^+$ is not complete under the ordinary metric, but, $\mathbb{R}^+$ is a complete multiplicative metric space, and the convergence of a sequence in $\mathbb{R}^+$ in both multiplicative metric space and ordinary metric space are equivalent. Of course they may be different in more general cases.

Proposition 2.6 ([17])**.** *Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1$ as $n, m \to \infty$.*

Definition 2.7 ([17])**.** A multiplicative metric space (X, d) is said to be *multiplicative complete* if every multiplicative Cauchy sequence in (X, d) is multiplicative convergent in X.

Definition 2.8 ([17])**.** Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces and $f : X \to Y$ be a mapping. If, for any $\epsilon > 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\delta(f(x))$, then we call f *multiplicative continuous* at $x \in X$.

Proposition 2.9 ([17])**.** *Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces, $f : X \to Y$ be a mapping and $\{x_n\}$ be any sequence in X. Then f is multiplicative continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ for every sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$.*

Proposition 2.10 ([17])**.** *Let (X, d_X) be a multiplicative metric spaces, $\{x_n\}$, $\{y_n\}$ be two sequences in X such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ and let $x, y \in X$. Then*

$$d(x_n, y_n) \to d(x, y) \ \ (n \to \infty).$$

Definition 2.11. The self-mappings f and g of a set X are said to be *commutative* or *commuting* on X if, for all $x \in X$,

$$fgx = gfx.$$

Definition 2.12 ([13, 20])**.** Suppose that f, g are two self-mappings of a multiplicative metric space (X, d). Two mappings f and g are said to be *weakly commutative* or *weakly commuting* on X if, for all $x \in X$,

$$d(fgx, gfx) \leq d(fx, gx).$$

Example 2.13. Let $X = \mathbb{R}$ and (X, d) be a multiplicative metric space defined by $d(x, y) = e^{|x-y|}$ for all $x, y \in X$. Let f and g be two self-mappings of X defined by $fx = x^3$ and $gx = 2 - x$ for all $x \in X$. Then

$$d(fx_n, gx_n) = e^{|x_n-1|\cdot|x_n^2+x_n+2|} \to 1 \ \text{ if and only if } \ x_n \to 1$$

and

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = \lim_{n\to\infty} e^{6|x_n-1|^2} = 1 \text{ if } x_n \to 1.$$

Thus f and g are compatible. Note that

$$d(fg0, gf0) = d(8,2) = e^6 > e^2 = d(0,2) = d(f0, g0)$$

and so two mappings f and g are not weakly commuting on X.

Definition 2.14 ([11, 12, 13, 18])**.** Let S and A be two mappings of a multiplicative metric space (X, d) into itself and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some $t \in X$. Then S and A are said to be:

(1) *compatible* if whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some $t \in X$;

(2) *compatible of type* (A) if

$$\lim_{n\to\infty} d(SAx_n, AAx_n) = 1, \quad \lim_{n\to\infty} d(ASx_n, SSx_n) = 1;$$

(3) *compatible of type* (B) if

$$\lim_{n\to\infty} d(SAx_n, AAx_n) \leq \left[\lim_{n\to\infty} d(SAx_n, St) \cdot \lim_{n\to\infty} d(St, SSx_n)\right]^{1/2}$$

and

$$\lim_{n\to\infty} d(ASx_n, SSx_n) \leq \left[\lim_{n\to\infty} d(ASx_n, At) \cdot \lim_{n\to\infty} d(At, AAx_n)\right]^{1/2}.$$

Proposition 2.15 ([13, 18])**.** *Let S and A be compatible mappings of type (A) in a multiplicative metric space (X, d). Then*

(1) *S and A are compatible mappings of type (B);*

(2) *S and A are compatible if one of S and A is a continuous mapping.*

Proposition 2.16 ([13, 18])**.** *Let S and A be continuous mappings of a multiplicative metric space (X, d) into itself. If S and A are compatible of type (B). Then*

(1) *S and A are compatible mappings of type (A);*

(2) *S and A are compatible mappings.*

Proposition 2.17 ([13, 18])**.** *Let S and A be continuous mappings of a multiplicative metric space (X, d) into itself. Then*

(1) *S and A are compatible if and only if they are compatible of type (B);*

(2) *S and A are compatible of type (A) if and only if they are compatible of type (B).*

Proposition 2.18 ([13, 18])**.** *Let S and A be compatible mappings of a multiplicative metric space (X, d) into itself. If $St = At$ for some $t \in X$, then*

$$SAt = SSt = AAt = ASt.$$

Proposition 2.19 ([13, 18])**.** *Let S and A be compatible mappings of a multiplicative metric space (X, d) into itself. Suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some $t \in X$. Then we have*

(1) $\lim_{n\to\infty} ASx_n = St$ *if S is continuous at t;*

(2) $\lim_{n\to\infty} SAx_n = At$ *if A is continuous at t;*

(3) *$SAt = ASt$ and $St = At$ if S and A are continuous at t.*

Proposition 2.20 ([13, 18])**.** *Let S and A be compatible mappings of type (B) of a multiplicative metric space (X, d) into itself. If $St = At$ for some $t \in X$, then*

$$SAt = SSt = AAt = ASt.$$

Proposition 2.21 ([13, 18])**.** *Let S and A be compatible mappings of type (B) of a multiplicative metric space (X, d) into itself. Suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some $t \in X$. Then we have the following:*

(1) $\lim_{n\to\infty} AAx_n = St$ *if S is continuous at t;*

(2) $\lim_{n\to\infty} SSx_n = At$ *if A is continuous at t;*

(3) $SAt = ASt$ *and* $St = At$ *if S and A are continuous at t.*

Example 2.22. Let $X = \mathbb{R}$ be the set of all real numbers with the usual multiplicative metric $d(x, y) = |x/y|$ and define two mappings $S, A : X \to X$ by

$$S(x) = \begin{cases} \frac{1}{x^4} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases} \quad A(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0 \end{cases}$$

for all $x \in X$. Then S and A are not continuous at $t = 0$. Consider a sequence $\{x_n\}$ in X defined by $x_n = n$ for each $n \geq 1$. Then, letting $n \to \infty$, we have

$$Sx_n = \frac{1}{n^4} \to t = 0, \quad Ax_n = \frac{1}{n^2} \to t = 0$$

and

$$\lim_{n\to\infty} d(SAx_n, ASx_n) = \lim_{n\to\infty} \left|\frac{n^8}{n^8}\right| = 1.$$

However, the following limits do not exist:

$$\lim_{n\to\infty} d(SAx_n, AAx_n) = \lim_{n\to\infty} \left|\frac{n^8}{n^4}\right| = \infty,$$

$$\begin{aligned}\left[\lim_{n\to\infty} d(SAx_n, S(0)) \cdot \lim_{n\to\infty} d(S(0), SSx_n)\right]^{1/2} &= \left[\lim_{n\to\infty} \left|\frac{n^8}{1}\right| \cdot \lim_{n\to\infty} \left|\frac{n^{16}}{1}\right|\right]^{1/2} \\ &= \infty\end{aligned}$$

and

$$\lim_{n\to\infty} d(ASx_n, SSx_n) = \lim_{n\to\infty} \left|\frac{n^{16}}{n^8}\right| = \infty,$$

$$\begin{aligned}\left[\lim_{n\to\infty} d(ASx_n, A(0)) \cdot \lim_{n\to\infty} d(A(0), AAx_n)\right]^{1/2} &= \left[\lim_{n\to\infty} \left|\frac{n^8}{2}\right| \cdot \lim_{n\to\infty} \left|\frac{n^4}{2}\right|\right]^{1/2} \\ &= \infty.\end{aligned}$$

Therefore, S and A are compatible, but they are not compatible of types (A) and (B).

Definition 2.23 ([17])**.** Lat (X, d) be a multiplicative metric space. A mapping $f : X \to X$ is called a *multiplicative contraction* if there exists a real constant $\lambda \in (0, 1]$ such that, for all $x, y \in X$,

$$d(fx, fy) \leq [d(x, y)]^\lambda.$$

In 2013, He *et al.* [8] proved the following result.

Theorem 2.24 ([8])**.** *Let S, T, A and B be self-mappings of a complete multiplicative metric space X satisfying the following conditions:*

(a) $S(X) \subset B(X)$ *and* $T(X) \subset A(X)$*;*

(b) *A and S are weakly commuting on X, B and T also are weakly commuting on X;*

(c) *one of S, T, A and B is continuous;*

(d) *there exists a real constant $\lambda \in (0, 1]$ such that, for all $x, y \in X$*

$$d(Sx, Ty) \leq [\max\{d(Ax, By), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ax, Ty)\}]^\lambda.$$

Then S, T, A and B have a unique common fixed point in X.

3. Main Results

In this section, we prove some common fixed point results for generalized contraction mappings satisfying compatible and compatibility of type (A) and (B) conditions.

Now, we improve Theorem 2.24 by introducing the following result.

Theorem 3.1. *Let S, T, A and B be four self-mappings of a complete multiplicative metric space X satisfying the following conditions:*

(a) *$S(X) \subset B(X)$ and $T(X) \subset A(X)$;*
(b) *the pairs (A, S) and (B, T) are compatible;*
(c) *one of S, T, A and B is continuous;*
(d) *for all $x, y \in X$,*

$$d(Sx, Ty) \quad \leq \Big[\varphi\Big(\max\Big\{d(Ax, By), \frac{d(Ax, Sx)d(By, Ty)}{1 + d(Ax, By)}, \frac{d(Ax, Ty)d(By, Ax)}{1 + d(Ax, By)}\Big\}\Big)\Big]^{\lambda},$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and monotone increasing function such that $\varphi(0) < t$ for all $t > 0$. Then S, T, A and B have a unique common fixed point in X.

Proof. Since $S(X) \subset B(X)$, we can consider a point $x_0 \in X$, there exists $x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$. Also, for this point x_1, there exists $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Continuing in this way, we can construct a sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1}, \quad y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}$$

for each $n \geq 0$. Now, we have to show that $\{y_n\}$ is a multiplicative Cauchy sequence in X. Indeed, it follows that, for all $n \geq 1$,

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \Big[\varphi\Big(\max\Big\{d(Ax_{2n}, Bx_{2n+1}), \frac{d(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Ax_{2n}, Bx_{2n+1})}, \\ &\qquad \frac{d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Ax_{2n})}{1 + d(Ax_{2n}, Bx_{2n+1})}\Big\}\Big)\Big]^{\lambda} \\ &\leq \Big[\varphi\Big(\max\Big\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n+1})\Big\}\Big)\Big]^{\lambda} \\ &\leq \Big[\varphi\Big(\max\Big\{d(y_{2n-1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1})\Big\}\Big)\Big]^{\lambda} \\ &\leq [d(y_{2n-1}, y_{2n})]^{\lambda} \cdot [d(y_{2n}, y_{2n+1})]^{\lambda}, \end{aligned}$$

which implies that

$$d(y_{2n}, y_{2n+1}) \leq [d(y_{2n-1}, y_{2n})]^{\frac{\lambda}{1-\lambda}} = [d(y_{2n-1}, y_{2n})]^{h},$$

where $h = \frac{\lambda}{1-\lambda} \in (0, 1)$. Similarly, we have

$$\begin{aligned} d(y_{2n+2}, y_{2n+1}) &= d(Sx_{2n+2}, Tx_{2n+1}) \\ &\leq \Big[\varphi\Big(\max\Big\{d(Ax_{2n+2}, Bx_{2n+1}), \frac{d(Ax_{2n+2}, Sx_{2n+2})d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Ax_{2n+2}, Bx_{2n+1})}, \\ &\qquad \frac{d(Ax_{2n+2}, Tx_{2n+1})d(Bx_{2n+1}, Ax_{2n+2})}{1 + d(A_{2n+2}, Bx_{2n+1})}\Big\}\Big)\Big]^{\lambda} \\ &\leq \Big[\varphi\Big(\max\Big\{d(y_{2n}, y_{2n+1}), d(y_{2n+2}, y_{2n+1}), d(y_{2n}, y_{2n+1})\Big\}\Big)\Big]^{\lambda} \\ &\leq \Big[\varphi\Big(\max\Big\{d(y_{2n}, y_{2n+1}) \cdot d(y_{2n+2}, y_{2n+1})\Big\}\Big)\Big]^{\lambda} \\ &\leq [d(y_{2n}, y_{2n+1})]^{\lambda} \cdot [d(y_{2n+2}, y_{2n+1})]^{\lambda}, \end{aligned}$$

which implies that

$$d(y_{2n+1}, y_{2n+2}) \leq [d(y_{2n}, y_{2n+1})]^{\frac{\lambda}{1-\lambda}} = [d(y_{2n}, y_{2n+1})]^h.$$

Thus it follows that, for all $n \geq 1$,

$$d(y_n, y_{n+1}) \leq [d(y_{n-1}, y_n)]^h \leq [d(y_{n-2}, y_{n-1})]^{h^2} \leq \cdots \leq [d(y_0, y_1)]^{h^n}.$$

Therefore, for all $n, m \in \mathbb{N}$ with $n < m$, by the multiplicative triangle inequality, we obtain

$$\begin{aligned}
d(y_n, y_m) &\leq d(y_n, y_{n+1}) \cdot d(y_{n+1}, y_{n+2}) \cdots d(y_{m-1}, y_m) \\
&\leq [d(y_0, y_1)]^{h^n} \cdot [d(y_0, y_1)]^{h^{n+1}} \cdots [d(y_0, y_1)]^{h^{m-1}} \\
&\leq [d(y_0, y_1)]^{\frac{h^n}{1-h}}.
\end{aligned}$$

This means that $d(y_n, y_m) \to 1$ as $n, m \to \infty$. Hence $\{y_n\}$ is a multiplicative Cauchy sequence in X. By the completeness of X, there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ also converge to a point $z \in X$.

Now, suppose that A is continuous. Then $\{AAx_{2n}\}$ and $\{ASx_{2n}\}$ converge to Az as $n \to \infty$. Since the mappings A and S are compatible on X, it follows from Proposition 2.19 that $\{SAx_{2n}\}$ converges to Az as $n \to \infty$.

Now, we claim that $z = Az$. Consider

$$\begin{aligned}
d(SAx_{2n}, Tx_{2n+1}) \leq \Big[\varphi\Big(\max\Big\{&d(A^2x_{2n}, Bx_{2n+1}), \frac{d(A^2x_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(A^2x_{2n}, Bx_{2n+1})} \\
&\frac{d(Ax^2x_{2n}, Tx_{2n+1})d(Bx_{2n+1}, A^2x_{2n})}{1 + d(A^2_{2n}, Bx_{2n+1})}\Big\}\Big)\Big]^\lambda.
\end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned}
d(Az, z) &\leq \Big[\varphi\Big(\max\Big\{d(Az, z), \frac{d(Az, Az)d(z, z)}{1 + d(Az, z)}, \frac{d(Az, z)d(z, Az)}{1 + d(Az, z)}\Big\}\Big)\Big]^\lambda \\
&\leq \Big[\varphi\Big(\max\Big\{d(Az, z), \frac{1}{d(Az, z)}, d(Az, z)\Big\}\Big)\Big]^\lambda \\
&= \varphi\Big([d(Az, z)]^\lambda\Big) = \Big[\varphi\Big(d(Az, z)\Big)\Big]^\lambda \\
&\leq [d(Az, z)]^\lambda.
\end{aligned}$$

Thus $d(Az, z) = 1$ and so $Az = z$. Again, from (d), we obtain

$$\begin{aligned}
d(Sz, Tx_{2n+1}) \leq \Big[\varphi\Big(\max\Big\{&d(Az, Bx_{2n+1}), \frac{d(Az, Sz)d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Az, Bx_{2n+1})} \\
&\frac{d(Az, Tx_{2n+1})d(Bx_{2n+1}, Az)}{1 + d(Az, Bx_{2n+1})}\Big\}\Big)\Big]^\lambda.
\end{aligned}$$

Letting $n \to \infty$ on both sides of the above inequality, we can obtain

$$\begin{aligned}
d(Sz, z) &\leq \Big[\varphi\Big(\max\Big\{d(Az, z), \frac{d(z, Sz)d(z, z)}{1 + d(Az, z)}, \frac{d(Az, z)d(z, Az)}{1 + d(z, z)}\Big\}\Big)\Big]^\lambda \\
&= \Big[\varphi\Big(d(Sz, z)\Big)\Big]^\lambda \\
&\leq [d(Sz, z)]^\lambda,
\end{aligned}$$

which implies that $Sz = z$. Thus, since $S(X) \subset B(X)$, there exists $u \in X$ such that $z = Sz = Bu$. By using (d) and $z = Sz = Az = Bu$, we can obtain

$$
\begin{aligned}
d(z,Tu) &= d(Sz,Tu)\\
&\le \Big[\varphi\Big(\max\Big\{d(Az,Bu),\frac{d(Az,Sz)d(Bu,Tu)}{1+d(Az,Bu)},\frac{d(Az,Tu)d(Bu,Az)}{1+d(Az,Bu)}\Big\}\Big)\Big]^{\lambda}\\
&\le \varphi\Big[\Big(d(z,Tu)\Big)\Big]^{\lambda}\\
&\le [d(z,Tu)]^{\lambda}.
\end{aligned}
$$

This implies that $z = Tu$. Since B and T are compatible on X and $Tu = z = Bu$, by Proposition 2.18, we have $BTu = TBu$ and hence $Bz = BTu = TBu = Tz$. Also, we have

$$
\begin{aligned}
d(z,Bz) &= d(z,Tz)\\
&\le \Big[\varphi\Big(\max\Big\{d(Az,Bz),\frac{d(Az,Sz)d(Bz,Tz)}{1+d(Az,Bz)},\frac{d(Az,Tz)d(Bz,Az)}{1+d(Az,Bz)}\Big\}\Big)\Big]^{\lambda}\\
&\le \Big[\varphi\Big(\max\Big\{d(z,Bz),\frac{1}{d(z,Bz)},d(z,Bz)\Big\}\Big)\Big]^{\lambda}\\
&= \Big[\varphi\Big(d(z,Bz)\Big)\Big]^{\lambda}\\
&\le [d(z,Bz)]^{\lambda}.
\end{aligned}
$$

This implies that $d(z,Bz) = 1$ and so $z = Bz$. Therefore, we obtain $z = Sz = Az = Tz = Bz$ and so z is a common fixed point of S, T, A and B. Similarly, we can also complete the proof when B is continuous.

Next, suppose that S is continuous. Then $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ converge to Az as $n \to \infty$. Since A and S are compatible on X, it follows from Proposition 2.19 that $\{ASx_{2n}\}$ converges to Az as $n \to \infty$. Now, we can consider

$$
\begin{aligned}
d(S^2x_{2n},Tx_{2n+1}) \le \Big[\varphi\Big(\max\Big\{&d(ASx_{2n},Bx_{2n+1}),\frac{d(ASx_{2n},S^2x_{2n})d(Bx_{2n+1},Tx_{2n+1})}{1+d(ASx_{2n},Bx_{2n+1})},\\
&\frac{d(ASx_{2n},Tx_{2n+1})d(Bx_{2n+1},ASx_{2n})}{1+d(ASx_{2n},Bx_{2n+1})}\Big\}\Big)\Big]^{\lambda}.
\end{aligned}
$$

Taking $n \to \infty$ on the both sides of the above inequality, we can obtain

$$
\begin{aligned}
d(Sz,z) &\le \Big[\varphi\Big(\max\Big\{d(Sz,z),\frac{d(Sz,Sz)d(z,z)}{1+d(Sz,z)},\frac{d(Sz,z)d(z,Sz)}{1+d(Sz,z)}\Big\}\Big)\Big]^{\lambda}\\
&\le \Big[\varphi\Big(\max\Big\{d(Sz,z),\frac{1}{d(Sz,z)},d(z,Sz)\Big\}\Big)\Big]^{\lambda}\\
&= \Big[\varphi\Big(d(Sz,z)\Big)\Big]^{\lambda} \le [d(Sz,z)]^{\lambda},
\end{aligned}
$$

which implies that $Sz = z$. Since $S(X) \subset B(X)$, there exists a point $v \in X$ such that $z = Sz = Bw$. Also, we have

$$
\begin{aligned}
d(S^2x_{2n},Tv) \le \Big[\varphi\Big(\max\Big\{&d(ASx_{2n},Bv),\frac{d(ASx_{2n},S^2x_{2n})d(Bv,Tv)}{1+d(ASx_{2n},Bv)},\\
&\frac{d(ASx_{2n},Tv)d(Bv,ASx)}{1+d(ASx_{2n},Bv)}\Big\}\Big)\Big]^{\lambda}.
\end{aligned}
$$

Letting $n \to \infty$ on both sides of the above inequality, we can obtain

$$
\begin{aligned}
d(z,Tv) &\le \Big[\varphi\Big(\max\Big\{d(Sz,z),\frac{d(Sz,Sz)d(z,Tv)}{1+d(Sz,z)},\frac{d(Sz,Tv)d(z,Sz)}{1+d(Sz,z)}\Big\}\Big)\Big]^{\lambda}\\
&= \Big[\varphi\Big(d(z,Tv)\Big)\Big]^{\lambda}\\
&\le d^{\lambda}(z,Tv),
\end{aligned}
$$

which implies that $z = Tv$. Since B and T are compatible on X and $Bv = Tv = z$, by Proposition 2.18, we have $BTv = TBv$ and hence $Bz = BTv = TBv = Tz$. Now, we have

$$d(Sx_{2n}, Tz) \le \Big[\varphi\Big(\max\Big\{d(Ax_{2n}, Bz), \frac{d(Ax_{2n}, Sx_{2n})d(Bz, Tz)}{1 + d(Ax_{2n}, Bz)}, \frac{d(Ax_{2n}, Tz)d(Bz, Ax_{2n})}{1 + d(Ax_{2n}, Bz)}\Big\}\Big)\Big]^{\lambda}.$$

Taking $n \to \infty$ on the two sides of the above inequality and using $Bz = Tz$, we can obtain

$$\begin{aligned} d(z, Tz) &\le \Big[\varphi\Big(\max\Big\{d(z, Tz), \frac{d(z, z)d(Tz, Tz)}{1 + d(z, Tz)}, \frac{d(z, Tz)d(Tz, z)}{1 + d(z, Tz)}\Big\}\Big)\Big]^{\lambda} \\ &\le \Big[\varphi\Big(\max\Big\{d(z, Tz), \frac{1}{d(z, Tz)}, d(Tz, z)\Big\}\Big)\Big]^{\lambda} \\ &= \Big[\varphi\Big(d(z, Tz)\Big)\Big]^{\lambda} \\ &\le [d(z, Tz)]^{\lambda}. \end{aligned}$$

which implies that $Tz = z$. Since $T(X) \subset A(X)$, there exists a point $w \in X$ such that $z = Tz = Aw$. Then we have

$$\begin{aligned} d(Sw, z) &= d(Sw, Tz) \\ &\le \Big[\varphi\Big(\max\Big\{d(Aw, Bz), \frac{d(Aw, Sw)d(Bz, Tz)}{1 + d(Aw, Bz)}, \frac{d(Aw, Tz)d(Bz, Aw)}{1 + d(Aw, Bz)}\Big\}\Big)\Big]^{\lambda} \\ &= \Big[\varphi\Big(d(Sw, z)\Big)\Big]^{\lambda} \\ &\le [d(Sw, z)]^{\lambda}, \end{aligned}$$

which implies that $Sw = z$. Since S and A are compatible on X and $Sw = Aw = z$, by Proposition 2.18, we have $ASw = SAw$ and hence $Az = ASw = SAw = Sz$. That is, $z = Az = Sz = Bz = Tz$. Therefore, z is common fixed point of S, T, A and B. Similarly, we can complete the proof when T is continuous.

Finally, for the proof of uniqueness of the common fixed point z, suppose that z and w are two common fixed points of S, T, A and B. Then, by using the condition (d), we have

$$\begin{aligned} d(z, w) &= d(Sz, Tw) \\ &\le \Big[\varphi\Big(\max\Big\{d(Az, Bw), \frac{d(Az, Sz)d(Bw, Tw)}{1 + d(Az, Bw)}, \frac{d(Az, Tw)d(Bz, Az)}{1 + d(Az, Bw)}\Big\}\Big)\Big]^{\lambda} \\ &= \Big[\varphi\Big(d(z, w)\Big)\Big]^{\lambda} \\ &\le [d(z, w)]^{\lambda}. \end{aligned}$$

This implies that $z = w$. Therefore, z is a unique common fixed point of S, T, A and B. This completes the proof. □

Here, we prove a common fixed point result for a generalized contractive mappings satisfying a compatibility of type (A).

Theorem 3.2. *Let S, T, A and B be four self-mappings of a complete multiplicative metric space (X, d) satisfying the conditions* (a), (c) *and* (d). *If the pairs (A, S) and (B, T) are compatible of type (A), then S, T, A and B have a unique common fixed point in X.*

Proof. Suppose that A is continuous on X. Since A and S are compatible of type (A). From Proposition 2.15(2), A and S are compatible and so the result easily follows from Theorem 3.1. Similarly, if B is continuous and B, T are compatible of type (A), then B and T are compatible and so the result easily follows from Theorem 3.1. We can get the same results when S or T is continuous. This completes the proof. □

Now, we prove the following result for a generalized contractive mappings satisfying a compatibility of type (B).

Theorem 3.3. *Let S, T, A and B be four self-mappings of a complete multiplicative metric space (X, d) satisfying the conditions* (a), (c) *and* (d). *If the pairs (A, S) and (B, T) are compatible of type (B), then S, T, A and B have a unique common fixed point in X.*

Proof. In the proof of Theorem 3.1, there is a multiplicative Cauchy sequence $\{y_n\}$ in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ also converge to a point $z \in X$.

Suppose that S is continuous. Then $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ converge to Sz as $n \to \infty$. Since the pair (A, S) is compatible of type (B), it follows from Proposition 2.21 that $\{AAx_{2n}\}$ converges to Sz as $n \to \infty$. Thus we have

$$\begin{aligned} d(SAx_{2n}, Tx_{2n+1}) \leq \Big[\varphi\Big(\max\Big\{ & d(A^2x_{2n}, Bx_{2n+1}), \frac{d(A^2x_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(A^2x_{2n}, Bx_{2n+1})}, \\ & \frac{d(Ax^2x_{2n}, Tx_{2n+1})d(Bx_{2n+1}, A^2x_{2n})}{1 + d(A^2_{2n}, Bx_{2n+1})}\Big\}\Big)\Big]^{\lambda}. \end{aligned}$$

Taking $n \to \infty$ on the both sides of the above inequality, we can obtain

$$\begin{aligned} d(Sz, z) &\leq \Big[\varphi\Big(\max\Big\{ d(Sz, z), \frac{d(Sz, Sz)d(z, z)}{1 + d(Sz, z)}, \frac{d(Sz, z)d(z, Sz)}{1 + d(Sz, z)}\Big\}\Big)\Big]^{\lambda} \\ &\leq \Big[\varphi\Big(\max\Big\{ d(Sz, z), \frac{1}{d(Sz, z)}, d(Sz, z)\Big\}\Big)\Big]^{\lambda} \\ &= \Big[\varphi\Big(d(Sz, z)\Big)\Big]^{\lambda} \\ &\leq [d(Sz, z)]^{\lambda}, \end{aligned}$$

which implies that $Sz = z$. Since $S(X) \subset B(X)$, there exists a point $u \in X$ such that $z = Sz = Bu$. Thus we have

$$\begin{aligned} d(SAx_{2n}, Tu) \leq \Big[\varphi\Big(\max\Big\{ & d(A^2x_{2n}, Bu), \frac{d(A^2x_{2n}, SAx_{2n})d(Bu, Tu)}{1 + d(A^2x_{2n}, Bu)}, \\ & \frac{d(Ax^2x_{2n}, Tu)d(Bu, A^2x_{2n})}{1 + d(A^2_{2n}, Bu)}\Big\}\Big)\Big]^{\lambda}. \end{aligned}$$

By taking $n \to \infty$ in both sides of the above inequality , we have

$$\begin{aligned} d(Sz, Tu) &\leq \Big[\varphi\Big(\max\Big\{ d(Sz, z)), \frac{d(Sz, Sz)d(Sz, Tu)}{1 + d(Sz, z)}, \frac{d(Sz, Tu)d(Sz, Sz)}{1 + d(Sz, z)}\Big\}\Big)\Big]^{\lambda} \\ &= \Big[\varphi\Big(d(Sz, Tz)\Big)\Big]^{\lambda} \\ &\leq [d(Sz, Tu)]^{\lambda}. \end{aligned}$$

This implies that $Tu = Sz$ and $z = Tu$. Since the pair (B, T) is compatible of type (B) and $Bu = z = Tu$, by Proposition 2.20, we have $TBu = BTu$ and so $Bz = BTu = TBu = Tz$. Now, we have

$$\begin{aligned} d(Sx_{2n}, Tz) \leq \Big[\varphi\Big(\max\Big\{ & d(Ax_{2n}, Bz), \frac{d(Ax_{2n}, Sx_{2n})d(Bz, Tz)}{1 + d(Ax_{2n}, Bz)}, \\ & \frac{d(Ax_{2n}, Tz)d(Bz, Ax_{2n})}{1 + d(Ax_{2n}, Bz)}\Big\}\Big)\Big]^{\lambda}. \end{aligned}$$

By taking the limit $n \to \infty$ on both sides of the above inequality, we have

$$\begin{aligned} d(z, Tz) &\leq \Big[\varphi\Big(\max\Big\{ d(z, Tz), \frac{d(z, z)d(Tz, Tz)}{1 + d(z, Tz)}, \frac{d(z, Tz)d(Tz, z)}{1 + d(z, Tz)}\Big\}\Big)\Big]^{\lambda} \\ &= \Big[\varphi\Big(d(z, Tz)\Big)\Big]^{\lambda} \\ &\leq [d(z, Tz)]^{\lambda}, \end{aligned}$$

which implies that $Tz = z$. Since $T(X) \subset A(X)$, there exists a point $v \in X$ such that $z = Tz = Av$. Thus we have

$$\begin{aligned} d(sv, z) &= d(Sv, Tz) \\ &\leq \Big[\varphi\Big(\max\Big\{d(Av, Bz), \frac{d(Av, Sv)d(Bz, Tz)}{1 + d(Av, Bz)}, \frac{d(Av, Tz)d(Bv, Av)}{1 + d(Av, Bz)}\Big\}\Big)\Big]^{\lambda} \\ &= \Big[\varphi\Big(d(Sv, z)\Big)\Big]^{\lambda} \\ &\leq [d(Sv, z)]^{\lambda}. \end{aligned}$$

This implies that $Sv = z$. Since the pair (A, S) is compatible of type (B) and $Sv = z = Av$, it follows from Proposition 2.20 that $Sz = SAv = ASv = Az$. Therefore, $Az = Bz = Sz = Tz = z$ and hence z is common fixed point of S, T, A and B.

Now, suppose that A is continuous. Then $\{AAx_{2n}\}$ and $\{ASx_{2n}\}$ converge to Az as $n \to \infty$. Since $\{A, S\}$ is compatible of type (B), it follows from Proposition 2.21 that $\{SSx_{2n}\}$ converges to Az as $n \to \infty$. Now, by the condition (d), we have

$$\begin{aligned} d(S^2x_{2n}, Tx_{2n+1}) \leq \Big[\varphi\Big(\max\Big\{&d(ASx_{2n}, Bx_{2n+1}), \frac{d(ASx_{2n}, S^2x_{2n})d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(ASx_{2n}, Bx_{2n+1})}, \\ &\frac{d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, ASx_{2n})}{1 + d(ASx_{2n}, Bx_{2n+1})}\Big\}\Big)\Big]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we have

$$d(Az, z) \leq [d(Az, z)]^{\lambda},$$

which implies $Az = z$. Also, by the condition (d), we have

$$\begin{aligned} d(Sz, Tx_{2n+1}) \leq \Big[\varphi\Big(\max\Big\{&d(Az, Bx_{2n+1}), \frac{d(Az, Sz)d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Az, Bx_{2n+1})}, \\ &\frac{d(Az, Tx_{2n+1})d(Bx_{2n+1}, Az)}{1 + d(Az, Bx_{2n+1})}\Big\}\Big)\Big]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we have

$$d(Sz, z) \leq [d(Sz, z)^{\lambda},$$

which implies $Sz = z$. Since $S(X) \subset B(X)$, there exists a point $w \in X$ such that $z = Sz = Bw$. Thus we have

$$\begin{aligned} d(z, Tw) &= d(Sz, Tu) \\ &\leq \Big[\varphi\Big(\max\Big\{d(Az, Bu)), \frac{d(Az, Sz)d(Bu, Tu)}{1 + d(Az, Bu)}, \frac{d(Az, Tu)d(Bu, Az)}{1 + d(Az, Bu)}\Big\}\Big)\Big]^{\lambda}. \end{aligned}$$

This implies that $z = Tw$. Since (B, T) is compatible of type (B) and $Bw = z = Tw$, from Proposition 2.20, it follows that $TBw = BTw$ and so $Bz = BTw = TBw = Tz$. Thus, by the condition (d), we have

$$d(Sz, Tz) \leq \Big[\varphi\Big(\max\Big\{d(z, Tz)), \frac{d(z, z)d(Bz, Tz)}{1 + d(z, Tz)}, \frac{d(z, Tz)d(Tz, z)}{1 + d(z, Tz)}\Big\}\Big)\Big]^{\lambda}.$$

This implies that $z = Tz$. Therefore, z is common fixed point of S, T, A and B. Similarly, we can complete the proof when B or T is continuous.

Finally, for the proof of uniqueness of the common fixed point z, if z and w are two common fixed points of S, T, A and B, then we have

$$\begin{aligned} d(z, w) &= d(Sz, Tw) \\ &\leq \Big[\varphi\Big(\max\Big\{d(Az, Bw), \frac{d(Az, Sz)d(Bw, Tw)}{1 + d(Az, Bw)}, \frac{d(Az, Tw)d(Bz, Az)}{1 + d(Az, Bw)}\Big\}\Big)\Big]^{\lambda} \\ &= \Big[\varphi\Big(d(z, w)\Big)\Big]^{\lambda} \\ &\leq [d(z, w)]^{\lambda}, \end{aligned}$$

which implies that $z = w$. Therefore, z is a unique common fixed point of S, T, A and B. This completes the proof. □

Now, we give an example to illustrate for the main results in this paper.

Example 3.4. Let $X = [1, \infty)$ with the usual multiplicative metric $d(x, y) = \left|\frac{x}{y}\right|$. Consider the following self-mappings S, T, A and B of X defined by $Sx = x$, $Tx = x^2$, $Bx = 2x^4 - 1$ and $Ax = 2x^2 - 1$ for all $x \in X$, respectively. Then we have the following:

(1) $S(X) = T(X) = B(X) = A(X) = X$;
(2) S, T, A and B are all continuous mappings;
(3) the pairs (A, S) and (B, T) are compatible and also, they are compatible mappings of type (A) and (B).

Consider a sequence $\{x_n\}$ defined in X by $x_n = 1 + \frac{1}{n}$ for each $n \geq 1$. Then $x_n \to 1$ as $n \to \infty$. Now, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 1 = t \in X$$

as $n \to \infty$. Also, we have

$$\begin{aligned} &\lim_{n\to\infty} d(ASx_n, SAx_n) = 1, \quad \lim_{n\to\infty} d(BTx_n, TBx_n) = 1, \\ &\lim_{n\to\infty} d(ASx_n, SSx_n) = 1, \quad \lim_{n\to\infty} d(SAx_n, AAx_n) = 1, \\ &\lim_{n\to\infty} d(BTx_n, TTx_n) = 1, \quad \lim_{n\to\infty} d(TBx_n, BBx_n) = 1. \end{aligned}$$

(4) For $\lambda = \frac{1}{3}$, we have

$$d(Sx, Ty) \leq \left[\varphi\Big(\max\Big\{d(Ax, By), \frac{d(Ax, Sx)d(By, Ty)}{1 + d(Ax, By)}, \frac{d(Ax, Ty)d(By, Ax)}{1 + d(Ax, By)}\Big\}\Big)\right]^{\lambda}$$

for all $x, y \in X$. Therefore, all the conditions of Theorems 3.1, 3.2 and 3.3 are satisfied and, in fact, 1 is a unique common fixed point of S, T, A and B.

If we take $\varphi(t) = kt$ in Theorem 3.1, then we can obtain the following result.

Corollary 3.5. *Let S, T, A and B be self-mappings of a complete multiplicative metric space X satisfying the following conditions:*

(a) *$S(X) \subset B(X)$ and $T(X) \subset A(X)$;*
(b) *the pairs (A, S) and (B, T) are compatible;*
(c) *one of S, T, A and B is continuous;*
(d) *for all $x, y \in X$, there exists $k \in [0, 1)$ such that*

$$d(Sx, Ty) \leq \Big(k \max\Big\{d(Ax, By), \frac{d(Ax, Sx)d(By, Ty)}{1 + d(Ax, By)}, \frac{d(Ax, Ty)d(By, Ax)}{1 + d(Ax, By)}\Big\}\Big)^{\lambda}.$$

Then S, T, A and B have a unique common fixed point in X.

If we take $\varphi(t) = kt$ in Theorem 3.2, then we have the following result.

Corollary 3.6. *Let S, T, A and B be four self-mappings of a complete multiplicative metric space (X, d) satisfying the conditions* (a), (c) *and* (d) *of the* Corollary 3.5. *If the pairs (A, S) and (B, T) are compatible of type (A), then S, T, A and B have a unique common fixed point in X.*

If we take $\varphi(t) = kt$ in Theorem 3.3, then we have the following result.

Corollary 3.7. *Let S, T, A and B be four self-mappings of a complete multiplicative metric space (X, d) satisfying the conditions* (a)*,* (c) *and* (d) *of* Theorem 3.3*. If the pairs (A, S) and (B, T) are compatible of type (B), then S, T, A and B have a unique common fixed point in X.*

If we take $A = B$ and $S = T$ in Corollary 3.5, then we have the following result.

Corollary 3.8. *Let T and A be two self-mappings of a complete multiplicative metric space X satisfying the following conditions:*

(a) *$T(X) \subset A(X)$;*
(b) *the pair T, A is compatible;*
(c) *one of T and A is continuous;*
(d) *for all $x, y \in X$, there exists $k \in [0, 1)$ such that*

$$d(Tx,Ty) \leq \Big(k \max\Big\{d(Ax,Ay), \frac{d(Ax,Tx)d(Ay,Ty)}{1+d(Ax,Ay)}, \frac{d(Ax,Ty)d(Ay,Ax)}{1+d(Ax,Ay)}\Big\}\Big)^{\lambda}.$$

Then T and A have a unique common fixed point in X.

In Theorems 3.1, 3.2 and 3.3, if we take $k = 1$, $S = T$ and $A = B = I_X$ (the identity mapping on X), then we have the following, which is Banach's Fixed Point Theorem in a complete multiplicative metric space (X, d).

Corollary 3.9. *Let T be a mapping of a complete multiplicative metric space (X, d) into itself satisfying the following condition:*

$$d(Tx, Ty) \leq [d(x, y)]^{\lambda}$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{2})$. Then T has a unique fixed point in X.

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