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Common fixed point results of generalized almost rational contraction mappings with an application

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Abstract

In this paper, we introduce the notion of generalized almost rational contraction with respect to a pair of self mappings on a complete metric space. Several common fixed point results for such mappings are proved. Our results extend and unify various results in the existing literature. An example and application to obtain the existence of a common solution of the system of functional equations arising in dynamic programming are also given in order to illustrate the effectiveness of the presented results.

Keywords: Point of coincidence, common fixed point, cyclic admissible mappings, almost contractions, weakly compatible mappings, functional equations.
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1. Introduction and Preliminaries

Fixed point theory plays a vital role in solving problems arising in various disciplines of mathematical analysis such as split feasibility problems, variational inequality problems, nonlinear optimization problems, equilibrium problems, complementarity problems, selection and matching problems, and problems of proving an existence of solution of integral and differential equations. One of the basic and the most widely applied

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result in metric fixed point theory is "Banach (or Banach-Cassioppoli) Contraction principle" due to Banach [8]. It states that if (X,d) is a complete metric space and $f : X \to X$ satisfies

$$d(fx, fy) \leq kd(x, y),$$

for all $x, y \in X$ with $k \in (0,1)$, then f has a unique fixed point. The basic idea of this principle rests in the use of successive approximations to establish the existence and uniqueness of solution of an operator equation $f(x) = x$, particularly it can be employed to prove the existence of solution of differential or integral equations. Due to its applications in mathematics and other related disciplines, Banach contraction principle has been generalized in many directions. Extensions of Banach contraction principle have been obtained either by generalizing the domain of the mapping or by extending the contractive condition on the mappings (see, [1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14, 15, 16, 18, 19, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 34, 35] and references therein).

In metric fixed point theory, contractive conditions on mappings play vital role in finding the solution of fixed point problems. It is a common practice to extend and generalize existing contractive conditions and then to employ it to obtain fixed point result in the framework of a metric space. Following this trend, Samet et al. [34] first introduced α-admissible mappings and then α-ψ-contractive type mappings to obtain some interesting generalizations of Banach contraction principle. For more results in this direction, we refer to [6, 13, 14, 16, 17, 20, 21, 22, 24, 28, 30, 32] and references mentioned therein. Recently, Alizadeh et al. [5] defined the concept of cyclic (α, β)-admissible mapping as follows:

Definition 1.1 ([5])**.** Let X be a nonempty set and $\alpha, \beta : X \longrightarrow [0, \infty)$. A self-mapping T on X is called cyclic (α, β)-admissible mapping if

(i) $\alpha(x) \geq 1$ for some x in X implies $\beta(Tx) \geq 1$,
(ii) $\beta(x) \geq 1$ for some x in X implies $\alpha(Tx) \geq 1$.

Definition 1.2 ([18, 25])**.** A pair (f, T) of self-mappings on a set X is said to be *weakly compatible* if f and T commute at their coincidence point (i.e. $fTx = Tfx,\ x \in X$ whenever $fx = Tx$).

A point $y \in X$ is called a *point of coincidence* of two self-mappings f and T on X if there exists a point $x \in X$such that $y = fx = Tx$. Also, $x \in X$ is called a *common fixed point* of mappings f and T if $x = fx = Tx$.

The notations $\mathcal{F}(f, T)$ and $\mathcal{C}(f, T)$ stand for the set of all common fixed points and the set of all coincidence points of f and T, respectively. In the sequel, we will indicate the set of all real numbers, the set of all non negative real numbers and the set of all natural numbers by the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively.

To obtain common fixed point results, we extend the definition of cyclic (α, β)-admissible mapping to a pair of two mappings as follows:

Definition 1.3. Let f, g, S and T be selfmaps of a nonempty set X and $\alpha, \beta : X \to \mathbb{R}^+$. Then the pair (f, g) is called cyclic (α, β)-admissible with respect to (S, T) (briefly, (f, g) is cyclic $(\alpha, \beta)_{(S,T)}$-admissible pair) if

(i) $\alpha(Sx) \geq 1$ for some $x \in X$ implies $\beta(fx) \geq 1$,
(ii) $\beta(Tx) \geq 1$ for some $x \in X$ implies $\alpha(gx) \geq 1$.

If we take $S = T = I_X$ (identity mapping on X), then Definition 1.3 reduces to following definition.

Definition 1.4. Let f and g be selfmaps of a nonempty set X and $\alpha, \beta : X \to \mathbb{R}^+$. Then the pair (f, g) is called cyclic (α, β)-admissible if

(i) $\alpha(x) \geq 1$ for some $x \in X$ implies $\beta(fx) \geq 1$,
(ii) $\beta(x) \geq 1$ for some $x \in X$ implies $\alpha(gx) \geq 1$.

On the other hand, Khan et al. [26] introduced and employed the notion of altering distance function to obtain some interesting fixed point results in metric spaces. Note that altering distance functions are continuous whereas Su [35] defined generalized altering distance function, not necessarily continuous, as follows:

Definition 1.5 ([35]). A mapping $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is called a generalized altering distance function if

(*i*) η is non-decreasing,
(*ii*) $\eta(t) = 0$ if and only if $t = 0$.

We set
$\digamma = \{\eta : \mathbb{R}^+ \to \mathbb{R}^+ : \eta$ is generalized altering distance$\}$.
$\Phi = \{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ : \varphi$ is a nondecreasing, right upper semi-continuous and for all $t > 0$, we have $\eta(t) > \varphi(t)$, where η is a generalized altering distance$\}$.
$\Theta = \{\theta : \mathbb{R}^+ \to \mathbb{R}^+ : \theta$ is continuous and $\theta(t) = 0$ iff $t = 0\}$.
Following the direction in [13], we denote set $\Psi_1 = \{\psi_1 : \mathbb{R}^{+^6} \to \mathbb{R}^+ : \psi_1$ satisfies (i)-$(iii)\}$, where

(*i*) ψ_1 is nondecreasing and continuous in each coordinate;
(*ii*) $\psi_1(t,t,t,t,t,t) \le t$ for all $t > 0$;
(*iii*) $\psi_1(t_1,t_2,t_3,t_4,t_5,t_6) = 0$ iff $t_i = 0$ for all $i \in \{1,2,3,4,5,6\}$.

$\Psi_2 = \{\psi_2 : \mathbb{R}^{+^4} \to \mathbb{R}^+ : \psi_2$ is continuous in each coordinate and if any one of the argument is zero, then $\psi_2(t_1,t_2,t_3,t_4) = 0\}$.
We now introduce generalized almost rational contraction mappings as follows:

Definition 1.6. Let f, g, S and T be selfmaps of a metric space (X,d), and (f,g) be a cyclic $(\alpha,\beta)_{(S,T)}$-admissible pair. We say that (f,g) is a generalized almost (S,T)-rational contraction pair if

$$\alpha(Sx)\beta(Ty) \ge 1 \text{ implies } \eta(d(fx,gy)) \le \varphi(M(x,y)) + L\theta(N(x,y)) \tag{1.1}$$

for all $x, y \in X$ and some $L \ge 0$, where $\eta \in \digamma$, $\varphi \in \Phi$, $\theta \in \Theta$ and

$$\begin{aligned} M(x,y) =& \psi_1\Bigg(d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{d(Sx,gy)+d(fx,Ty)}{2}, \\ & \frac{d(Ty,gy)[1+d(Sx,fx)]}{1+d(Sx,Ty)}, \frac{d(fx,Ty)[1+d(Sx,gy)]}{1+d(Sx,Ty)}\Bigg), \\ N(x,y) =& \psi_2(d(Sx,fx), d(Ty,gy), d(Sx,gy), d(fx,Ty)), \end{aligned}$$

with $\psi_1 \in \Psi_1$ and $\psi_2 \in \Psi_2$.

In this paper, we obtain some common fixed point results of generalized almost rational contraction pairs. Our results extend, generalize and unify comparable results in the existing literature. An example is presented to support the results obtained herein. We employ our results to give common fixed points of cyclic mappings on complete metric spaces. As an application of our results, the existence of common bounded solutions of a system of functional equations arising in dynamic programming are also investigated.

2. Main Results

Our main result is stated as follows.

Theorem 2.1. *Let f, g, S and T be selfmaps of a complete metric space (X,d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f,g) be a generalized almost (S,T)-rational contraction pair. Suppose that:*

(*a*) *there exists $x_0 \in X$ such that $\alpha(Sx_0) \ge 1$ and $\beta(Tx_0) \ge 1$;*

(b) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n) \geq 1$, $\beta(x_n) \geq 1$ *for all* n *and* $x_n \to x$ *as* $n \to \infty$, *then* $\alpha(x) \geq 1$ *and* $\beta(x) \geq 1$.

Then the pairs (f, S) *and* (g, T) *have a point of coincidence in* X. *Moreover, if*

(i) $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible,*
(ii) $\alpha(Su) \geq 1$ *and* $\beta(Tv) \geq 1$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have a common fixed point.*

Proof. Let x_0 be a given point in X such that $\alpha(Sx_0) \geq 1$ and $\beta(Tx_0) \geq 1$. Since $fX \subset TX$, we can choose a point $x_1 \in X$ such that $fx_0 = Tx_1$. Also, since $gX \subset SX$, there exists a point $x_2 \in X$ such that $gx_1 = Sx_2$. Continuing this way, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = gx_{2n+1} = Sx_{2n+2}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Since (f, g) is a cyclic $(\alpha, \beta)_{(S,T)}$-admissible pair and $\alpha(Sx_0) \geq 1$, we have $\beta(fx_0) = \beta(Tx_1) \geq 1$ which further implies $\alpha(gx_1) = \alpha(Sx_2) \geq 1$. Continuing this way, we obtain that $\alpha(Sx_{2n}) \geq 1$ and $\beta(Tx_{2n+1}) \geq 1$ for all $n \in \mathbb{N}_0$. Similarly, by $\beta(Tx_0) \geq 1$, we have $\beta(Tx_{2n}) \geq 1$ and $\alpha(Sx_{2n+1}) \geq 1$ for all $n \in \mathbb{N}_0$. This means that

$$\alpha(Sx_n) \geq 1 \text{ and } \beta(Tx_n) \geq 1, \quad \text{for all } n \in \mathbb{N}_0. \tag{2.2}$$

If $y_{2n} = y_{2n+1}$, by simple procedures, the proof is finished. Suppose that $y_{2n} \neq y_{2n+1}$ for all $n \in \mathbb{N}_0$. Now we show that

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0. \tag{2.3}$$

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (1.1) and using (2.1) and (2.2), we obtain

$$\begin{aligned} \eta(d(y_{2n}, y_{2n+1})) &= \eta(d(fx_{2n}, gx_{2n+1})) \\ &\leq \varphi(M(x_{2n}, x_{2n+1})) + L\theta(N(x_{2n}, x_{2n+1})), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) = &\psi_1\Bigg(d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1}), \\ &\frac{d(Sx_{2n}, gx_{2n+1}) + d(fx_{2n}, Tx_{2n+1})}{2}, \\ &\frac{d(Tx_{2n+1}, gx_{2n+1})[1 + d(Sx_{2n}, fx_{2n})]}{1 + d(Sx_{2n}, Tx_{2n+1})}, \\ &\frac{d(fx_{2n}, Tx_{2n+1})[1 + d(Sx_{2n}, gx_{2n+1})]}{1 + d(Sx_{2n}, Tx_{2n+1})}\Bigg) \\ = &\psi_1\Bigg(d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ &\frac{d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})}{2}, \\ &\frac{d(y_{2n}, y_{2n+1})[1 + d(y_{2n-1}, y_{2n})]}{1 + d(y_{2n-1}, y_{2n})}, \\ &\frac{d(y_{2n}, y_{2n})[1 + d(y_{2n-1}, y_{2n+1})]}{1 + d(y_{2n-1}, y_{2n})}\Bigg) \\ = &\psi_1\Bigg(d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \end{aligned}$$

$$
\begin{aligned}
&\left.\frac{d\left(y_{2n-1},y_{2n+1}\right)}{2},d\left(y_{2n},y_{2n+1}\right),0\right)\\
\leq&\psi_1\Bigg(d\left(y_{2n-1},y_{2n}\right),d\left(y_{2n-1},y_{2n}\right),d\left(y_{2n},y_{2n+1}\right),\\
&\frac{d\left(y_{2n-1},y_{2n}\right)+d\left(y_{2n},y_{2n+1}\right)}{2},d(y_{2n},y_{2n+1}),d(y_{2n},y_{2n+1})\Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
N\left(x_{2n},x_{2n+1}\right)&=\psi_2\left(d\left(Sx_{2n},fx_{2n}\right),d\left(Tx_{2n+1},gx_{2n+1}\right),d\left(Sx_{2n},gx_{2n+1}\right),d\left(fx_{2n},Tx_{2n+1}\right)\right)\\
&=\psi_2\left(d\left(y_{2n-1},y_{2n}\right),d\left(y_{2n},y_{2n+1}\right),d\left(y_{2n-1},y_{2n+1}\right),d\left(y_{2n},y_{2n}\right)\right)=0.
\end{aligned}
$$

If $d\left(y_{2n-1},y_{2n}\right)\leq d\left(y_{2n},y_{2n+1}\right)$ for some $n\in\mathbb{N}$, then by (2.4), we have

$$
\begin{aligned}
\eta\left(d\left(y_{2n},y_{2n+1}\right)\right)&\leq\varphi\left(M\left(x_{2n},x_{2n+1}\right)\right)\\
&\leq\varphi\Bigg(\psi_1\Bigg(d\left(y_{2n-1},y_{2n}\right),d\left(y_{2n-1},y_{2n}\right),d\left(y_{2n},y_{2n+1}\right),\\
&\qquad\frac{d\left(y_{2n-1},y_{2n}\right)+d\left(y_{2n},y_{2n+1}\right)}{2},d\left(y_{2n},y_{2n+1}\right),d\left(y_{2n},y_{2n+1}\right)\Bigg)\Bigg)\\
&\leq\varphi(\psi_1(d\left(y_{2n},y_{2n+1}\right),d\left(y_{2n},y_{2n+1}\right),d\left(y_{2n},y_{2n+1}\right),\\
&\qquad d\left(y_{2n},y_{2n+1}\right),d\left(y_{2n},y_{2n+1}\right),d\left(y_{2n},y_{2n+1}\right)))\\
&\leq\varphi\left(d\left(y_{2n},y_{2n+1}\right)\right),
\end{aligned}
$$

a contradiction to the fact $y_{2n}\neq y_{2n+1}$. So for all $n\in\mathbb{N}$, we have $d\left(y_{2n},y_{2n+1}\right)<d\left(y_{2n-1},y_{2n}\right)$.
From (2.4), we deduce

$$
\eta\left(d\left(y_{2n},y_{2n+1}\right)\right)\leq\varphi\left(d\left(y_{2n-1},y_{2n}\right)\right). \tag{2.5}
$$

Putting $x=x_{2n+1}$ and $y=x_{2n+2}$ in (1.1) and following arguing similar to those given above, we get

$$
\eta\left(d\left(y_{2n+1},y_{2n+2}\right)\right)\leq\varphi\left(d\left(y_{2n},y_{2n+1}\right)\right). \tag{2.6}
$$

From (2.5) and (2.6), we conclude

$$
\eta\left(d\left(y_{n},y_{n+1}\right)\right)\leq\varphi\left(d\left(y_{n-1},y_{n}\right)\right). \tag{2.7}
$$

It follows that the sequence $\{d\left(y_n,y_{n+1}\right)\}$ is decreasing and bounded below. Hence, there exists $r\geq 0$ such that $\lim_{n\to\infty}d\left(y_n,y_{n+1}\right)=r$. If $r>0$, then taking limit as $n\to\infty$ on both sides of (2.7), we have

$$
\begin{aligned}
\eta\left(r\right)&\leq\lim_{n\to\infty}\eta\left(d\left(y_n,y_{n+1}\right)\right)\\
&\leq\lim_{n\to\infty}\varphi\left(d\left(y_{n-1},y_n\right)\right)\leq\varphi\left(r\right),
\end{aligned}
$$

a contradiction and hence $r=0$, that is, the equation (2.3) holds.

Now we show that $\{y_n\}$ is a Cauchy sequence in X. For that, it is sufficient to show that the sequence $\{y_{2n}\}$ is Cauchy in X. Assume on contrary that $\{y_{2n}\}$ is not a Cauchy sequence. Then, there exists some $\varepsilon>0$ for which we can find two subsequences $\{y_{2m_k}\}$ and $\{y_{2n_k}\}$ of $\{y_{2n}\}$ such that n_k is the smallest index satisfying $n_k>m_k>k$ and

$$
d\left(y_{2n_k},y_{2m_k}\right)\geq\varepsilon\quad\text{and}\quad d\left(y_{2n_k-1},y_{2m_k}\right)<\varepsilon. \tag{2.8}
$$

Using the triangular inequality and (2.8),

$$
\varepsilon\leq d\left(y_{2n_k},y_{2m_k}\right)\leq d\left(y_{2n_k},y_{2n_k-1}\right)+d\left(y_{2n_k-1},y_{2m_k}\right)
$$

$$< d\left(y_{2n_k}, y_{2n_k-1}\right) + \varepsilon.$$

Letting $n \to \infty$ in the above inequality and using (2.3), we obtain

$$\lim_{k\to\infty} d\left(y_{2n_k}, y_{2m_k}\right) = \varepsilon. \tag{2.9}$$

Also, from the triangular inequality, we have

$$\left|d\left(y_{2n_k}, y_{2m_k+1}\right) - d\left(y_{2n_k}, y_{2m_k}\right)\right| \leq d\left(y_{2m_k}, y_{2m_k+1}\right).$$

On taking limit as $k \to \infty$ on both sides of above inequality and using (2.3) and (2.9), we get

$$\lim_{k\to\infty} d\left(y_{2n_k}, y_{2m_k+1}\right) = \varepsilon. \tag{2.10}$$

Similarly, it is easy to show that

$$\lim_{k\to\infty} d\left(y_{2n_k-1}, y_{2m_k}\right) = \lim_{k\to\infty} d\left(y_{2n_k-1}, y_{2m_k+1}\right) = \varepsilon. \tag{2.11}$$

Now, since

$$\begin{aligned}
M\left(x_{2n_k}, x_{2m_k+1}\right) =&\psi_1\Bigg(d\left(Sx_{2n_k}, Tx_{2m_k+1}\right), d\left(Sx_{2n_k}, fx_{2n_k}\right), d\left(Tx_{2m_k+1}, gx_{2m_k+1}\right),\\
&\frac{d\left(Sx_{2n_k}, gx_{2m_k+1}\right) + d\left(fx_{2n_k}, Tx_{2m_k+1}\right)}{2},\\
&\frac{d\left(Tx_{2m_k+1}, gx_{2m_k+1}\right)\left[1 + d\left(Sx_{2n_k}, fx_{2n_k}\right)\right]}{1 + d\left(Sx_{2n_k}, Tx_{2m_k+1}\right)},\\
&\frac{d\left(fx_{2n_k}, Tx_{2m_k+1}\right)\left[1 + d\left(Sx_{2n_k}, gx_{2m_k+1}\right)\right]}{1 + d\left(Sx_{2n_k}, Tx_{2m_k+1}\right)}\Bigg)\\
=&\psi_1\Bigg(d\left(y_{2n_k-1}, y_{2m_k}\right), d\left(y_{2n_k-1}, y_{2n_k}\right), d\left(y_{2m_k}, y_{2m_k+1}\right),\\
&\frac{d\left(y_{2n_k-1}, y_{2m_k+1}\right) + d\left(y_{2n_k}, y_{2m_k}\right)}{2},\\
&\frac{d\left(y_{2m_k}, y_{2m_k+1}\right)\left[1 + d\left(y_{2n_k-1}, y_{2n_k}\right)\right]}{1 + d\left(y_{2n_k-1}, y_{2m_k}\right)},\\
&\frac{d\left(y_{2n_k}, y_{2m_k}\right)\left[1 + d\left(y_{2n_k-1}, y_{2m_k+1}\right)\right]}{1 + d\left(y_{2n_k-1}, y_{2m_k}\right)}\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
N\left(x_{2n_k}, x_{2m_k+1}\right) =&\psi_2(d\left(Sx_{2n_k}, fx_{2n_k}\right), d\left(Tx_{2m_k+1}, gx_{2m_k+1}\right), d\left(Sx_{2n_k}, gx_{2m_k+1}\right), d\left(fx_{2n_k}, Tx_{2m_k+1}\right))\\
=&\psi_2\left(d\left(y_{2n_k-1}, y_{2n_k}\right), d\left(y_{2m_k}, y_{2m_k+1}\right), d\left(y_{2n_k-1}, y_{2m_k+1}\right), d\left(y_{2n_k}, y_{2m_k}\right)\right),
\end{aligned}$$

then, letting $n \to \infty$, we deduce that

$$\begin{aligned}
\lim_{k\to\infty} M\left(x_{2n_k}, x_{2m_k+1}\right) =&\psi_1\left\{\varepsilon, 0, 0, \varepsilon, 0, \varepsilon\right\} \leq \varepsilon, \text{ and}\\
\lim_{k\to\infty} N\left(x_{2n_k}, x_{2m_k+1}\right) =&\psi_2\left\{0, 0, \varepsilon, \varepsilon\right\} = 0.
\end{aligned}$$

From (2.2), we have $\alpha\left(Sx_{2n_k}\right)\beta\left(Tx_{2m_k+1}\right) \geq 1$. Substituting $x = x_{2n_k}$ and $y = x_{2m_k+1}$ in (1.1), we get

$$\begin{aligned}
\eta\left(d\left(y_{2n_k}, y_{2m_k+1}\right)\right) =&\eta\left(d\left(fx_{2n_k}, gx_{2m_k+1}\right)\right)\\
\leq&\varphi\left(M\left(x_{2n_k}, x_{2m_k+1}\right)\right) + L\theta\left(N\left(x_{2n_k}, x_{2m_k+1}\right)\right).
\end{aligned}$$

On taking limit as $k \to \infty$, we have

$$\begin{aligned}\eta(\varepsilon) &\leq \lim_{k\to\infty} \eta\left(d\left(y_{2n_k}, y_{2m_k+1}\right)\right)\\ &\leq \lim_{k\to\infty} \varphi\left(M\left(x_{2n_k}, x_{2m_k+1}\right)\right) + L \lim_{k\to\infty} \theta\left(N\left(x_{2n_k}, x_{2m_k+1}\right)\right)\\ &= \lim_{k\to\infty} \varphi\left(M\left(x_{2n_k}, x_{2m_k+1}\right)\right) \leq \varphi(\varepsilon),\end{aligned}$$

a contradiction and so $\{y_{2n}\}$ is a Cauchy sequence in X. Thus, from the completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to\infty} y_n = z. \tag{2.12}$$

From (2.1) and (2.12), we obtain

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{2.13}$$

We now show that z is a common fixed point of f,g,S and T.

Since $g(X) \subset S(X)$, we can choose a point u in X such that $z = Su$. Suppose that $d(z, fu) \neq 0$.

By (2.2), (2.13) and the condition (b), we have $\alpha(Su)\beta(Tx_{2n+1}) \geq 1$. Then, putting $x = u$ and $y = x_{2n+1}$ in (1.1), we get

$$\eta\left(d\left(fu, gx_{2n+1}\right)\right) \leq \varphi\left(M\left(u, x_{2n+1}\right)\right) + L\theta\left(N\left(u, x_{2n+1}\right)\right), \tag{2.14}$$

where

$$\begin{aligned}M\left(u, x_{2n+1}\right) = \psi_1\Bigg(& d\left(Su, Tx_{2n+1}\right), d\left(Su, fu\right), d\left(Tx_{2n+1}, gx_{2n+1}\right),\\ & \frac{d\left(Su, gx_{2n+1}\right) + d\left(fu, Tx_{2n+1}\right)}{2},\\ & \frac{d\left(Tx_{2n+1}, gx_{2n+1}\right)\left[1 + d\left(Su, fu\right)\right]}{1 + d\left(Su, Tx_{2n+1}\right)},\\ & \frac{d\left(fu, Tx_{2n+1}\right)\left[1 + d\left(Su, gx_{2n+1}\right)\right]}{1 + d\left(Su, Tx_{2n+1}\right)}\Bigg)\\ \to \psi_1 &\left(0, d(z, fu), 0, \frac{d(fu, z)}{2}, 0, d(fu, z)\right),\end{aligned}$$

and

$$\begin{aligned}N\left(u, x_{2n+1}\right) &= \psi_2\left(d\left(Su, fu\right), d\left(Tx_{2n+1}, gx_{2n+1}\right), d\left(Su, gx_{2n+1}\right), d\left(fu, Tx_{2n+1}\right)\right)\\ &\to \psi_2\left(d(z, fu), 0, 0, d(fu, z)\right),\end{aligned}$$

as $n \to \infty$. From (2.14), we have

$$\begin{aligned}\eta\left(d(fu, z)\right) &\leq \lim_{n\to\infty} \eta\left(d\left(fu, gx_{2n+1}\right)\right)\\ &\leq \lim_{n\to\infty} \varphi\left(M\left(u, x_{2n+1}\right)\right) + L \lim_{n\to\infty} \theta\left(N\left(u, x_{2n+1}\right)\right)\\ &= \lim_{n\to\infty} \varphi\left(M\left(u, x_{2n+1}\right)\right)\\ &\leq \varphi\left(\psi_1\left(0, d(z, fu), 0, \frac{d(fu, z)}{2}, 0, d(fu, z)\right)\right)\\ &\leq \varphi\left(d(fu, z)\right),\end{aligned}$$

a contradiction and hence $d(fu, z) = 0$, that is $fu = z$, and so $u \in \mathcal{C}(f, S)$.

Similarly, since $f(X) \subset T(X)$, we can choose a point v in X such that $z = Tv$. Suppose that $d(z, gv) \neq 0$.

By (2.2), (2.13) and the condition (b), we have $\alpha(Sx_{2n})\beta(Tv) \geq 1$. Then, substituting $x = x_{2n}$ and $y = v$ in (1.1), we deduce

$$\eta(d(fx_{2n}, gv)) \leq \varphi(M(x_{2n}, v)) + L\theta(N(x_{2n}, v)), \tag{2.15}$$

where

$$\begin{aligned} M(x_{2n}, v) = \psi_1\Bigg(& d(Sx_{2n}, Tv), d(Sx_{2n}, fx_{2n}), d(Tv, gv), \\ & \frac{d(Sx_{2n}, gv) + d(fx_{2n}, Tv)}{2}, \frac{d(Tv, gv)[1 + d(Sx_{2n}, fx_{2n})]}{1 + d(Sx_{2n}, Tv)}, \\ & \frac{d(fx_{2n}, Tv)[1 + d(Sx_{2n}, gv)]}{1 + d(Sx_{2n}, Tv)}\Bigg) \\ \to \psi_1 & \left(0, 0, d(z, gv), \frac{d(z, gv)}{2}, d(z, gv), 0\right), \end{aligned}$$

and

$$\begin{aligned} N(x_{2n}, v) &= \psi_2(d(Sx_{2n}, fx_{2n}), d(Tv, gv), d(Sx_{2n}, gv), d(fx_{2n}, Tv)) \\ &\to \psi_2(0, d(z, gv), d(z, gv), 0), \end{aligned}$$

as $n \to \infty$. Now by (2.15), we have

$$\begin{aligned} \eta(d(z, gv)) &\leq \lim_{n\to\infty} \eta(d(fx_{2n}, gv)) \\ &\leq \lim_{n\to\infty} \varphi(M(x_{2n}, v)) + L \lim_{n\to\infty} \theta(N(x_{2n}, v)) \\ &= \lim_{n\to\infty} \varphi(M(x_{2n}, v)) \\ &\leq \varphi\left(\psi_1\left(0, 0, d(z, gv), \frac{d(z, gv)}{2}, d(z, gv), 0\right)\right) \\ &\leq \varphi(d(z, gv)), \end{aligned}$$

a contradiction and hence $d(z, gv) = 0$, that is $z = gv$, and so $v \in \mathcal{C}(g, T)$.

Thus, $z = fu = Su = gv = Tv$. By the weak compatibility of the pairs (f, S) and (g, T), we obtain that $fz = Sz$ and $gz = Tz$.

Since $z \in \mathcal{C}(f, S)$ and $v \in \mathcal{C}(g, T)$, by (ii), we have $\alpha(Sz)\beta(Tv) \geq 1$ and so, from (1.1)

$$\begin{aligned} \eta(d(fz, z)) &= \eta(d(fz, gv)) \\ &\leq \varphi(M(z, v)) + L\theta(N(z, v)), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} M(z, v) = \psi_1\Bigg(& d(Sz, Tv), d(Sz, fz), d(Tv, gv), \\ & \frac{d(Sz, gv) + d(fz, Tv)}{2}, \frac{d(Tv, gv)[1 + d(Sz, fz)]}{1 + d(Sz, Tv)}, \\ & \frac{d(fz, Tv)[1 + d(Sz, gv)]}{1 + d(Sz, Tv)}\Bigg) \\ = \psi_1 & (d(fz, z), 0, 0, d(fz, z), 0, d(fz, z)) \leq d(fz, z), \end{aligned}$$

and

$$N(z, v) = \psi_2(d(Sz, fz), d(Tv, gv), d(Sz, gv), d(fz, Tv))$$

$$= \psi_2 (0, 0, d (fz, z), d (fz, z)) = 0.$$

By (2.16), we get

$$\eta (d (fz, z)) \leq \varphi (d (fz, z)),$$

which implies that $z = fz$, and so $z = fz = Sz$. Similarly, it can be shown that $z = gz = Tz$. This completes the proof. □

Corollary 2.2. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f, g) be a cyclic $(\alpha, \beta)_{(S,T)}$-admissible pair such that*

$$\alpha (Sx) \beta (Ty) \eta (d (fx, gy)) \leq \varphi (M_{\max} (x, y)) + L\theta (N_{\min} (x, y)), \tag{2.17}$$

for all $x, y \in X$ and some $L \geq 0$, where $\eta \in \digamma$, $\varphi \in \Phi$, $\theta \in \Theta$ and

$$M_{\max} (x, y) = \max \Bigg(d (Sx, Ty), d (Sx, fx), d (Ty, gy), \frac{d (Sx, gy) + d (fx, Ty)}{2}, \frac{d (Ty, gy) [1 + d (Sx, fx)]}{1 + d (Sx, Ty)}, \frac{d (fx, Ty) [1 + d (Sx, gy)]}{1 + d (Sx, Ty)} \Bigg)$$

and

$$N_{\min} (x, y) = \min (d (Sx, fx), d (Ty, gy), d (Sx, gy), d (fx, Ty)).$$

Assume that the conditions (a) and (b) in Theorem 2.1 *are satisfied. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the conditions (i) and (ii) in* Theorem 2.1 *hold, then f, g, S and T have a common fixed point.*

Proof. Let $\alpha (Sx) \beta (Ty) \geq 1$ for $x, y \in X$. If we take $\psi_1 (t_1, t_2, t_3, t_4, t_5, t_6) = \max \{t_1, t_2, t_3, t_4, t_5, t_6\}$ and $\psi_2 (t_1, t_2, t_3, t_4) = \min \{t_1, t_2, t_3, t_4\}$ in Theorem 2.1, then by (2.17), we have

$$\eta (d (fx, gy)) \leq \varphi (M (x, y)) + L\theta (N (x, y)).$$

Hence the result follows from Theorem 2.1. □

If we take $\alpha (Sx) = \beta (Ty) = 1$, and $\eta (t) = t$, $\varphi (t) = \delta t$ and $\theta (t) = t$ in Corollary 2.2, we have a generalized version of Theorem 1 in [10],

Theorem 2.3. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$ and $g(X) \subset S(X)$. Suppose that there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that*

$$d (fx, gy) \leq \delta M_{\max} (x, y) + LN_{\min} (x, y), \tag{2.18}$$

for all $x, y \in X$. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a common fixed point.

If we take $L = 0$ in Corollary 2.2, we have the following result.

Corollary 2.4. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f, g) be a cyclic $(\alpha, \beta)_{(S,T)}$-admissible pair such that*

$$\alpha (Sx) \beta (Ty) \eta (d (fx, gy)) \leq \varphi (M_{\max} (x, y)), \tag{2.19}$$

for all $x, y \in X$, where $\eta \in \digamma$ and $\varphi \in \Phi$. Assume that the conditions (a) and (b) in Theorem 2.1 *are satisfied. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the conditions (i) and (ii) in* Theorem 2.1 *hold, then f, g, S and T have a common fixed point.*

If we take $\varphi (t) = \eta (t) - \phi (t)$ in Corollary 2.4, we have the following corollary.

Corollary 2.5. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f, g) be a cyclic $(\alpha, \beta)_{(S,T)}$-admissible pair such that*

$$\alpha(Sx)\,\beta(Ty)\,\eta(d(fx, gy)) \leq \eta(M_{\max}(x, y)) - \phi(M_{\max}(x, y)), \tag{2.20}$$

for all $x, y \in X$, where $\eta \in F$ and $\phi \in \Phi$. Assume that the conditions (a) and (b) in Theorem 2.1*are satisfied. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the conditions (i) and (ii) in* Theorem 2.1 *hold, then f, g, S and T have a common fixed point.*

If we take $\alpha(Sx) = \beta(Ty) = 1$ in Corollary 2.5, we have a generalized version of Theorem 2.1 in [2],

Theorem 2.6. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$. Suppose that for any $x, y \in X$, there exist $\eta \in F$ and $\phi \in \Phi$ such that*

$$\eta(d(fx, gy)) \leq \eta(M_{\max}(x, y)) - \phi(M_{\max}(x, y)). \tag{2.21}$$

Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a common fixed point.

If we take $\eta(t) = t$ in Corollary 2.4, we have the following result.

Corollary 2.7. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f, g) be a cyclic $(\alpha, \beta)_{(S,T)}$-admissible pair such that*

$$\alpha(Sx)\,\beta(Ty)\,d(fx, gy) \leq \varphi(M_{\max}(x, y)), \tag{2.22}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Assume that the conditions (a) and (b) in Theorem 2.1 *are satisfied. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the conditions (i) and (ii) in* Theorem 2.1 *hold, then f, g, S and T have a common fixed point.*

For the uniqueness of the fixed point of a generalized almost (S, T)-rational contraction, we will consider the following hypothesis.

(H) For all $x, y \in \mathcal{F}(f, g, S, T)$, we have $\alpha(Sx) \geq 1$ and $\beta(Ty) \geq 1$.

Theorem 2.8. *Adding condition (H) to the hypotheses of* Theorem 2.1*, we obtain the uniqueness of the common fixed point of f, g, S and T.*

Proof. Suppose that $x = fx = gx = Sx = Tx$ and $y = fy = gy = Sy = Ty$. Then, from (H), since $\alpha(Sx)\,\beta(Ty) \geq 1$, applying (1.1), we obtain

$$\begin{aligned} \eta(d(x, y)) =& \eta(d(fx, gy)) \\ \leq& \varphi(M(x, y)) + L\theta(N(x, y)), \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} M(x, y) =& \psi_1\Big(d(Sx, Ty), d(Sx, fx), d(Ty, gy), \\ & \frac{d(Sx, gy) + d(fx, Ty)}{2}, \frac{d(Ty, gy)[1 + d(Sx, fx)]}{1 + d(Sx, Ty)}, \\ & \frac{d(fx, Ty)[1 + d(Sx, gy)]}{1 + d(Sx, Ty)} \Big) \\ =& \psi_1(d(x, y), 0, 0, d(x, y), 0, d(x, y)) \leq d(x, y), \end{aligned}$$

and

$$\begin{aligned}N(x,y) &= \psi_2(d(Sx,fx), d(Ty,gy), d(Sx,gy), d(fx,Ty))\\ &= \psi_2(0,0,d(x,y), d(x,y)) = 0.\end{aligned}$$

From (2.23), we have

$$\eta(d(x,y)) \leq \varphi(d(x,y)),$$

which implies that $d(x,y) = 0$, that is, $x = y$. □

Remark 2.9. Adding condition (H) to the hypotheses of Corollaries 2.2, 2.4, 2.5 and 2.7, we obtain the uniqueness of the common fixed point of f, g, S and T.

If we choose $S = T = I_X$ in Corollary 2.2, we have the following result.

Corollary 2.10. *Let f and g be selfmaps of a complete metric space (X,d) and (f,g) be a cyclic (α,β)-admissible pair such that*

$$\alpha(x)\beta(y)\eta(d(fx,gy)) \leq \varphi(M_{fg}(x,y)) + L\theta(N_{fg}(x,y))$$

for all $x, y \in X$ and some $L \geq 0$, where $\eta \in F$, $\varphi \in \Phi$, $\theta \in \Theta$ and

$$\begin{aligned}M_{fg}(x,y) &= \max\left(d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(fx,y)}{2},\right.\\ &\left.\frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(fx,y)[1+d(x,gy)]}{1+d(x,y)}\right),\\ N_{fg}(x,y) &= \min(d(x,fx), d(y,gy), d(x,gy), d(fx,y)).\end{aligned}$$

Assume also that the following conditions are satisfied:

(a) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;*
(b) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\alpha(x) \geq 1$ and $\beta(x) \geq 1$.*

Then f and g have a common fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ whenever $x, y \in \mathcal{F}(f,g)$, then f and g have a unique common fixed point.

Now, we furnish the following example which illustrates Theorem 2.1 as well as Theorem 2.8.

Example 2.11. Let $X = \mathbb{R}$ be endowed with the usual metric and $\eta, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\eta(t) = t$ and $\varphi(t) = \frac{7t}{8}$. Also, let $\psi_1(t_1,t_2,t_3,t_4,t_5,t_6) = \max\{t_1,t_2,t_3,t_4,t_5,t_6\}$ for all $t_1,t_2,t_3,t_4,t_5,t_6 \geq 0$. Define the self-mappings f, g, S and T on X by

$$fx = \begin{cases} -\frac{2x}{5} & \text{if } x \in \left[-\frac{1}{2}, 0\right], \\ \frac{7x}{10} & \text{if } x \in \mathbb{R}\setminus\left[-\frac{1}{2}, 0\right], \end{cases} \quad \text{and} \quad gx = \begin{cases} -\frac{2x}{15} & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \frac{7x}{10} & \text{if } x \in \mathbb{R}\setminus\left[0, \frac{1}{2}\right], \end{cases}$$

$$Sx = \begin{cases} -\frac{4x}{5} & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \frac{x}{5} & \text{if } x \in \mathbb{R}\setminus\left[0, \frac{1}{2}\right], \end{cases} \quad \text{and} \quad Tx = \begin{cases} -\frac{4x}{5} & \text{if } x \in \left[-\frac{1}{2}, 0\right], \\ \frac{x}{10} & \text{if } x \in \mathbb{R}\setminus\left[-\frac{1}{2}, 0\right]. \end{cases}$$

Note that $f(X) \subset T(X)$ and $g(X) \subset S(X)$, $\{f, S\}$ and $\{g, T\}$ are weakly compatible.
Define $\alpha, \beta : X \to \mathbb{R}^+$ as

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in \left[-\frac{2}{5}, 0\right], \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{2}{5}\right], \\ 0 & \text{otherwise}. \end{cases}$$

If there exists $x \in X$ such that $\alpha(Sx) \geq 1$, then $Sx \in \left[-\frac{2}{5}, 0\right]$ and hence $x \in \left[0, \frac{1}{2}\right]$. By the definitions of f and β, we have $fx \in \left[0, \frac{2}{5}\right]$ and so $\beta(fx) \geq 1$. If for some $x \in X$, we have $\beta(Tx) \geq 1$ then $Tx \in \left[0, \frac{2}{5}\right]$ and hence $x \in \left[-\frac{1}{2}, 0\right]$. By the definitions of g and α, we have $gx \in \left[-\frac{2}{5}, 0\right]$ and so $\alpha(gx) \geq 1$. Therefore, (f, g) is a cyclic $(\alpha, \beta)_{(S,T)}$-admissible pair. Moreover, $\alpha(Sx_0) \geq 1$ and $\beta(Tx_0) \geq 1$ holds for $x_0 = 0$.

If $\{x_n\}$ is any sequence in X such that $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then by the definition of α and β, we have $x_n \in \left[-\frac{2}{5}, 0\right] \cap \left[0, \frac{2}{5}\right] = \{0\}$ for all $n \in \mathbb{N}$ and so $x \in \{0\}$ which implies that $\alpha(x) \geq 1$ and $\beta(x) \geq 1$.

Now, we prove that (f, g) is a generalized almost (S, T)-rational contraction pair. Let $\alpha(Sx)\beta(Ty) \geq 1$. Then $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left[-\frac{1}{2}, 0\right]$, and so

$$\begin{aligned}
\eta(d(fx, gy)) = |fx - gy| = & \left|\frac{7x}{10} - \frac{7y}{10}\right| \\
= & \frac{7}{8} \cdot \frac{4}{5}|x - y| = \frac{7}{8} d(Sx, Ty) \\
= & \varphi(d(Sx, Ty)) \leq \varphi(M(x, y)) \\
\leq & \varphi(M(x, y)) + L\theta(N(x, y)),
\end{aligned}$$

for some $L \geq 0$ and $\theta \in \Theta$. Note that assumption (ii) of Theorem 2.1 and the condition (H) also hold. Thus, by Theorems 2.1 and 2.8, f, g, S and T have a unique common fixed point which is 0.

3. Common fixed points of cyclic mappings

Let A and B be two nonempty subsets of a set X. A mapping $f : X \to X$ is said to be cyclic (with respect to A and B) if $f(A) \subseteq B$ and $f(B) \subseteq A$.

The fixed point theory of cyclic contractive mappings is a recent development. Kirk et al. [27] in 2003 introduced a class of mappings which satisfy contraction condition for points x and y where $x \in A$ and $y \in B$. For more work in this direction, we refer to [19, 31, 33].

Definition 3.1. The mappings $f, g, S, T : A \cup B \to A \cup B$ are called cyclic if $fA \subseteq TB$ and $gB \subseteq SA$, where A, B are nonempty subsets of a metric space (X, d).

As an application of our results in the previous section, we obtain some fixed point results of cyclic mappings in the setting of complete metric spaces.

Theorem 3.2. *Let A and B be two closed subsets of complete metric space X such that $A \cap B \neq \emptyset$ and $f, g, S, T : A \cup B \to A \cup B$ with $fA \subseteq TB$ and $gB \subseteq SA$. Assume that*

$$\eta(d(fx, gy)) \leq \varphi(M(x, y)) + L\theta(N(x, y)) \tag{3.1}$$

for any $x \in A$ and $y \in B$ and some $L \geq 0$, where $\eta \in F$, $\varphi \in \Phi$ and $\theta \in \Theta$. If S and T are one to one then the pairs (f, S) and (g, T) have a coincidence point in $A \cap B$. If $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a common fixed point in $A \cap B$.

Proof. Define $\alpha, \beta : X \to \mathbb{R}^+$ by

$$\alpha(x) = \begin{cases} 1, & x \in SA, \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \beta(x) = \begin{cases} 1, & x \in TB, \\ 0, & \text{otherwise} \end{cases}.$$

Let $\alpha(Sx)\beta(Ty) \geq 1$. Then $Sx \in SA$ and $Ty \in TB$. Since S and T are one to one, we have $x \in A$ and $y \in B$. From (3.1), we obtain that

$$\eta(d(fx, gy)) \leq \varphi(M(x, y)) + L\theta(N(x, y)).$$

Let $\alpha(Sx) \geq 1$ for some $x \in X$, so $Sx \in SA$ and then $x \in A$. Hence, $fx \in TB$ and so $\beta(fx) \geq 1$. Again, let $\beta(Tx) \geq 1$ for some $x \in X$. Then $Tx \in TB$ and so $x \in B$. Hence, $gx \in SA$ and $\alpha(gx) \geq 1$. Therefore, (f, g) is a cyclic $(\alpha, \beta)_{(S,T)}$-admissible pair.

There exists an $x_0 \in A \cap B$, as $A \cap B$ is nonempty. This implies that $Sx_0 \in SA$ and $Tx_0 \in TB$ and so $\alpha(Sx_0) \geq 1$ and $\beta(Tx_0) \geq 1$.

Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. Then $x_n \in SA \cap TB$ for all $n \in \mathbb{N}$ and so $x \in SA \cap TB$. This implies that $\alpha(x) \geq 1$ and $\beta(x) \geq 1$.

Thus, the conditions (a) and (b) of Theorem 2.1 hold. Hence, there exist $u, v, z \in A \cup B$ such that $z = fu = Su = gv = Tv$. Moreover, since $Su \in SA$ and $Tv \in TB$, we deduce that $\alpha(Su) \geq 1$ and $\beta(Tv) \geq 1$. Thus, the hypothesis (ii) in Theorem 2.1 is also satisfied.

On the other hand, since S and T are one to one, there exist $u_1, v_1 \in A$ and $u_2, v_2 \in B$ such that $Su_1 = Su_2 = z$ and $Tv_1 = Tv_2 = z$ which implies that $u_1 = u_2 = u$ and $v_1 = v_2 = v$. Therefore, $z = fu = Su$ and $z = gv = Tv$ for $u, v \in A \cap B$.

Finally, suppose that $\{f, S\}$ and $\{g, T\}$ are weakly compatible. Following arguments similar to those in proof of Theorem 2.1, we have $z = fz = gz = Sz = Tz$. □

4. An application in dynamic programming

The existence and uniqueness of solutions of functional equations and system of functional equations arising in dynamic programming have been studied by using different fixed point results (see, [1, 7, 22, 30]).

Throughout this section, we assume that U and V are Banach spaces, $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space. We now prove the existence of the common solution of the following system of functional equations:

$$p_i(x) = \sup_{y \in D} \{q(x, y) + Q_i(x, y, p_i(\tau(x, y)))\}, \qquad x \in W \tag{4.1}$$

where $\tau : W \times D \to W$, $q : W \times D \to \mathbb{R}$ and $Q_i : W \times D \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2\}$. It is well known that equation of the type (4.1) provides useful tools for mathematical optimization, computer and dynamic programming (see, [9, 12]).

Let $B(W)$ denote the space of all bounded real-valued functions defined on the set W, where $B(W)$ is endowed with the metric $d(h, k) = sup_{x \in W}|hx - kx|$ for all $h, k \in B(W)$. Note that $B(W)$ is a complete metric space.

We consider the operators $f_i : B(W) \to B(W)$ given by

$$f_i h_i(x) = \sup_{y \in D} \{q(x, y) + Q_i(x, y, h_i(\tau(x, y)))\},$$

for $x \in W$, $h_i \in B(W)$, where $i \in \{1, 2\}$; these operators are well-defined if the functions q_i and G_i are bounded.

Suppose that the following conditions hold.

- (A) $p, q : W \times D \to \mathbb{R}$ and $G, K : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded;
- (B) there exist $\xi, \zeta : X \to \mathbb{R}$ such that if $\xi(h) \geq 0$ and $\zeta(k) \geq 0$ for all $h, k \in B(W)$, then for every $(x, y) \in W \times D$ and $t \in W$, we have

$$|Q_1(x, y, h(x)) - Q_2(x, y, k(x))| \leq \ln(1 + M(h, k)), \tag{4.2}$$

where

$$M(h, k) = \max\Bigg(d(h(t), k(t)), d(h(t), f_1 h(t)), d(k(t), f_2 k(t)),$$
$$\frac{d(h(t), f_2 k(t)) + d(f_1 h(t), k(t))}{2},$$

$$\frac{d\left(k\left(t\right),f_{2}k\left(t\right)\right)\left[1+d\left(h\left(t\right),f_{1}h\left(t\right)\right)\right]}{1+d\left(h\left(t\right),k\left(t\right)\right)},$$
$$\left.\frac{d\left(f_{1}h\left(t\right),k\left(t\right)\right)\left[1+d\left(h\left(t\right),f_{2}k\left(t\right)\right)\right]}{1+d\left(h\left(t\right),k\left(t\right)\right)}\right);$$

(C)

$$\xi\left(h\right)\geq 0 \text{ for some } h\in X \text{ implies } \zeta\left(f_{1}h\right)\geq 0,$$

and

$$\zeta\left(h\right)\geq 0 \text{ for some } h\in X \text{ implies } \xi\left(f_{2}h\right)\geq 0;$$

(D) if $\{h_n\}$ is a sequence in $B(W)$ such that $\xi\left(h_n\right)\geq 0$ and $\zeta\left(h_n\right)\geq 0$ for all $n\in\mathbb{N}_0$ and $h_n\to h^*$ as $n\to\infty$, then $\xi\left(h^*\right)\geq 0$ and $\zeta\left(h^*\right)\geq 0$;

(E) there exists $h_0\in B(W)$ such that $\xi\left(h_0\right)\geq 0$ and $\zeta\left(h_0\right)\geq 0$.

Theorem 4.1. *Assume that the conditions (A)-(E) are satisfied. Then the system of functional equations (4.1) has a common bounded solution in W.*

Proof. Let λ be an arbitrary positive number, $x\in W$ and $h_1,h_2\in B(W)$ such that $\xi\left(h_1\right)\geq 0$ and $\zeta\left(h_2\right)\geq 0$. Then there exist $y_1,y_2\in D$ such that

$$f_1h_1\left(x\right)<q\left(x,y_1\right)+Q_1\left(x,y_1,h_1\left(\tau\left(x,y_1\right)\right)\right)+\lambda, \tag{4.3}$$

$$f_2h_2\left(x\right)<q\left(x,y_2\right)+Q_2\left(x,y_2,h_2\left(\tau\left(x,y_2\right)\right)\right)+\lambda, \tag{4.4}$$

$$f_1h_1\left(x\right)\geq q\left(x,y_2\right)+Q_1\left(x,y_2,h_1\left(\tau\left(x,y_2\right)\right)\right), \tag{4.5}$$

$$f_2h_2\left(x\right)\geq q\left(x,y_1\right)+Q_2\left(x,y_1,h_2\left(\tau\left(x,y_1\right)\right)\right). \tag{4.6}$$

From (4.3) and (4.6), we have

$$\begin{aligned}f_1h_1\left(x\right)-f_2h_2\left(x\right)&<Q_1\left(x,y_1,h_1\left(\tau\left(x,y_1\right)\right)\right)-Q_2\left(x,y_1,h_2\left(\tau\left(x,y_1\right)\right)\right)+\lambda\\&\leq\left|Q_1\left(x,y_1,h_1\left(\tau\left(x,y_1\right)\right)\right)-Q_2\left(x,y_1,h_2\left(\tau\left(x,y_1\right)\right)\right)\right|+\lambda\\&\leq\ln\left(1+M\left(h_1,h_2\right)\right)+\lambda.\end{aligned} \tag{4.7}$$

Similarly, from (4.4) and (4.5), we obtain that

$$f_2h_2\left(x\right)-f_1h_1\left(x\right)<\ln\left(1+M\left(h_1,h_2\right)\right)+\lambda. \tag{4.8}$$

By (4.7) and (4.8), we have

$$\left|f_1h_1\left(x\right)-f_2h_2\left(x\right)\right|<\ln\left(1+M\left(h_1,h_2\right)\right)+\lambda$$

or, equivalently,

$$d\left(f_1h_1,f_2h_2\right)\leq\ln\left(1+M\left(h_1,h_2\right)\right)+\lambda.$$

Since $\lambda>0$ is arbitrary, we get

$$d\left(f_1h_1,f_2h_2\right)\leq\ln\left(1+M\left(h_1,h_2\right)\right).$$

Define $\alpha,\beta:B(W)\to\mathbb{R}^+$ by

$$\alpha\left(h\right)=\begin{cases}1 & \text{if } \xi\left(h\right)\geq 0 \text{ where } h\in B(W),\\ 0 & \text{otherwise}\end{cases}$$

and

$$\beta(h) = \begin{cases} 1 & \text{if } \zeta(h) \geq 0 \text{ where } h \in B(W), \\ 0 & \text{otherwise.} \end{cases}$$

Also, define $\eta, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\eta(t) = t$ and $\varphi(t) = \ln(1+t)$. Thus, we have

$$\begin{aligned} \alpha(h_1)\beta(h_2)\eta(d(f_1h_1, f_2h_2)) &\leq \varphi(M(h_1,h_2)) \\ &\leq \varphi(M(h_1,h_2)) + L\theta(N(h_1,h_2)), \end{aligned}$$

where $L \geq 0$, $\theta \in \Theta$ and

$$N(h_1,h_2) = \min(d(h(t), f_1h(t)), d(k(t), f_2k(t)), d(h(t), f_2k(t)), d(f_1h(t), k(t))).$$

If $f = f_1$ and $g = f_2$, it is easy to observe that all the hypotheses of Corollary 2.10 are satisfied. Therefore f_1 and f_2 have a common fixed point and hence the system of functional equations (4.1) has a bounded common solution. □

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