Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2316–2330
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Dynamics of an almost periodic facultative mutualism model with time delays

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Communicated by C. Park

Abstract

By using some new analytical techniques, modified inequalities and Mawhin's continuation theorem of coincidence degree theory, some sufficient conditions for the existence of at least one positive almost periodic solution of a kind of two-species model of facultative mutualism with time delays are obtained. Further, the global asymptotic stability of the positive almost periodic solution of this model is also considered. Some examples and numerical simulations are provided to illustrate the main results of this paper. Finally, a conclusion is also given to discuss how the parameters of the system influence the existence and globally asymptotic stability of positive almost periodic oscillations.

Keywords: Almost periodic solution, coincidence degree, facultative mutualism, stability.
2010 MSC: 34K13, 34K20, 47H10, 92D25.

1. Introduction

Mutualism is a common occurrence in nature, and is found in many types of communities. Such interactions are well documented in this field. Obvious examples include the algal-fungal associations of lichens [8], plant-pollinator interactions [12], seed dispersal systems that rely on animal vectors [19], the legume nitrogen-fixing bacteria interactions [1, 13], and damselfish-sea anemone interactions [23]. Mutualism may be obligate or facultative, but models of obligate mutualism have qualitatively different stability properties from those of facultative ones (see [24]). An obligate mutualist is a species which requires the presence of another species for its survival, e.g., some species of *Acacia* require the ant *Pseudomyrmex* in order to survive (see [11]). A facultative mutualist is one which benefits in some way from the association with

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Received 2015-11-16

another species but will survive in its absence, e.g., blue-green algae can grow and reproduce in the absence of zooplankton grazers, but growth and reproduction are enhanced by the presence of the zooplankton (see [22]). Despite the fact that mutualisms are common in nature, attempts to model such interactions mathematically are somewhat scant in the literature, in other words, in theoretical population biology mutualism has received very little attention compared to that given to predator-prey interactions or competition among species (see [2, 17, 18, 25, 26]).

Inspired by a delayed single-species population growth model with so-called hereditary effect (see [3, 6, 21]) as follows:

$$y'(t) = y(t)\big[r(t) - a(t)y(t) + b(t)y(t-\mu(t))\big],$$

where the net birth rate $r(t)$, the self-inhibition rate $a(t)$, the reproduction rate $b(t)$, and the delay $\mu(t)$ are nonnegative continuous functions. Clearly, such a system involves a positive feedback term $b(t)y(t-\mu(t))$, which is due to gestation (see [5, 16]).

Suppose further that the dynamics of two species with respective densities y_1 and y_2 governed by the uncoupled system of delayed differential equations and when these two species are allowed to cohabit a common habitat, then each species enhances the average growth rate of the other, Liu et al. [18] proposed a delayed two-species system modelling “facultative mutualism” as follows:

$$\begin{cases} y_1'(t) = y_1(t)\big[r_1(t) - a_1(t)y_1(t) + b_1(t)y_1(t-\mu_1(t)) + c_1(t)y_2(t-\nu_1(t))\big], \\ y_2'(t) = y_2(t)\big[r_2(t) - a_2(t)y_2(t) + b_2(t)y_2(t-\mu_2(t)) + c_2(t)y_1(t-\nu_2(t))\big], \end{cases} \tag{1.1}$$

where r_i, a_i, b_i, c_i, μ_i and ν_i are continuous periodic functions, $i = 1, 2$. There are situations in which the interaction of two species is mutually beneficial, for example, plant-pollinator systems. The interaction may be facultative, meaning that the two species could survive separately, or obligatory, meaning that each species will become extinct without the assistance of the other. In [18], some sufficient conditions are derived for the existence and global asymptotic stability of positive periodic solutions of system (1.1) by using Mawhin's continuation theorem of coincidence degree theory and constructing a suitable Lyapunov functional.

It is well known that the non-autonomous case of ecosystem is more suitable, since all species are suffering to the fluctuation of the environment. In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits and harvesting, etc. So it is usual to assume the periodicity of parameters in the systems. However, in applications, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples, see Example 1.1) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity. Recently, there are many scholars concerning the almost periodic oscillations of the ecosystems, see [4, 9, 10, 14, 15, 20, 27, 29, 30, 31, 32, 33] and the references cited therein.

Example 1.1. Let us consider the following simple fishing model:

$$\begin{cases} y_1'(t) = y_1(t)\big[|\sin(\sqrt{2}t)| + 1 - y_1(t)\big], \\ y_2'(t) = y_2(t)\big[|\sin(\sqrt{3}t)| + 1 - y_2(t)\big]. \end{cases} \tag{1.2}$$

In system (1.2), $|\sin(\sqrt{2}t)|$ is $\frac{\sqrt{2}\pi}{2}$-periodic function and $|\sin(\sqrt{3}t)|$ is $\frac{\sqrt{3}\pi}{3}$-periodic function, which imply that system (1.2) is with incommensurable periods. Then there is no a priori reason to expect the existence of positive periodic solutions of system (1.2). Thus, it is significant to study the existence of positive almost periodic solutions of system (1.2).

In recent years, on the basis of permanence result, many scholars studied the existence, uniqueness and global asymptotic stability of the positive almost periodic solution for some kinds of non-linear ecosystems by using almost periodic theory [4, 9]. For more details, we refer to [10, 14, 15, 20, 27, 29, 30, 31, 32, 33].

However, by today's literature, there are few people obtaining the permanence of system (1.1), i.e., it is difficult to find two certain positive constants α and β such that $\alpha \leq \liminf_{t\to+\infty} y_i(t) \leq \limsup_{t\to+\infty} y_i(t) \leq \beta$, $i = 1, 2$. Therefore, to the best of the author's knowledge, so far, there are scarcely any papers concerning the existence of positive almost periodic solutions of system (1.1). Motivated by the above reason, the main purpose of this paper is to establish some new sufficient conditions on the existence of positive almost periodic solutions of system (1.1) by using Mawhin's continuation theorem of coincidence degree theory.

Let $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{N}^+$ denote the sets of real numbers, integers and positive integers, respectively, $C(\mathbb{X}, \mathbb{Y})$ and $C^1(\mathbb{X}, \mathbb{Y})$ be the space of continuous functions and continuously differential functions which map $\mathbb{X}$ into $\mathbb{Y}$, respectively. Especially, $C(\mathbb{X}) := C(\mathbb{X}, \mathbb{X})$, $C^1(\mathbb{X}) := C^1(\mathbb{X}, \mathbb{X})$. Related to a continuous bounded function f, we use the following notations:

$$f^l = \inf_{s\in\mathbf{R}} f(s), \quad f^u = \sup_{s\in\mathbf{R}} f(s), \quad |f|_\infty = \sup_{s\in\mathbf{R}} |f(s)|, \quad \bar{f} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s)\,\mathrm{d}s.$$

Throughout this paper, we always make the following assumption for system (1.1):

(H_0) r_i, a_i, b_i, c_i, μ_i and ν_i are continuous nonnegative almost periodic functions, $i = 1, 2$.

The paper is organized as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, we obtain sufficient condition for the existence of at least one positive almost periodic solution of system (1.1) by way of Mawhin's continuation theorem of coincidence degree theory. In Section 4, we consider the global asymptotic stability of a unique positive almost periodic solution to system (1.1) by means of Lyapunov functional. Some examples are also given to illustrate our main results.

2. Definitions and lemmas

Definition 2.1 ([4, 9])**.** Let $x \in C(\mathbf{R}) = C(\mathbf{R}, \mathbf{R})$. x is said to be almost periodic on $\mathbf{R}$, if for every $\epsilon > 0$, the set

$$T(x, \epsilon) = \{\tau : |x(t + \tau) - x(t)| < \epsilon, \forall t \in \mathbf{R}\}$$

is relatively dense, i.e., for every $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval length l, there exists a number $\tau = \tau(\epsilon) \in T(x, \epsilon)$ in this interval such that

$$|x(t + \tau) - x(t)| < \epsilon, \ \ \forall t \in \mathbf{R}.$$

τ is called to the ϵ-almost period of x, $T(x, \epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x, \epsilon)$.

Let $AP(\mathbf{R})$ denote the set of all real valued almost periodic functions on $\mathbf{R}$ and

$$AP(\mathbf{R}, \mathbf{R}^n) = \left\{(x_1, x_2, \ldots, x_n)^T : x_i \in AP(\mathbf{R}), i = 1, 2, \ldots, n, n \in \mathbf{N}^+\right\}.$$

Lemma 2.2 ([4, 9])**.** *If $x \in AP(\mathbf{R})$, then x is bounded and uniformly continuous on $\mathbf{R}$.*

Lemma 2.3 ([31])**.** *Assume that $x \in AP(\mathbf{R}) \cap C^1(\mathbf{R})$ with $x' \in C(\mathbf{R})$, for every $\epsilon > 0$, we have the following conclusions*:

(1) *there is a point $\xi_\epsilon \in [0, +\infty)$ such that $x(\xi_\epsilon) \in [x^* - \epsilon, x^*]$ and $x'(\xi_\epsilon) = 0$;*

(2) *there is a point $\eta_\epsilon \in [0, +\infty)$ such that $x(\eta_\epsilon) \in [x_*, x_* + \epsilon]$ and $x'(\eta_\epsilon) = 0$.*

Lemma 2.4 ([32]). *Assume that $x \in AP(\mathbf{R})$. Then for arbitrary interval $[a,b]$ with $b-a=\omega>0$, there exist $\xi_0 \in [a,b]$, $\xi_1 \in (-\infty,a]$ and $\xi_2 \in [\xi_1+\omega,+\infty)$ such that*

$$x(\xi_1)=x(\xi_2) \quad \text{and} \quad x(\xi_0) \le x(s), \ \ \forall s \in [\xi_1,\xi_2].$$

Lemma 2.5 ([32]). *Assume that $x \in AP(\mathbf{R})$. Then for arbitrary interval $[a,b]$ with $b-a=\omega>0$, there exist $\eta_0 \in [a,b]$, $\eta_1 \in (-\infty,a]$ and $\eta_2 \in [\eta_1+\omega,+\infty)$ such that*

$$x(\eta_1)=x(\eta_2) \quad \text{and} \quad x(\eta_0) \ge x(s), \ \ \forall s \in [\eta_1,\eta_2].$$

Lemma 2.6 ([9]). *Assume that $f \in AP(\mathbf{R})$ and $\bar{f}=m(f)>0$. Then for all $a \in \mathbf{R}$, there exists a positive constant T_0 independent of a such that*

$$\frac{1}{T}\int_a^{a+T} f(s)\,\mathrm{d}s \in \left[\frac{\bar{f}}{2},\frac{3\bar{f}}{2}\right], \quad \forall T \ge T_0.$$

3. Almost periodic solution

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin [7].

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\mathrm{dimKer}L = \mathrm{codimIm}L < +\infty$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I-Q)$. It follows that $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : (I-P)\mathbb{X} \to \mathrm{Im}L$ is invertible and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Lemma 3.1 ([7]). *Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. If all the following conditions hold:*

- *(a)* $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0,1)$;
- *(b)* $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;
- *(c)* $\deg\{JQN, \Omega\cap\mathrm{Ker}L, 0\} \neq 0$, *where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is an isomorphism.*

Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

For $f \in AP(\mathbf{R})$, we denote by

$$\Lambda(f) = \left\{\varpi \in \mathbf{R} : \lim_{T\to\infty}\frac{1}{T}\int_0^T f(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s \neq 0\right\},$$

$$\mathrm{mod}(f) = \left\{\sum_{j=1}^m n_j\varpi_j : n_j \in \mathbf{Z}, m \in \mathbf{N}, \varpi_j \in \Lambda(f), j=1,2\ldots,m\right\},$$

the set of Fourier exponents and the module of f, respectively.

Now we are in the position to present and prove our result on the existence of at least one positive almost periodic solution for system (1.1).

Theorem 3.2. *Assume that (H_0) and the following conditions hold:*

(H_1) $r_i^l > 0$, $i=1,2$,

(H_2) $a_i^l > b_i^u$, $i=1,2$,

(H_3) $(a_1^l - b_1^u)(a_2^l - b_2^u) > c_1^u c_2^u$,

then system (1.1) *admits at least one positive almost periodic solution.*

Proof. Under the invariant transformation $(y_1, y_2)^T = (e^{x_1}, e^{x_2})^T$, system (1.1) reduces to

$$\begin{cases} x_1'(t) = r_1(t) - a_1(t)e^{x_1(t)} + b_1(t)e^{x_1(t-\mu_1(t))} + c_1(t)e^{x_2(t-\nu_1(t))} := F_1(t), \\ x_2'(t) = r_2(t) - a_2(t)e^{x_2(t)} + b_2(t)e^{x_2(t-\mu_2(t))} + c_2(t)e^{x_1(t-\nu_2(t))} := F_2(t). \end{cases} \tag{3.1}$$

It is easy to see that if system (3.1) has one almost periodic solution $(x_1, x_2)^T$, then $(y_1, y_2)^T = (e^{x_1}, e^{x_2})^T$ is a positive almost periodic solution of system (1.1). Therefore, to complete the proof it suffices to show that system (3.1) has one almost periodic solution.

Take $\mathbb{X} = \mathbb{Y} = \mathbf{V}_1 \bigoplus \mathbf{V}_2$, where

$$\mathbf{V}_1 = \Big\{ z = (x_1, x_2)^T \in AP(\mathbf{R}, \mathbf{R}^2) : \mathrm{mod}(x_i) \subseteq \mathrm{mod}(L_i), \forall \varpi \in \Lambda(x_1) \cup \Lambda(x_2), |\varpi| \geq \theta_0 \Big\},$$

$$\mathbf{V}_2 = \big\{ z = (x_1, x_2)^T \equiv (k_1, k_2)^T, k_1, k_2 \in \mathbf{R} \big\},$$

where

$$L_1 = L_1(t, \varphi) = r_1(t) - a_1(t)e^{\varphi_1(0)} + b_1(t)e^{\varphi_1(-\mu_1(0))} + c_1(t)e^{\varphi_2(-\nu_1(0))},$$
$$L_2 = L_2(t, \varphi) = r_2(t) - a_2(t)e^{\varphi_2(0)} + b_2(t)e^{\varphi_2(-\mu_2(0))} + c_2(t)e^{\varphi_1(-\nu_2(0))},$$

$\varphi = (\varphi_1, \varphi_2)^T \in C([-\tau, 0], \mathbf{R}^2)$, $\tau = \max_{i=1,2}\{\mu_i^u, \nu_i^u\}$, θ_0 is a given positive constant. Define the norm

$$\|z\| = \max\left\{ \sup_{s\in\mathbf{R}} |x_1(s)|, \sup_{s\in\mathbf{R}} |x_2(s)| \right\}, \quad \forall z = (x_1, x_2)^T \in \mathbb{X} = \mathbb{Y},$$

then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces with the norm $\|\cdot\|$. Set

$$L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}, \quad Lz = L(x_1, x_2)^T = (x_1', x_2')^T,$$

where $\mathrm{Dom}L = \{z = (x_1, x_2)^T \in \mathbb{X} : x_1, x_2 \in C^1(\mathbf{R}), x_1', x_2' \in C(\mathbf{R})\}$ and

$$N : \mathbb{X} \to \mathbb{Y}, \quad Nz = N \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}.$$

With these notations system (3.1) can be written in the form

$$Lz = Nz, \quad \forall z \in \mathbb{X}.$$

It is not difficult to verify that $\mathrm{Ker}L = \mathbf{V}_2$, $\mathrm{Im}L = \mathbf{V}_1$ is closed in $\mathbb{Y}$ and $\dim \mathrm{Ker}L = 2 = \mathrm{codim}\,\mathrm{Im}L$. Therefore, L is a Fredholm mapping of index zero (see [28]). Now define two projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ as

$$Pz = P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} m(x_1) \\ m(x_2) \end{bmatrix} = Qz, \quad \forall z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{X} = \mathbb{Y}.$$

Then P and Q are continuous projectors such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I-Q)$. Furthermore, through an easy computation we find that the inverse $K_P : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{Dom}L$ of L_P has the form

$$K_P z = K_P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \int_0^t x_1(s)\,\mathrm{d}s - m\left[\int_0^t x_1(s)\,\mathrm{d}s\right] \\ \int_0^t x_2(s)\,\mathrm{d}s - m\left[\int_0^t x_2(s)\,\mathrm{d}s\right] \end{bmatrix}, \quad \forall z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathrm{Im}L.$$

Then $QN : \mathbb{X} \to \mathbb{Y}$ and $K_P(I - Q)N : \mathbb{X} \to \mathbb{X}$ read

$$QNz = QN \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} m(F_1) \\ m(F_2) \end{bmatrix},$$

$$K_P(I-Q)Nz = K_P(I-Q)N \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f[x_1(t)] - Qf[x_1(t)] \\ f[x_2(t)] - Qf[x_2(t)] \end{bmatrix}, \quad \forall z \in \mathrm{Im}L,$$

where $f(x)$ is defined by $f[x(t)] = \int_0^t \big[Nx(s) - QNx(s)\big]\, \mathrm{d}s$. Then N is L-compact on $\bar{\Omega}$ (see [28]).

In order to apply Lemma 3.1, we need to search for an appropriate open-bounded subset Ω.

Corollaryresponding to the operator equation $Lz = \lambda z$, $\lambda \in (0,1)$, we have

$$\begin{cases} x_1'(t) = \lambda\big[r_1(t) - a_1(t)e^{x_1(t)} + b_1(t)e^{x_1(t-\mu_1(t))} + c_1(t)e^{x_2(t-\nu_1(t))}\big], \\ x_2'(t) = \lambda\big[r_2(t) - a_2(t)e^{x_2(t)} + b_2(t)e^{x_2(t-\mu_2(t))} + c_2(t)e^{x_1(t-\nu_2(t))}\big]. \end{cases} \tag{3.2}$$

Suppose that $(x_1, x_2)^T \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of system (3.2) for some $\lambda \in (0,1)$. By Lemma 2.3, for every $\epsilon \in (0,1)$, there are two points $\xi_\epsilon^{(1)}, \xi_\epsilon^{(2)} \in [0, +\infty)$ such that

$$x_1'(\xi_\epsilon^{(1)}) = 0,\ x_1(\xi_\epsilon^{(1)}) \in [x_1^* - \epsilon, x_1^*]; \quad x_2'(\xi_\epsilon^{(2)}) = 0,\ x_2(\xi_\epsilon^{(2)}) \in [x_2^* - \epsilon, x_2^*], \tag{3.3}$$

where $x_1^* = \sup_{s\in\mathbf{R}} x_1(s)$ and $x_2^* = \sup_{s\in\mathbf{R}} x_2(s)$.

Further, in view of (H_2)-(H_3), we may assume the above ϵ is small enough so that

$$a_2^l > e^\epsilon b_2^u \quad \text{and} \quad (a_1^l - e^\epsilon b_1^u)(a_2^l - e^\epsilon b_2^u) > e^{2\epsilon} c_1^u c_2^u.$$

It follows from (3.2) and (3.3) that

$$0 = r_1(\xi_\epsilon^{(1)}) - a_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)})} + b_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)} - \mu_1(\xi_\epsilon^{(1)}))} + c_1(\xi_\epsilon^{(1)})e^{x_2(\xi_\epsilon^{(1)} - \nu_1(\xi_\epsilon^{(1)}))}, \tag{3.4}$$

$$0 = r_2(\xi_\epsilon^{(2)}) - a_2(\xi_\epsilon^{(2)})e^{x_2(\xi_\epsilon^{(2)})} + b_2(\xi_\epsilon^{(2)})e^{x_2(\xi_\epsilon^{(2)} - \mu_2(\xi_\epsilon^{(2)}))} + c_2(\xi_\epsilon^{(2)})e^{x_1(\xi_\epsilon^{(2)} - \nu_2(\xi_\epsilon^{(2)}))}. \tag{3.5}$$

In view of (3.4), we have from (3.3) that

$$\begin{aligned} a_1^l e^{x_1^* - \epsilon} &\le a_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)})} \\ &\le r_1(\xi_\epsilon^{(1)}) + b_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)} - \mu_1(\xi_\epsilon^{(1)}))} + c_1(\xi_\epsilon^{(1)})e^{x_2(\xi_\epsilon^{(1)} - \nu_1(\xi_\epsilon^{(1)}))} \\ &\le r_1^u + b_1^u e^{x_1^*} + c_1^u e^{x_2^*}. \end{aligned}$$

That is,

$$(a_1^l - e^\epsilon b_1^u)e^{x_1^*} \le e^\epsilon r_1^u + e^\epsilon c_1^u e^{x_2^*}. \tag{3.6}$$

Similarly, we obtain from (3.5) that

$$(a_2^l - e^\epsilon b_2^u)e^{x_2^*} \le e^\epsilon r_2^u + e^\epsilon c_2^u e^{x_1^*}. \tag{3.7}$$

Substituting (3.7) into (3.6) leads to

$$(a_1^l - e^\epsilon b_1^u)(a_2^l - e^\epsilon b_2^u)e^{x_1^*} \le e^\epsilon r_1^u(a_2^l - e^\epsilon b_2^u) + e^\epsilon c_1^u[e^\epsilon r_2^u + e^\epsilon c_2^u e^{x_1^*}],$$

which implies that

$$\big[(a_1^l - e^\epsilon b_1^u)(a_2^l - e^\epsilon b_2^u) - e^{2\epsilon} c_1^u c_2^u\big]e^{x_1^*} \le e^\epsilon r_1^u(a_2^l - e^\epsilon b_2^u) + e^{2\epsilon} c_1^u r_2^u$$

is equivalent to

$$x_1^* \le \ln\left[\frac{e^\epsilon r_1^u(a_2^l - e^\epsilon b_2^u) + e^{2\epsilon} c_1^u r_2^u}{(a_1^l - e^\epsilon b_1^u)(a_2^l - e^\epsilon b_2^u) - e^{2\epsilon} c_1^u c_2^u}\right].$$

Letting $\epsilon \to 0$ in the above inequality, we obtain

$$x_1^* \le \ln\left[\frac{r_1^u(a_2^l - b_2^u) + c_1^u r_2^u}{(a_1^l - b_1^u)(a_2^l - b_2^u) - c_1^u c_2^u}\right] := \rho_1. \tag{3.8}$$

Substituting (3.8) into (3.7) leads to

$$(a_2^l - e^\epsilon b_2^u)e^{x_2^*} \le e^\epsilon r_2^u + e^\epsilon c_2^u e^{\rho_1}.$$

Letting $\epsilon \to 0$ in the above inequality, we obtain

$$x_2^* \le \ln\left[\frac{r_2^u + c_2^u e^{\rho_1}}{a_2^l - b_2^u}\right] := \rho_2. \tag{3.9}$$

Also, by Lemma 2.3, for every $\epsilon \in (0,1)$, there are two points $\eta_\epsilon^{(1)}, \eta_\epsilon^{(2)} \in [0,+\infty)$ such that

$$x_1'(\eta_\epsilon^{(1)}) = 0,\ x_1(\eta_\epsilon^{(1)}) \in [x_{1*}, x_{1*}+\epsilon]; \quad x_2'(\eta_\epsilon^{(2)}) = 0,\ x_2(\eta_\epsilon^{(2)}) \in [x_{2*}, x_{2*}+\epsilon], \tag{3.10}$$

where $x_{1*} = \inf_{s\in\mathbf{R}} x_1(s)$ and $x_{2*} = \inf_{s\in\mathbf{R}} x_2(s)$. From system (3.2), it follows from (3.10) that

$$0 = r_1(\eta_\epsilon^{(1)}) - a_1(\eta_\epsilon^{(1)})e^{x_1(\eta_\epsilon^{(1)})} + b_1(\eta_\epsilon^{(1)})e^{x_1(\eta_\epsilon^{(1)} - \mu_1(\eta_\epsilon^{(1)}))} + c_1(\eta_\epsilon^{(1)})e^{x_2(\eta_\epsilon^{(1)} - \nu_1(\eta_\epsilon^{(1)}))}, \tag{3.11}$$

$$0 = r_2(\eta_\epsilon^{(2)}) - a_2(\eta_\epsilon^{(2)})e^{x_2(\eta_\epsilon^{(2)})} + b_2(\eta_\epsilon^{(2)})e^{x_2(\eta_\epsilon^{(2)} - \mu_2(\eta_\epsilon^{(2)}))} + c_2(\eta_\epsilon^{(2)})e^{x_1(\eta_\epsilon^{(2)} - \nu_2(\eta_\epsilon^{(2)}))}. \tag{3.12}$$

In view of (3.11), we have from (3.10) that

$$r_1^l \le r_1(\eta_\epsilon^{(1)}) < a_1(\eta_\epsilon^{(1)})e^{x_1(\eta_\epsilon^{(1)})} \le a_1^u e^{x_{1*}+\epsilon},$$

which yields that

$$x_{1*} \ge \ln\frac{r_1^l}{a_1^u e^\epsilon}.$$

Letting $\epsilon \to 0$ in the above inequality, we obtain

$$x_{1*} \ge \ln\frac{r_1^l}{a_1^u} := \rho_3. \tag{3.13}$$

Similarly, we can easily obtain from the second equation of system (3.2) that

$$x_{2*} \ge \ln\frac{r_2^l}{a_2^u} := \rho_4. \tag{3.14}$$

Set $C = |\rho_1| + |\rho_2| + |\rho_3| + |\rho_4| + 1$. Clearly, C is independent of $\lambda \in (0,1)$. Consider the algebraic equations $QNz_0 = 0$ for $z_0 = (x_1^0, x_2^0)^T \in \mathbf{R}^2$ as follows:

$$\begin{cases} 0 = m(r_1) - m(a_1)e^{x_1^0} + m(b_1)e^{x_1^0} + m(c_1)e^{x_2^0}, \\ 0 = m(r_2) - m(a_2)e^{x_2^0} + m(b_2)e^{x_2^0} + m(c_2)e^{x_1^0}. \end{cases}$$

Similar to the arguments as that in (3.8)-(3.9) and (3.13)-(3.14), we can easily obtain that

$$\rho_3 \le x_1^0 \le \rho_1, \quad \rho_4 \le x_2^0 \le \rho_2.$$

Then $\|z_0\| = |x_1^0| + |x_2^0| < C$. Let $\Omega = \{z \in \mathbb{X} : \|z\| < C\}$, then Ω satisfies conditions (a) and (b) of Lemma 3.1.

Finally, we will show that condition (c) of Lemma 3.1 is satisfied. Let us consider the homotopy

$$H(\iota, w) = \iota QNw + (1-\iota)Fw, \ \ (\iota, w) \in [0,1] \times \mathbf{R}^2,$$

where

$$Fw = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \bar{r}_1 - \bar{a}_1 e^{x_1} + \bar{b}_1 e^{x_1} \\ \bar{r}_2 - \bar{a}_2 e^{x_2} + \bar{b}_2 e^{x_2} \end{pmatrix}.$$

From the above discussion it is easy to verify that $H(\iota, w) \neq 0$ on $\partial\Omega \cap \mathrm{Ker}L$. By the invariance property of homotopy, we have

$$\deg\big(JQN, \Omega \cap \mathrm{Ker}L, 0\big) = \deg\big(QN, \Omega \cap \mathrm{Ker}L, 0\big) = \deg\big(F, \Omega \cap \mathrm{Ker}L, 0\big),$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$.

Note that the equations of the following system

$$\begin{cases} \bar{r}_1 - \bar{a}_1 e^{x_1} + \bar{b}_1 e^{x_1} = 0, \\ \bar{r}_2 - \bar{a}_2 e^{x_2} + \bar{b}_2 e^{x_2} = 0 \end{cases}$$

has a solution:

$$(u, v) = \left(\ln\left[\frac{\bar{r}_1}{\bar{b}_1 - \bar{a}_1}\right], \ln\left[\frac{\bar{r}_2}{\bar{b}_2 - \bar{a}_2}\right] \right) \in \Omega.$$

It follows from (H_3) that

$$\begin{aligned} \deg\big(JQN, \Omega \cap \mathrm{Ker}L, 0\big) &= \deg\big(F, \Omega \cap \mathrm{Ker}L, 0\big) \\ &= \mathrm{sign}\begin{vmatrix} (\bar{b}_1 - \bar{a}_1)e^{x_1} & 0 \\ 0 & (\bar{b}_2 - \bar{a}_2)e^{x_2} \end{vmatrix}_{(x_1,x_2)=(u,v)} \\ &= \mathrm{sign}[(\bar{b}_1 - \bar{a}_1)(\bar{b}_2 - \bar{a}_2)] \\ &= 1. \end{aligned}$$

Obviously, all the conditions of Lemma 3.1 are satisfied. Therefore, system (3.1) has one almost periodic solution, that is, system (1.1) has at least one positive almost periodic solution. This completes the proof. □

Example 3.3. Consider the following model with different periodic coefficients:

$$\begin{cases} y_1'(t) = y_1(t)\big[|\sin(\sqrt{2}t)| + 1 - y_1(t) + 0.1|\sin(\sqrt{5}t)|y_1(t-2) + 0.2y_2(t-1)\big], \\ y_2'(t) = y_2(t)\big[|\sin(\sqrt{3}t)| + 1 - y_2(t) + 0.1|\cos(\sqrt{2}t)|y_2(t-1) + 0.3y_1(t-3)\big]. \end{cases} \tag{3.15}$$

Obviously, $r_1^l = r_2^l = 1$, $r_1^u = r_2^u = 2$, $a_1^l = a_1^u = a_2^l = a_2^u = 1$, $c_1^l = c_1^u = 0.2$, $c_2^l = c_2^u = 0.3$, $b_1^l = b_2^l = 0$, $b_1^u = b_2^u = 0.1$. It is easy to see that

$$a_1^l = 1 > 0.1 = b_1^u, \quad (a_1^l - b_1^u)(a_2^l - b_2^u) = 0.81 > 0.06 = c_1^u c_2^u.$$

So (H_0)-(H_3) hold. Therefore, all the conditions in Theorem 3.2 are satisfied. By Theorem 3.2, system (3.15) admits at least one positive almost periodic solution.

In order to broaden condition (H_1) in Theorem 3.2, we give the following result:

Theorem 3.4. *Assume that* (H_0), (H_2)-(H_3) *and the following condition hold:*

(H_4) $\bar{r}_i > 0$, $i = 1, 2$,

then system (1.1) *admits at least one positive almost periodic solution.*

Proof. By the same arguments as that in Theorem 3.2, we have (3.8)-(3.9). From (H_4) and Lemma 2.6, $\forall k \in \mathbf{R}$, there exists a constant $\pi_0 \in (0, +\infty)$ independent of k such that

$$\frac{1}{T}\int_k^{k+T} r_i(s)\,\mathrm{d}s \in \left[\frac{\bar{r}_i}{2}, \frac{3\bar{r}_i}{2}\right], \quad \forall T \geq \pi_0,\ i = 1, 2. \tag{3.16}$$

$\forall n_0 \in \mathbf{Z}$, by lemma 2.5, we can conclude that there exist $\eta_{x_1}^{n_0} \in [n_0\pi_0, n_0\pi_0 + \pi_0]$, $\eta_1^{n_0} \in (-\infty, n_0\pi_0]$ and $\eta_2^{n_0} \in [n_0\pi_0 + \pi_0, +\infty)$ such that

$$x_1(\eta_1^{n_0}) = x_1(\eta_2^{n_0}) \quad \text{and} \quad x_1(\eta_{x_1}^{n_0}) \geq x_1(s),\ \ \forall s \in [\eta_1^{n_0}, \eta_2^{n_0}]. \tag{3.17}$$

Integrating the first equation of system (3.2) from $\eta_1^{n_0}$ to $\eta_2^{n_0}$ leads to

$$\int_{\eta_1^{n_0}}^{\eta_2^{n_0}} \left[r_1(s) - a_1(s)e^{x_1(s)} + b_1(s)e^{x_1(s-\mu_1(s))} + c_1(s)e^{x_2(s-\nu_1(s))}\right]\mathrm{d}s = 0,$$

which yields from (3.16)-(3.17) that

$$\frac{\bar{r}_1}{2} \leq \frac{1}{\eta_2^{n_0} - \eta_1^{n_0}}\int_{\eta_1^{n_0}}^{\eta_2^{n_0}} r_1(s)\,\mathrm{d}s \leq \frac{1}{\eta_2^{n_0} - \eta_1^{n_0}}\int_{\eta_1^{n_0}}^{\eta_2^{n_0}} a_1(s)e^{x_1(s)}\,\mathrm{d}s \leq a_1^u e^{x_1(\eta_{x_1}^{n_0})},$$

which yields that

$$x_1(\eta_{x_1}^{n_0}) \geq \ln\frac{\bar{r}_1}{2a_1^u}. \tag{3.18}$$

Further, we obtain from system (3.2) that

$$\begin{aligned}\int_{n_0\pi_0}^{n_0\pi_0+\pi_0} |x_1'(s)|\,\mathrm{d}s &= \int_{n_0\pi_0}^{n_0\pi_0+\pi_0}\left|r_1(s) - a_1(s)e^{x_1(s)} + b_1(s)e^{x_1(s-\mu_1(s))} + c_1(s)e^{x_2(s-\nu_1(s))}\right|\mathrm{d}s \\ &\leq \left[r_1^u + a_1^u e^{\rho_1} + b_1^u e^{\rho_1} + c_1^u e^{\rho_2}\right]\pi_0 := \Theta_1.\end{aligned} \tag{3.19}$$

It follows from (3.18)-(3.19) that

$$\begin{aligned}x_1(t) &\geq x_1(\eta_{x_1}^{n_0}) - \int_{n_0\pi_0}^{n_0\pi_0+\pi_0} |x_1'(s)|\,\mathrm{d}s \\ &\geq \ln\frac{\bar{r}_1}{2a_1^u} - \Theta_1 := \vec{\rho}_3, \quad \forall t \in [n_0\pi_0, n_0\pi_0 + \pi_0].\end{aligned} \tag{3.20}$$

Obviously, $\vec{\rho}_3$ is a constant independent of n_0. So it follows from (3.20) that

$$x_{1*} = \inf_{s\in\mathbf{R}} x_1(s) = \inf_{n_0\in\mathbf{Z}}\Big\{\min_{s\in[n_0\pi_0, n_0\pi_0+\pi_0]} x_1(s)\Big\} \geq \inf_{n_0\in\mathbf{Z}}\{\vec{\rho}_3\} = \vec{\rho}_3. \tag{3.21}$$

Similar to the argument as that in (3.21), there must exist a constant $\vec{\rho}_4$ so that

$$x_{2*} \geq \vec{\rho}_4.$$

The remaining proof is similar to Theorem 3.2, so we omit it. This completes the proof. □

Example 3.5. Consider the following model:

$$\begin{cases} y_1'(t) = y_1(t)\big[|\sin(\sqrt{2}t)| - y_1(t) + 0.1|\sin(\sqrt{5}t)|y_1(t-2) + 0.2y_2(t-1)\big], \\ y_2'(t) = y_2(t)\big[|\sin(\sqrt{3}t)| - y_2(t) + 0.1|\cos(\sqrt{2}t)|y_2(t-1) + 0.3y_1(t-3)\big]. \end{cases} \tag{3.22}$$

Obviously, $r_1^l = r_2^l = 0$, $r_1^u = r_2^u = 1$, $\bar{r}_1 = \bar{r}_2 = \frac{2}{\pi}$, $a_1^l = a_1^u = a_2^l = a_2^u = 1$, $c_1^l = c_1^u = 0.2$, $c_2^l = c_2^u = 0.3$, $b_1^l = b_2^l = 0$, $b_1^u = b_2^u = 0.1$. It is easy to see that

$$a_1^l = 1 > 0.1 = b_1^u, \quad (a_1^l - b_1^u)(a_2^l - b_2^u) = 0.81 > 0.06 = c_1^u c_2^u.$$

So (H_2)-(H_4) hold. Therefore, all the conditions in Theorem 3.4 are satisfied. By Theorem 3.4, system (3.22) admits at least one positive almost periodic solution.

Remark 3.6. Since $r_1^l = r_2^l = 0$ in system (3.22), it is impossible to obtain the existence of positive almost periodic solutions to this system by Theorem 3.2.

From the proofs of Theorems 3.2-3.4, we can show that,

Corollary 3.7. *Assume that* (H_1)-(H_3) *hold. Suppose further that* r_i, a_i, b_i, c_i, μ_i *and* ν_i *of system* (1.1) *are continuous nonnegative periodic functions with periods* α_i, β_i, γ_i, σ_i, ψ_i *and* ω_i, *respectively,* $i = 1, 2$, *then system* (1.1) *admits at least one positive almost periodic solution.*

Corollary 3.8. *Assume that* (H_2)-(H_4) *hold. Suppose further that* r_i, a_i, b_i, c_i, μ_i *and* ν_i *of system* (1.1) *are continuous nonnegative periodic functions with periods* α_i, β_i, γ_i, σ_i, ψ_i *and* ω_i, *respectively,* $i = 1, 2$, *then system* (1.1) *admits at least one positive almost periodic solution.*

Remark 3.9. In system (3.15) or (3.22), corresponding to Corollaries 3.7-3.8, $\alpha_1 = \frac{\pi}{\sqrt{2}}$, $\alpha_2 = \frac{\pi}{\sqrt{3}}$, β_i, σ_i, ψ_i and ω_i are arbitrary constants, $i = 1, 2$, $\gamma_1 = \frac{\pi}{\sqrt{5}}$, $\gamma_2 = \frac{\pi}{\sqrt{2}}$. To the best of our knowledge, through all coefficients of system (3.15) or (3.22) are periodic functions, it is impossible to sure the existence of positive periodic solutions of system (3.15) or (3.22) by today's literature. However, by Corollaries 3.7-3.8, we obtain the existence of positive almost periodic solutions of system (3.15) or (3.22).

In Corollaries 3.10-3.11, let $\alpha_i = \beta_i = \gamma_i = \sigma_i = \psi_i = \omega_i = \omega$, $i = 1, 2$, then we obtain that,

Corollary 3.10. *Assume that* (H_1)-(H_3) *hold. Suppose further that* r_i, a_i, b_i, c_i, μ_i *and* ν_i *of system* (1.1) *are continuous nonnegative* ω*-periodic functions,* $i = 1, 2$, *then system* (1.1) *admits at least one positive* ω*-periodic solution.*

Corollary 3.11. *Assume that* (H_2)-(H_4) *hold. Suppose further that* r_i, a_i, b_i, c_i, μ_i *and* ν_i *of system* (1.1) *are continuous nonnegative* ω*-periodic functions,* $i = 1, 2$, *then system* (1.1) *admits at least one positive* ω*-periodic solution.*

Remark 3.12. In [18], Liu et al. obtained Corollaries 3.10, but they couldn't obtain Corollaries 3.8. Therefore, our main result extends their work.

4. Global asymptotic stability

Theorem 4.1. *Assume that* (H_0)-(H_3) *hold. Suppose further that*

(H_5) $\mu_i, \nu_i \in C^1(\mathbf{R})$ *and* $\sup_{s\in\mathbf{R}}\{\mu_i'(s), \nu_i'(s)\} < 1$, $i = 1, 2$.

(H_6) *there exist two positive constants* λ_1 *and* λ_2 *such that*

$$\Theta_1 = \inf_{s\in\mathbf{R}} \left\{ \lambda_1 a_1(t) - \lambda_1 \frac{b_1(\varphi_1(t))}{1 - \mu_1'(\varphi_1(t))} - \lambda_2 \frac{c_2(\phi_2(t))}{1 - \nu_2'(\phi_2(t))} \right\} > 0,$$

$$\Theta_2 = \inf_{s\in\mathbf{R}} \left\{ \lambda_2 a_2(t) - \lambda_1 \frac{c_1(\phi_1(t))}{1 - \nu_1'(\phi_1(t))} - \lambda_2 \frac{b_2(\varphi_2(t))}{1 - \mu_2'(\varphi_2(t))} \right\} > 0,$$

where φ_1, φ_2, ϕ_1 *and* ϕ_2 *are the inverse functions of* $t - \mu_1(t)$, $t - \mu_2(t)$, $t - \nu_1(t)$ *and* $t - \nu_2(t)$, *respectively.*

Then the almost periodic solution of system (1.1) *is globally asymptotically stable.*

Proof. From Theorem 3.2, we know that system (1.1) has at least one positive almost periodic solution $(y_1, y_2)^T$. Suppose that $(\bar{y}_1, \bar{y}_2)^T$ is another solution of system (1.1).

Let $(x_1, x_2)^T = (\ln y_1, \ln y_2)^T$ and $(\bar{x}_1, \bar{x}_2)^T = (\ln \bar{y}_1, \ln \bar{y}_2)^T$, then system (1.1) is transformed into

$$\begin{cases} x_1'(t) = r_1(t) - a_1(t)y_1(t) + b_1(t)y_1(t-\mu_1(t)) + c_1(t)y_2(t-\nu_1(t)), \\ x_2'(t) = r_2(t) - a_2(t)y_2(t) + b_2(t)y_2(t-\mu_2(t)) + c_2(t)y_1(t-\nu_2(t)), \\ \bar{x}_1'(t) = r_1(t) - a_1(t)\bar{y}_1(t) + b_1(t)\bar{y}_1(t-\mu_1(t)) + c_1(t)\bar{y}_2(t-\nu_1(t)), \\ \bar{x}_2'(t) = r_2(t) - a_2(t)\bar{y}_2(t) + b_2(t)\bar{y}_2(t-\mu_2(t)) + c_2(t)\bar{y}_1(t-\nu_2(t)). \end{cases} \tag{4.1}$$

Define

$$V(t) = V_0(t) + V_1(t) + V_2(t) + V_3(t) + V_4(t),$$

where

$$V_0(t) = \lambda_1|x_1(t) - \bar{x}_1(t)| + \lambda_2|x_2(t) - \bar{x}_2(t)|,$$

$$V_1(t) = \lambda_1 \int_{t-\mu_1(t)}^{t} \frac{b_1(\varphi_1(s))}{1-\mu_1'(\varphi_1(s))}|y_1(s) - \bar{y}_1(s)|\,\mathrm{d}s,$$

$$V_2(t) = \lambda_1 \int_{t-\nu_1(t)}^{t} \frac{c_1(\phi_1(s))}{1-\nu_1'(\phi_1(s))}|y_2(s) - \bar{y}_2(s)|\,\mathrm{d}s,$$

$$V_3(t) = \lambda_2 \int_{t-\mu_2(t)}^{t} \frac{b_2(\varphi_2(s))}{1-\mu_2'(\varphi_2(s))}|y_2(s) - \bar{y}_2(s)|\,\mathrm{d}s,$$

$$V_4(t) = \lambda_2 \int_{t-\nu_2(t)}^{t} \frac{c_2(\phi_2(s))}{1-\nu_2'(\phi_2(s))}|y_1(s) - \bar{y}_1(s)|\,\mathrm{d}s.$$

By calculating the upper right derivative of V_1 along system (4.1), it follows from the mean value theorem for multivariate function and the monotone property of the function $f(t) = \frac{t}{d_1 t + d_2}$ $(d_1, d_2 \in \mathbf{R})$ that

$$\begin{aligned} D^+V_0(t) =& \lambda_1 \mathrm{sgn}[x_1(t) - \bar{x}_1(t)][x_1'(t) - \bar{x}_1'(t)] + \lambda_2 \mathrm{sgn}[x_2(t) - \bar{x}_2(t)][x_2'(t) - \bar{x}_2'(t)] \\ \le& -\lambda_1 a_1(t)|y_1(t) - \bar{y}_1(t)| - \lambda_2 a_2(t)|y_2(t) - \bar{y}_2(t)| \\ &+ \lambda_1 b_1(t)|y_1(t-\mu_1(t)) - \bar{y}_1(t-\mu_1(t))| \\ &+ \lambda_1 c_1(t)|y_2(t-\nu_1(t)) - \bar{y}_2(t-\nu_1(t))| \\ &+ \lambda_2 b_2(t)|y_2(t-\mu_2(t)) - \bar{y}_2(t-\mu_2(t))| \\ &+ \lambda_2 c_2(t)|y_1(t-\nu_2(t)) - \bar{y}_1(t-\nu_2(t))|. \end{aligned} \tag{4.2}$$

Further, by calculating the upper right derivative of V_1, V_2 and V_3 along system (4.1), it follows that

$$D^+V_1(t) = \lambda_1 \frac{b_1(\varphi_1(t))}{1-\mu_1'(\varphi_1(t))}|y_1(t) - \bar{y}_1(t)| - \lambda_1 b_1(t)\big|y_1(t-\mu_1(t)) - \bar{y}_1(t-\mu_1(t))\big|, \tag{4.3}$$

$$D^+V_2(t) = \lambda_1 \frac{c_1(\phi_1(t))}{1-\nu_1'(\phi_1(t))}|y_2(t) - \bar{y}_2(t)| - \lambda_1 c_1(t)|y_2(t-\nu_1(t)) - \bar{y}_2(t-\nu_1(t))|, \tag{4.4}$$

$$D^+V_3(t) = \lambda_2 \frac{b_2(\varphi_2(t))}{1-\mu_2'(\varphi_2(t))}|y_2(t) - \bar{y}_2(t)| - \lambda_2 b_2(t)|y_2(t-\mu_2(t)) - \bar{y}_2(t-\mu_2(t))|, \tag{4.5}$$

$$D^+V_4(t) = \lambda_2 \frac{c_2(\phi_2(t))}{1-\nu_2'(\phi_2(t))}|y_1(t) - \bar{y}_1(t)| - \lambda_2 c_2(t)|y_1(t-\nu_2(t)) - \bar{y}_1(t-\nu_2(t))|. \tag{4.6}$$

Together with (4.2)-(4.6), it follows that

$$D^+V(t) \le -\left\{\lambda_1 a_1(t) - \lambda_1 \frac{b_1(\varphi_1(t))}{1-\mu_1'(\varphi_1(t))} - \lambda_2 \frac{c_2(\phi_2(t))}{1-\nu_2'(\phi_2(t))}\right\}|y_1(t) - \bar{y}_1(t)|$$

$$
\begin{aligned}
&-\left\{\lambda_2 a_2(t) - \lambda_1 \frac{c_1(\phi_1(t))}{1-\nu_1'(\phi_1(t))} - \lambda_2 \frac{b_2(\varphi_2(t))}{1-\mu_2'(\varphi_2(t))}\right\}|y_2(t)-\bar{y}_2(t)| \\
&\leq -\Gamma_1|y_1(t)-\bar{y}_1(t)| - \Gamma_2|y_2(t)-\bar{y}_2(t)|, \ \ \forall t \in \mathbf{R}.
\end{aligned}
$$

Therefore, V is non-increasing. Integrating of the last inequality from 0 to t leads to

$$
V(t) + \Gamma_1 \int_0^t |y_1(s)-\bar{y}_1(s)|\,\mathrm{d}s + \Gamma_2 \int_0^t |y_2(s)-\bar{y}_2(s)|\,\mathrm{d}s \leq V(0) < +\infty, \ \ \forall t \geq 0,
$$

that is,

$$
\int_0^{+\infty} |y_1(s)-\bar{y}_1(s)|\,\mathrm{d}s < +\infty, \ \ \int_0^{+\infty} |y_2(s)-\bar{y}_2(s)|\,\mathrm{d}s < +\infty,
$$

which implies that

$$
\lim_{s\to+\infty} |y_1(s)-\bar{y}_1(s)| = \lim_{s\to+\infty} |y_2(s)-\bar{y}_2(s)| = 0.
$$

This completes the proof. □

Remark 4.2. Under the conditions of Theorem 3.2, if system (1.1) is globally asymptotically stable, then it has only one positive almost periodic solution.

Corollary 4.3. *Assume that* (H_0)-(H_3), (H_5)-(H_6) *hold, then system* (1.1) *admits a unique globally asymptotically stable positive almost periodic solution.*

From Theorem 3.4 and Theorem 4.1, one has,

Corollary 4.4. *Assume that* (H_0), (H_2)-(H_6) *hold, then system* (1.1) *admits a unique globally asymptotically stable positive almost periodic solution.*

From Theorem 4.1 and Corollaries 3.7-3.8, we obtain that,

Corollary 4.5. *Assume that* (H_1)-(H_3), (H_5)-(H_6) *hold. Suppose further that* r_i, a_i, b_i, c_i, μ_i *and* ν_i *of system* (1.1) *are continuous nonnegative periodic functions with periods* α_i, β_i, γ_i, σ_i, ψ_i *and* ω_i, *respectively,* $i = 1, 2$, *then system* (1.1) *admits a unique globally asymptotically stable positive almost periodic solution.*

Corollary 4.6. *Assume that* (H_2)-(H_6) *hold. Suppose further that* r_i, a_i, b_i, c_i, μ_i *and* ν_i *of system* (1.1) *are continuous nonnegative periodic functions with periods* α_i, β_i, γ_i, σ_i, ψ_i *and* ω_i, *respectively,* $i = 1, 2$, *then system* (1.1) *admits a unique globally asymptotically stable positive almost periodic solution.*

In Corollaries 4.5-4.6, let $\alpha_i = \beta_i = \gamma_i = \sigma_i = \psi_i = \omega_i = \omega$, $i = 1, 2$, then we obtain that,

Corollary 4.7. *Assume that* (H_1)-(H_3), (H_5)-(H_6) *hold. Suppose further that* r_i, a_i, b_i, c_i, μ_i *and* ν_i *of system* (1.1) *are continuous nonnegative* ω*-periodic functions,* $i = 1, 2$, *then system* (1.1) *admits a unique globally asymptotically stable positive* ω*-periodic solution.*

Corollary 4.8. *Assume that* (H_2)-(H_6) *hold. Suppose further that* r_i, a_i, b_i, c_i, μ_i *and* ν_i *of system* (1.1) *are continuous nonnegative* ω*-periodic functions,* $i = 1, 2$, *then system* (1.1) *admits a unique globally asymptotically stable positive* ω*-periodic solution.*

Example 4.9. Consider the following model:

$$
\begin{cases}
y_1'(t) = y_1(t)\big[|\sin(\sqrt{2}t)| - y_1(t) + 0.01|\sin(\sqrt{5}t)|y_1(t-0.1\cos^2 t) + 0.2y_2(t-1)\big], \\
y_2'(t) = y_2(t)\big[|\sin(\sqrt{3}t)| - y_2(t) + 0.2|\cos(\sqrt{2}t)|y_2(t-1) + 0.03y_1(t-0.3\sin^2 t)\big].
\end{cases} \tag{4.7}
$$

In (H_6), let $\lambda_1 = \lambda_2 = 1$, it is clear that all of the conditions in Corollary 4.5 are satisfied. By Corollary 4.5, system (4.7) admits a unique globally asymptotically stable positive almost periodic solution (see Figures 1-3).

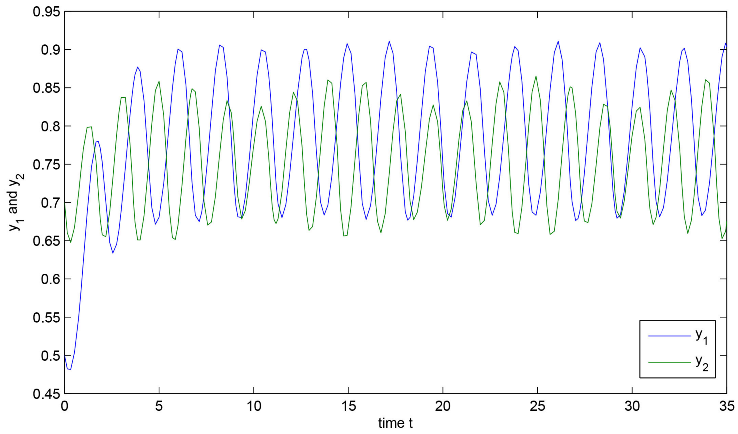

Figure 1 Almost periodic oscillations of system (4.7)

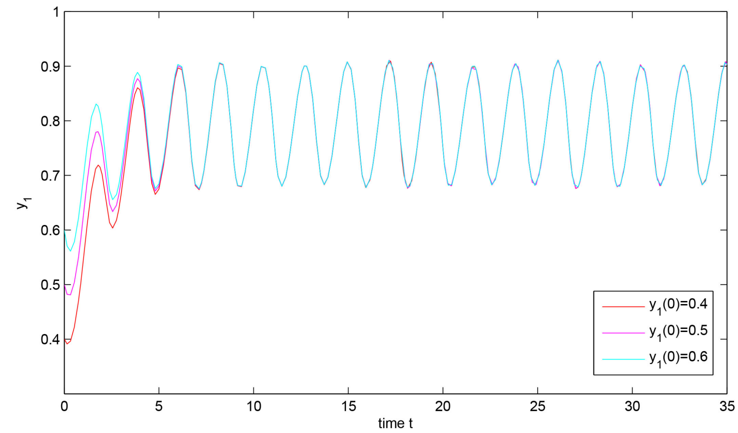

Figure 2 Global asymptotical stability of state variable y_1 of system (4.7)

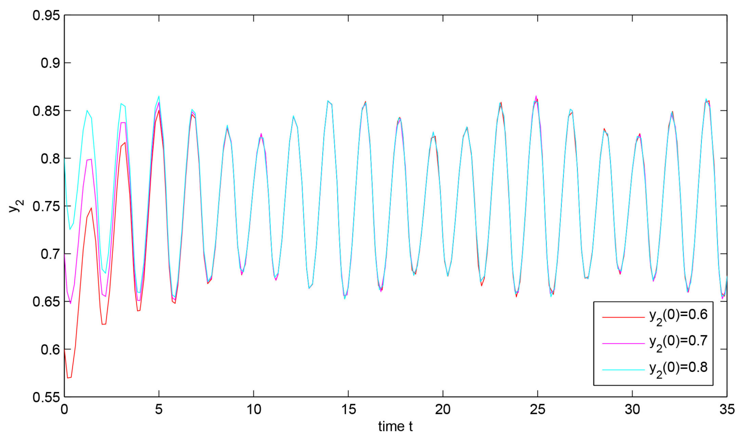

Figure 3 Global asymptotical stability of state variable y_2 of system (4.7)

5. Conclusions

Mutualism is the interaction of two species of organisms that benefits both. In general, mutualism may be obligate or facultative. Obligate mutualist may survive only by association; facultative mutualist, while benefiting from the presence of each other, may also survive in the absence of each other. Despite the fact that mutualism are not uncommon in nature, attempts to model such interactions mathematically are somewhat scant in the literature. In this paper we study an almost periodic nonautonomous delayed two-species system modeling "facultative mutualism", and this motivation comes from a nonautonomous delayed single-species population growth model (see [5]). We obtain easily verifiable sufficient criteria for the existence and globally asymptotic stability of positive almost periodic solutions of the above system. By comparing Theorem 4.1 with the corollaryresponding results in [5, 18], we can see that the existence and globally asymptotic stability of "positive almost periodic solution" of almost periodic nonautonomous delayed system (1.1) corollaryrespond to the existence, globally asymptotic stability of "positive periodic solution" of corollaryrespond periodic nonautonomous delayed system [18] and the existence, globally asymptotic stability of "positive equilibrium" of corollaryresponding autonomous undelayed system [5].

The conditions in Theorem 3.2 indicate that the positive almost periodic solution of system (1.1) is existence if system (1.1) satisfies

- the birth rate of species exceeds zero;
- the undelayed intra-specific competition dominates the delayed intra-specific reproduction, and the intra-specific competition is more significant than the inter-specific cooperation;
- the reproduction rate of species is small enough.

The condition (H_4) in Theorem 3.4 implies that when the birth rate of species is not strictly positive, the consumer-resource system (1.1) may also have a positive almost periodic oscillation. In view of Theorems 3.2-3.4, the time delays of system (1.1) have no effect on the existence of positive (almost) periodic solutions, which is different from the periodic case as that in [18].

In order to ensuring the global asymptotic stability of system (1.1), Theorem 4.1 gives us some implications as follows:

- the changing rate of the gestation or maturation period (i.e., the time delays) is important to ensure the global asymptotic stability of the system;
- the self-inhibition rate of species is more significant than the reproduction rate of all species;
- the birth rate of species has no effect on the global asymptotic stability of the system.

In order to obtain a more accurate description of the ecological system perturbed by human exploitation activities such as planting and harvesting and so on, we need to consider the impulsive differential equations. In this paper, we only studied system (1.1) without impulses, whether system (1.1) with impulses can be discussed in the same methods or not are still open problems. We will continue to study these problems in the future.

Acknowledgements

This work is supported by the Project of High School Educational Reform in Shanxi Province under Grant J2015118 and the Program for the (Reserved) Discipline Leaders of Taiyuan Institute of Technology. The authors thank the referee for his/her careful reading of the original manuscript and valuable comments and suggestions that greatly improved the presentation of this work.

References

[1] R. C. Burns, R. W. Hardy, *Nitrogen fixation in bacteria and higher plants*, Springer-Verlag, New York (1975). 1

[2] F. D. Chen, J. H. Yang, L. J. Chen, X. D. Xie, *On a mutualism model with feedback controls*, Appl. Math. Comput., **214** (2009), 581–587. 1

[3] M. Fan, K. Wang, *Periodic solutions of single population model with hereditary effect*, Math. Appl., **13** (2000), 58–61. 1

[4] A. M. Fink, *Almost Periodic Differential Equations*, Springer, Berlin, (1974). 1, 1, 2.1, 2.2

[5] H. I. Freedman, J. H. Wu, *Periodic solution of single species models with periodic delay*, SIAM J. Math. Anal., **23** (1992), 689–701. 1, 5

[6] H. Fujimoto, *Dynamical behaviours for population growth equations with delays*, Nonlinear Anal. TMA, **31** (1998), 549–558. 1

[7] R. E. Gaines, J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer, Berlin, (1977). 3, 3.1

[8] M. E. Hale, *The biology of lichens*, second ed., Arnold, London, (1974). 1

[9] C. Y. He, *Almost Periodic Differential Equations*, Higher Education Publishing House, Beijing, (1992). 1, 1, 2.1, 2.2, 2.6

[10] X. Z. He, *Almost periodic solutions of a competition system with dominated infinite delays*, Tóhoku Math. J., **50** (1998), 71–89. 1, 1

[11] D. H. Janzen, *Coevolution of mutualism between ants and acacias in Central America*, Evolution, **20** (1966), 249–275. 1

[12] D. H. Janzen, *Euglossine bees as long-distance pollinators of tropical plants*, Science, **171** (1971), 203–205. 1

[13] R. T. Lange, *Bacterial symbiosis with plants*, in: S. M. Henry (Ed.), Symbiosis, vol. I, Academic Press, New York, (1967), 99–170. 1

[14] Y. K. Li, P. Wang, B. Li, *Permanence and almost periodic solutions for a single-species system with impulsive effects on time scales*, J. Nonlinear Sci. Appl., **9** (2016), 1019–1034. 1, 1

[15] X. Lin, F. D. Chen, *Almost periodic solution for a Volterra model with mutual interference and Beddington-DeAngelis functional response*, Appl. Math. Comput., **214** (2009), 548–556. 1, 1

[16] Z. J. Liu, L. S. Chen, *Periodic solution of neutral Lotka-Volterra system with periodic delays*, J. Math. Anal. Appl., **324** (2006), 435–451. 1

[17] Z. J. Liu, R. H. Tan, Y. P. Chen, L. S. Chen, *On the stable periodic solutions of a delayed two-species model of facultative mutualism*, Appl. Math. Comput., **196** (2008), 105–117. 1

[18] Z. J. Liu, J. H. Wu, R. H. Tan, Y. P. Chen, *Modeling and analysis of a periodic delayed two-species model of facultative mutualism*, Appl. Math. Comput., **217** (2010), 893–903. 1, 1, 3.12, 5

[19] D. Mckey, *The ecology of coevolved seed dispersal systems*, in: L. E. Gilbert, P. E. Raven (Eds.), Coevolution of Animals and Plants, University of Texas Press, Austin, (1975), 159–191. 1

[20] X. Z. Meng, L. S. Chen, *Almost periodic solution of non-autonomous Lotka-Volterra predator-prey dispersal system with delays*, J. Theor. Biol., **243** (2006), 562–574. 1, 1

[21] R. K. Miler, *On Volterra's population equation*, SIAM J. Appl. Math., **14** (1996), 446–452. 1

[22] K. G. Porter, *Enhancement of algal growth and productivity by grazing zooplankton*, Science, **192** (1976), 1332–1334. 1

[23] J. Roughgarden, *Evolution of marine symbiosis-A simple cost-benefit model*, Ecology, **56** (1975), 1201–1208. 1

[24] J. H. Vandermeer, D. H. Boucher, *Varieties of mutualistic interactions in population models*, J. Theoret. Biol., **74** (1978), 549–558. 1

[25] C. Y. Wang, S. Wang, F. P. Yang, L. R. Li, *Global asymptotic stability of positive equilibrium of three-species Lotka-Volterra mutualism models with diffusion and delay effects*, Appl. Math. Model., **34** (2010), 4278–4288. 1

[26] Y. H. Xia, J. D. Cao, S. S. Cheng, *Periodic solutions for a Lotka-Volterra mutualism system with several delays*, Appl. Math. Model., **31** (2007), 1960–1969. 1

[27] Y. H. Xia, J. D. Cao, H. Y. Zhang, F. D. Chen, *Almost periodic solutions of n-species competitive system with feedback controls*, J. Math. Anal. Appl., **294** (2004), 503–522. 1, 1

[28] Y. Xie, X. G. Li, *Almost periodic solutions of single population model with hereditary effects*, Appl. Math. Comput., **203** (2008), 690–697. 3

[29] Z. J. Yao, *Uniqueness and global exponential stability of almost periodic solution for Hematopoiesis model on time scales*, J. Nonlinear Sci. Appl., **8** (2015), 142–152. 1, 1

[30] Z. J. Yao, *Almost periodicity of impulsive Hematopoiesis model with infinite delay*, J. Nonlinear Sci. Appl., **8** (2015), 856–865. 1, 1

[31] T. W. Zhang, *Multiplicity of positive almost periodic solutions in a delayed Hassell-Varley type predator-prey model with harvesting on prey*, Math. Methods Appl. Sci., **37** (2014), 686–697. 1, 1, 2.3

[32] T. W. Zhang, *Almost periodic oscillations in a generalized Mackey-Glass model of respiratory dynamics with several delays*, Int. J. Biomath., **7** (2014), 1–22. 1, 1, 2.4, 2.5

[33] T. W. Zhang, Y. K. Li, Y. Ye, *Persistence and almost periodic solutions for a discrete fishing model with feedback control*, Commun. Nonlinear Sci. Numer. Simul., **16** (2011), 1564–1573. 1, 1