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Sharp estimates on the solutions to combined fractional boundary value problems on the half-line

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Abstract

We prove the existence and the uniqueness of a positive solution to the following combined fractional boundary value problem on the half-line

$$\begin{cases} D^{\alpha}u(t) + a_1(t)u^{\sigma_1} + a_2(t)u^{\sigma_2} = 0, \ \ t \in (0,\infty), \ \ 1 < \alpha < 2, \\ \lim_{t\to 0} t^{2-\alpha}u(t) = 0, \ \lim_{t\to\infty} t^{1-\alpha}u(t) = 0, \end{cases}$$

where D^{α} is the standard Riemann–Liouville fractional derivative, $\sigma_1, \sigma_2 \in (-1, 1)$, and a_1, a_2 are non-negative continuous functions on $(0,\infty)$, which may be singular at $t = 0$ and satisfying some convenient assumptions related to the Karamata regular variation theory. We also give sharp estimates on such solution.

Keywords: Riemann–Liouville fractional derivative, Green's function, Karamata regular variation theory, positive solution, fixed point theorem.
2010 MSC: 34B40, 34B18, 34B27.

1. Introduction

The question of existence and uniqueness of solutions subject to fractional differential equations on the half-line has been studied by many authors; see for example [1, 3, 4, 5, 12, 17, 19, 24, 25] and the references therein. Such equations arise in various fields of science and engineering (see for instance [14, 16, 18] and references therein).

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Zhao and Ge [25] considered the following fractional boundary value problem (FBVP) on the half-line

$$\begin{cases} D^{\alpha}u(t)+f(t,u)=0, \quad t\in(0,\infty),\ 1<\alpha<2,\\ u(0)=0,\ \lim_{t\to\infty}D^{\alpha-1}u(t)=\beta u(\xi),\end{cases}$$

where D^{α} is the standard Riemann–Liouville fractional derivative (see Definition 2.1 below), $\beta\in\mathbb{R}$ and $0<\xi<\infty$. The function f is required to be nonnegative and continuous on $[0,\infty)\times\mathbb{R}$ and satisfying some adequate growth conditions. By means of the Leray–Schauder nonlinear alternative theorem (see [2]), they have proved the existence of solutions to the above boundary value problem.

On the other hand, Su and Zhang [24] studied the following FBVP

$$\begin{cases} D^{\alpha}u(t)=f(t,u,D^{\alpha-1}u), \quad t\in J:=[0,\infty), 1<\alpha\leq 2,\\ u(0)=0,\ \ \lim_{t\to\infty}D^{\alpha-1}u(t)=u_{\infty},\ \ u_{\infty}\in\mathbb{R},\end{cases}$$

where $f\in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. They have established an appropriate compactness criterion, in order to use Schauder's fixed point theorem on an unbounded domain to obtain the existence result for solutions.

Recently, in [6], we proved the existence and the uniqueness of a positive solution to the following FBVP

$$\begin{cases} D^{\alpha}u(t)+a(t)u^{\sigma}=0, \quad t\in(0,\infty),\ \ 1<\alpha<2,\\ \lim_{t\to 0}t^{2-\alpha}u(t)=0,\ \ \lim_{t\to\infty}t^{1-\alpha}u(t)=0,\end{cases}$$

where $\sigma\in(-1,1)$ and $a\in C^{+}((0,\infty))$. Here the function a is allowed to be singular at $t=0$.

In this article, we extend the results obtained in [6], by studying the existence, uniqueness and sharp estimates of a positive solution to the following *combined* FBVP

$$\begin{cases} D^{\alpha}u(t)+a_1(t)u^{\sigma_1}+a_2(t)u^{\sigma_2}=0, \quad t\in(0,\infty),\ \ 1<\alpha<2,\\ \lim_{t\to 0}t^{2-\alpha}u(t)=0,\ \ \lim_{t\to\infty}t^{1-\alpha}u(t)=0,\end{cases}\tag{1.1}$$

where, for $i\in\{1,2\}$, $\sigma_i\in(-1,1)$ and $a_i\in C^{+}((0,\infty))$ satisfy some appropriate assumptions related to the following subclasses of slowly varying functions $\mathcal{K}$ and $\mathcal{K}^{\infty}$.

Definition 1.1. A positive function L defined on $(0,\eta]$ (for some $\eta>1$) belongs to the Karamata class $\mathcal{K}$, if

$$L(t):=c\exp\left(\int_t^{\eta}\frac{z(s)}{s}ds\right),$$

where $c>0$ and $z\in C([0,\eta])$ such that $z(0)=0$.

Definition 1.2. A positive function $\widetilde{L}$ defined on $[1,\infty)$ belongs to the Karamata class $\mathcal{K}^{\infty}$ if

$$\widetilde{L}(t):=c\exp\left(\int_1^{t}\frac{z(s)}{s}ds\right),$$

where $c>0$ and $z\in C([1,\infty))$ such that $\lim_{t\to\infty}z(t)=0$.

The theory of slowly varying functions was initiated by Karamata in the fundamental paper [15]. We also point that out the first use of the Karamata theory in the study of the growth rate of solutions near the boundary was done in the paper of Cirstea and Rădulescu [13].

One can easily verify the following results.

Proposition 1.3.

(i) *$L\in\mathcal{K}$ if and only if $L\in C^1((0,\eta])$ for some $\eta>1$, L positive and* $\lim_{t\to 0^+}\frac{tL'(t)}{L(t)}=0$.

(ii) *$\widetilde{L} \in \mathcal{K}^\infty$ if and only if $\widetilde{L} \in C^1([1,\infty))$, $\widetilde{L}$ positive and $\lim_{t\to\infty} \frac{t\widetilde{L}'(t)}{\widetilde{L}(t)} = 0$.*

(iii) *Let $\widetilde{L} \in \mathcal{K}^\infty$; then there exists an $m \geq 0$ such that for every $c > 0$ and $t \geq 1$, we have*

$$(1+c)^{-m}\widetilde{L}(t) \leq \widetilde{L}(c+t) \leq (1+c)^m \widetilde{L}(t).$$

As an example of function belonging to $\mathcal{K}$ and $\mathcal{K}^\infty$ (see [8, 20, 23]), we quote

$$L(t) = \prod_{k=1}^{m} (\log_k(\frac{\omega}{t}))^{\xi_k}, \quad \widetilde{L}(t) = 2 + \sin(\log_2(\omega t)) \quad \text{and} \quad \widetilde{L}(t) = \exp\left\{ \prod_{k=1}^{m} (\log_k(\omega t))^{\tau_k} \right\},$$

where $\log_k x = \log \circ \log \circ \cdots \circ \log x$ (k times), $\xi_k \in \mathbb{R}$, $\tau_k \in (0,1)$ and $\omega > 0$ sufficiently large such that L (resp. $\widetilde{L}$) is defined and positive on $(0,\eta]$ ($\eta > 1$) (resp. on $[1,\infty)$).

In the sequel, we denote by $B^+((0,\infty))$ (resp. $C^+((0,\infty))$), the set of nonnegative Borel measurable functions (resp. nonnegative continuous functions) in $(0,\infty)$ and by $C_{2-\alpha}([0,\infty))$, the set of all functions f such that $t \mapsto t^{2-\alpha} f(t)$ is continuous on $[0,\infty)$.

We also denote by $C_0([0,\infty))$ the set of continuous functions v on $[0,\infty)$ such that $\lim_{t\to\infty} v(t) = 0$. It is known that $C_0([0,\infty))$ is a Banach space equipped with the supremum norm $\|v\|_\infty = \sup_{t\geq 0} |v(t)|$.

For $s,t \in \mathbb{R}$, we write $\min(s,t) = s \wedge t$ and $\max(s,t) = s \vee t$. For $f,g \in B^+((0,\infty))$, the notation $f(t) \approx g(t)$, $t \in (0,\infty)$, means that there exists a $c > 0$ such that $\frac{1}{c} f(t) \leq g(t) \leq c f(t)$ for all $t > 0$.

For $1 < \alpha < 2$, we denote by $G_\alpha(t,s)$, Green's function of the operator $u \mapsto -D^\alpha u$ on $(0,\infty)$ with the boundary conditions $\lim_{t\to 0} t^{2-\alpha} u(t) = 0$ and $\lim_{t\to\infty} t^{1-\alpha} u(t) = 0$. We define the potential kernel V on $B^+((0,\infty))$ by

$$Vf(t) := \int_0^\infty G_\alpha(t,s) f(s) ds, \quad t > 0.$$

To simplify our statements, we introduce the following. Let $1 < \alpha < 2$, $\sigma \in (-1,1)$, $\lambda \leq 2 + (\alpha-2)\sigma$ and $\mu \geq 1 + (\alpha-1)\sigma$. Let $L \in \mathcal{K}$ be defined on $(0,\eta]$ ($\eta > 1$) and $\widetilde{L} \in \mathcal{K}^\infty$ such that

$$\int_0^\eta \frac{L(s)}{s^{\lambda-(\alpha-2)\sigma-1}} ds < \infty \text{ and } \int_1^\infty \frac{\widetilde{L}(s)}{s^{\mu-(\alpha-1)\sigma}} ds < \infty.$$

Define the function $\Psi_{L,\lambda,\sigma}(t)$ for $t \in (0,\eta)$ by

$$\Psi_{L,\lambda,\sigma}(t) = \begin{cases} \left(\int_0^t \frac{L(s)}{s} ds \right)^{\frac{1}{1-\sigma}}, & \text{if } \lambda = 2 + (\alpha-2)\sigma, \\ (L(t))^{\frac{1}{1-\sigma}}, & \text{if } 1 + (\alpha-1)\sigma < \lambda < 2 + (\alpha-2)\sigma, \\ \left(\int_t^\eta \frac{L(s)}{s} ds \right)^{\frac{1}{1-\sigma}}, & \text{if } \lambda = 1 + (\alpha-1)\sigma, \\ 1, & \text{if } \lambda < 1 + (\alpha-1)\sigma, \end{cases} \tag{1.2}$$

and $\widetilde{\Psi}_{\widetilde{L},\mu,\sigma}(t)$ for $t \in [1,\infty)$ by

$$\widetilde{\Psi}_{\widetilde{L},\mu,\sigma}(t) = \begin{cases} \left(\int_t^\infty \frac{\widetilde{L}(s)}{s} ds \right)^{\frac{1}{1-\sigma}}, & \text{if } \mu = 1 + (\alpha-1)\sigma, \\ \left(\widetilde{L}(t)\right)^{\frac{1}{1-\sigma}}, & \text{if } 1 + (\alpha-1)\sigma < \mu < 2 + (\alpha-2)\sigma, \\ \left(\int_1^{t+1} \frac{\widetilde{L}(s)}{s} ds \right)^{\frac{1}{1-\sigma}}, & \text{if } \mu = 2 + (\alpha-2)\sigma, \\ 1, & \text{if } \mu > 2 + (\alpha-2)\sigma. \end{cases} \tag{1.3}$$

Throughout this article, assume the following condition:

(**H**) For $i \in \{1,2\}$, $a_i \in C^+((0,\infty))$ such that

$$a_i(t) \approx t^{-\lambda_i}(1+t)^{\lambda_i-\mu_i} L_i(1\wedge t)\widetilde{L}_i(1\vee t), \quad t>0, \tag{1.4}$$

where $1<\alpha<2$, $\sigma_i \in (-1,1)$, $\lambda_i \le 2+(\alpha-2)\sigma_i$ and $\mu_i \ge 1+(\alpha-1)\sigma_i$ the functions $L_i \in \mathcal{K}$ defined on $(0,\eta]$ $(\eta>1)$ and $\widetilde{L}_i \in \mathcal{K}^\infty$ are such that

$$\int_0^\eta \frac{L_i(s)}{s^{\lambda_i-(\alpha-2)\sigma_i-1}}ds < \infty \text{ and } \int_1^\infty \frac{\widetilde{L}_i(s)}{s^{\mu_i-(\alpha-1)\sigma_i}}ds < \infty. \tag{1.5}$$

As it will be seen, for $i \in \{1,2\}$, the numbers

$$\nu_i = \min\Big(1, \frac{2-\lambda_i+(\alpha-2)\sigma_i}{1-\sigma_i}\Big) \text{ and } \zeta_i = \max\Big(0, \frac{2-\mu_i+(\alpha-2)\sigma_i}{1-\sigma_i}\Big) \tag{1.6}$$

will play an important role in the study of asymptotic behavior. Without loss of generality, we may assume that

$$\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1} \le \frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2}, \frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1} \le \frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}.$$

We introduce the function θ defined on $(0,\infty)$ by

$$\theta(t) = \begin{cases} t^{\nu_1}(1+t)^{\zeta_2-\nu_1}\Psi_{L_1,\lambda_1,\sigma_1}(1\wedge t)\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}(1\vee t) & \text{if } \nu_1<\nu_2 \text{ and } \zeta_1<\zeta_2, \\ t^{\nu_1}(1+t)^{\zeta_2-\nu_1}\Psi_{L_1,\lambda_1,\sigma_1}(1\wedge t)\left(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}\right)(1\vee t) & \text{if } \nu_1<\nu_2 \text{ and } \zeta_1=\zeta_2, \\ t^{\nu_1}(1+t)^{\zeta_2-\nu_1}\left(\Psi_{L_1,\lambda_1,\sigma_1}+\Psi_{L_2,\lambda_2,\sigma_2}\right)(1\wedge t)\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}(1\vee t) & \text{if } \nu_1=\nu_2 \text{ and } \zeta_1<\zeta_2, \\ t^{\nu_1}(1+t)^{\zeta_2-\nu_1}\left(\Psi_{L_1,\lambda_1,\sigma_1}+\Psi_{L_2,\lambda_2,\sigma_2}\right)(1\wedge t)\left(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}\right)(1\vee t) & \text{if } \nu_1=\nu_2 \text{ and } \zeta_1=\zeta_2. \end{cases}$$

Observe that for $t \in (0,1]$

$$\theta(t) \approx \begin{cases} t^{\nu_1}\Psi_{L_1,\lambda_1,\sigma_1}(t) & \text{if } \nu_1<\nu_2, \\ t^{\nu_1}(\Psi_{L_1,\lambda_1,\sigma_1}(t)+\Psi_{L_2,\lambda_2,\sigma_2}(t)) & \text{if } \nu_1=\nu_2, \end{cases} \tag{1.7}$$

and for $t \ge 1$

$$\theta(t) \approx \begin{cases} t^{\zeta_2}\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}(t) & \text{if } \zeta_1<\zeta_2, \\ t^{\zeta_2}(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}(t)+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}(t)) & \text{if } \zeta_1=\zeta_2. \end{cases} \tag{1.8}$$

Our main results are the following two theorems.

Theorem 1.4. *Let $1<\alpha<2$, $\sigma_1,\sigma_2 \in (-1,1)$ and assume that hypothesis (H) is satisfied. Then one has for $t \in (0,\infty)$,*

$$t^{2-\alpha}V(\omega)(t) \approx \theta(t), \tag{1.9}$$

where $\omega(t) := a_1(t)t^{(\alpha-2)\sigma_1}\theta^{\sigma_1}(t)+a_2(t)t^{(\alpha-2)\sigma_2}\theta^{\sigma_2}(t)$.

Theorem 1.5. *Let $1<\alpha<2$, $\sigma_1,\sigma_2 \in (-1,1)$ and assume that hypothesis (H) is satisfied. Then problem (1.1) has a unique positive solution $u \in C_{2-\alpha}([0,\infty))$ satisfying for $t \in (0,\infty)$,*

$$u(t) \approx t^{\alpha-2}\theta(t). \tag{1.10}$$

Remark 1.6. The conclusion of Theorem 1.5 extends the one obtained in [6, Theorem 5] and also remains valid for the case $\alpha=2$ and $\sigma_1,\sigma_2<1$ (see [7]).

The paper is organized as follows. In Section 2, we present some properties of Green's function $G_\alpha(t,s)$ of the operator $u \mapsto -D^\alpha u$ on $(0,\infty)$ with boundary conditions $\lim_{t\to 0} t^{2-\alpha}u(t)=0$ and $\lim_{t\to\infty} t^{1-\alpha}u(t)=0$. Next, we collect some fundamental properties of the two Karamata classes $\mathcal{K}$ and $\mathcal{K}^\infty$. In Section 3, we establish our main results.

2. Preliminaries

2.1. On Green's function

We first recall the definition of the Riemann–Liouville fractional derivative of order $\beta > 0$ (denoted by D^{β}).

Definition 2.1 ([21, 22])**.** Let $\beta > 0$ and h be a real function defined on $(0, \infty)$. Then

$$D^{\beta}h(t) := \frac{1}{\Gamma(n-\beta)}(\frac{d}{dt})^{n}\int_0^t (t-s)^{n-\beta-1}h(s)ds, \quad t>0,$$

where $n = [\beta]+1$, $[\beta]$ denotes the integer part of the number β, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Next, we collect some lemmas that will be used in the proofs of our main results.

Lemma 2.2 ([6])**.** *Let $1 < \alpha < 2$ and $f \in L^1(0, \infty)$. The unique solution of*

$$\begin{cases} D^{\alpha}u(t) + f(t) = 0, \quad t > 0 \\ \lim_{t\to 0} t^{2-\alpha}u(t) = 0 \quad and \quad \lim_{t\to\infty} t^{1-\alpha}u(t) = 0, \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^{\infty} G_{\alpha}(t,s)f(s)ds,$$

where

$$G_{\alpha}(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & if \ 0 < s \le t < \infty, \\ t^{\alpha-1}, & if \ 0 < t \le s < \infty, \end{cases} \tag{2.2}$$

is Green's function for the FBVP (2.1)*.*

Lemma 2.3 ([6])**.** *Let $1 < \alpha < 2$.*

(*i*) *On $(0,\infty)\times(0,\infty)$, one has*

$$G_{\alpha}(t,s) \approx t^{\alpha-2}\min(t,s).$$

(*ii*) *Let f be a function such that the map $s \mapsto \min(1,s)f(s)$ is continuous and integrable on $(0,\infty)$. Then Vf is the unique solution in $C_{2-\alpha}([0,\infty))$ of the FBVP*

$$\begin{cases} D^{\alpha}u(t) + f(t) = 0, \quad t > 0 \\ \lim_{t\to 0} t^{2-\alpha}u(t) = 0 \ and \ \lim_{t\to\infty} t^{1-\alpha}u(t) = 0. \end{cases}$$

2.2. On the Karamata classes $\mathcal{K}$ and $\mathcal{K}^{\infty}$.

We summarize here some basic properties of functions belong to the Karamata classes $\mathcal{K}$ and $\mathcal{K}^{\infty}$ which will be useful.

Proposition 2.4 ([20, 23])**.**

(*i*) *Let $L_1, L_2 \in \mathcal{K}$ (resp. $\mathcal{K}^{\infty}$) and $p \in \mathbb{R}$. Then, the functions $L_1 + L_2$, L_1L_2 and L_1^p belong to the class $\mathcal{K}$ (resp. $\mathcal{K}^{\infty}$).*

(*ii*) *Let $L \in \mathcal{K}$ (resp. $\widetilde{L} \in \mathcal{K}^{\infty}$) and $\varepsilon > 0$. Then*

$$\lim_{t\to 0^+} t^{\varepsilon}L(t) = 0 \quad (resp. \ \lim_{t\to\infty} t^{-\varepsilon}\widetilde{L}(t) = 0).$$

Proposition 2.5 ([20, 23])**.**

(*i*) *Let $\gamma \in \mathbb{R}$ and $L \in \mathcal{K}$ defined on $(0,\eta]$. Then, the following hold.*

(*a*) *If $\gamma < -1$, then the integral $\int_0^\eta s^\gamma L(s)ds$ diverges and*

$$\int_t^\eta s^\gamma L(s)ds \underset{t\to 0^+}{\sim} -\frac{t^{\gamma+1}L(t)}{\gamma+1}.$$

(*b*) *If $\gamma > -1$, then the integral $\int_0^\eta s^\gamma L(s)ds$ converges and*

$$\int_0^t s^\gamma L(s)ds \underset{t\to 0^+}{\sim} \frac{t^{\gamma+1}L(t)}{\gamma+1}.$$

(*ii*) *Let $\gamma \in \mathbb{R}$ and $\widetilde{L} \in \mathcal{K}^\infty$. Then, the following hold.*

(*a*) *If $\gamma > -1$, then the integral $\int_1^\infty s^\gamma \widetilde{L}(s)ds$ diverges and*

$$\int_1^t s^\gamma \widetilde{L}(s)ds \underset{t\to\infty}{\sim} \frac{t^{\gamma+1}\widetilde{L}(t)}{\gamma+1}.$$

(*b*) *If $\gamma < -1$, then the integral $\int_1^\infty s^\gamma \widetilde{L}(s)ds$ converges and*

$$\int_t^\infty s^\gamma \widetilde{L}(s)ds \underset{t\to\infty}{\sim} -\frac{t^{\gamma+1}\widetilde{L}(t)}{\gamma+1}.$$

Lemma 2.6 ([11])**.** *Let $L \in \mathcal{K}$ defined on $(0,\eta]$. Then*

$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s}ds} = 0.$$

In particular, $t \mapsto \int_t^\eta \frac{L(s)}{s}ds \in \mathcal{K}$.

If further the integral $\int_0^\eta \dfrac{L(s)}{s}ds$ converges, then

$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s}ds} = 0.$$

In particular, $t \mapsto \int_0^t \frac{L(s)}{s}ds \in \mathcal{K}$.

Similar properties related to the class $\mathcal{K}^\infty$ are stated in the next lemma. We refer to [10] for the proof.

Lemma 2.7. *Let $\widetilde{L}$ be a function in $\mathcal{K}^\infty$. Then,*

$$\lim_{t\to\infty} \frac{\widetilde{L}(t)}{\int_1^t \frac{\widetilde{L}(s)}{s}ds} = 0.$$

In particular, $t \mapsto \int_1^{t+1} \frac{\widetilde{L}(s)}{s}ds \in \mathcal{K}^\infty$.

If further the integral $\int_1^\infty \dfrac{\widetilde{L}(s)}{s}ds$ converges, then

$$\lim_{t\to\infty} \frac{\widetilde{L}(t)}{\int_t^\infty \frac{\widetilde{L}(s)}{s}ds} = 0.$$

In particular, $t \mapsto \int_t^\infty \frac{\widetilde{L}(s)}{s}ds \in \mathcal{K}^\infty$.

Proposition 2.8 ([6]). *Let $L_0 \in \mathcal{K}$ defined on $(0,\eta]$ for $\eta > 1$ and $\widetilde{L}_0 \in \mathcal{K}^\infty$. Let $\beta \leq 2$ and $\gamma \geq 1$ such that*

$$\int_0^\eta s^{1-\gamma} L_0(s) ds < \infty \quad \text{and} \quad \int_1^\infty s^{-\gamma} \widetilde{L}_0(s) ds < \infty.$$

Put

$$b(t) = t^{-\beta} (1+t)^{\beta-\gamma} L_0(1 \wedge t) \widetilde{L}_0(1 \vee t), \quad t > 0.$$

Then, for $t > 0$,

$$t^{2-\alpha} V b(t) \approx \psi_\beta(1 \wedge t) \phi_\gamma(1 \vee t),$$

where, for $r \in (0,1]$,

$$\psi_\beta(r) = \begin{cases} \int_0^r \frac{L_0(s)}{s} ds & \text{if } \beta = 2, \\ r^{2-\beta} L_0(r) & \text{if } 1 < \beta < 2, \\ r \int_r^\eta \frac{L_0(s)}{s} ds & \text{if } \beta = 1, \\ r & \text{if } \beta < 2. \end{cases}$$

and for $r \geq 1$,

$$\phi_\gamma(r) = \begin{cases} r \int_r^\infty \frac{\widetilde{L}_0(s)}{s} ds & \text{if } \gamma = 1, \\ r^{2-\gamma} \widetilde{L}_0(r) & \text{if } 1 < \gamma < 2, \\ \int_1^{r+1} \frac{\widetilde{L}_0(s)}{s} ds & \text{if } \gamma = 2, \\ 1 & \text{if } \gamma > 2. \end{cases}$$

3. Proof of the main results

We recall that for $i \in \{1,2\}$, $a_i \in C^+((0,\infty))$ such that

$$a_i(t) \approx t^{-\lambda_i} (1+t)^{\lambda_i - \mu_i} L_i(1 \wedge t) \widetilde{L}_i(1 \vee t), t > 0,$$

where $1 < \alpha < 2$, $\sigma_i \in (-1,1)$, $\lambda_i \leq 2 + (\alpha-2)\sigma_i$, $\mu_i \geq 1 + (\alpha-1)\sigma_i$, $L_i \in \mathcal{K}$ defined on $(0,\eta]$ for $\eta > 1$ and $\widetilde{L} \in \mathcal{K}^\infty$ satisfying

$$\int_0^\eta \frac{L_i(s)}{s^{\lambda_i - (\alpha-2)\sigma_i - 1}} ds < \infty \text{ and } \int_1^\infty \frac{\widetilde{L}_i(s)}{s^{\mu_i - (\alpha-1)\sigma_i}} ds < \infty.$$

First, we will give sharp estimates of the function θ on $(0,1]$ and on $[1,\infty)$, respectively. To this end, let L, M and N be the nonnegative functions defined on $(0,1]$ by

$$\begin{aligned} L(t) &:= (L_1(t))^{\frac{1}{1-\sigma_1}} + (L_2(t))^{\frac{1}{1-\sigma_2}}, \\ M(t) &:= \left(\int_t^\eta \frac{L_1(s)}{s} ds\right)^{\frac{1}{1-\sigma_1}} + \left(\int_t^\eta \frac{L_2(s)}{s} ds\right)^{\frac{1}{1-\sigma_2}}, \\ N(t) &:= \left(\int_0^t \frac{L_1(s)}{s} ds\right)^{\frac{1}{1-\sigma_1}} + \left(\int_0^t \frac{L_2(s)}{s} ds\right)^{\frac{1}{1-\sigma_2}}, \text{ if } \int_0^\eta \frac{L_i(s)}{s} ds < \infty, \end{aligned}$$

and $\widetilde{L}$, $\widetilde{M}$ and $\widetilde{N}$ the nonnegative functions defined on $[1,\infty)$ by

$$\widetilde{L}(t) := (\widetilde{L}_1(t))^{\frac{1}{1-\sigma_1}} + (\widetilde{L}_2(t))^{\frac{1}{1-\sigma_2}},$$

$$\widetilde{M}(t) := \Big(\Big(\int_1^{t+1} \frac{\widetilde{L}_1(s)}{s}ds\Big)^{\frac{1}{1-\sigma_1}} + \Big(\int_1^{t+1} \frac{\widetilde{L}_2(s)}{s}ds\Big)^{\frac{1}{1-\sigma_2}}\Big),$$

$$\widetilde{N}(t) := \Big(\int_t^{\infty} \frac{\widetilde{L}_1(s)}{s}ds\Big)^{\frac{1}{1-\sigma_1}} + \Big(\int_t^{\infty} \frac{\widetilde{L}_2(s)}{s}ds\Big)^{\frac{1}{1-\sigma_2}}, \text{ if} \int_1^{\infty} \frac{\widetilde{L}_i(s)}{s}ds < \infty.$$

Remark 3.1. Let $L_0 \in \mathcal{K}$ be defined on $(0, \eta]$ for $\eta > 1$, $\widetilde{L}_0 \in \mathcal{K}^\infty$ and $\sigma \in (-1, 1)$. Then,

(i) $1 + \Big(\int_t^{\eta} \frac{L_0(s)}{s}ds\Big)^{\frac{1}{1-\sigma}} \approx \Big(\int_t^{\eta} \frac{L_0(s)}{s}ds\Big)^{\frac{1}{1-\sigma}}$ on $(0, 1]$;

(ii) $1 + \Big(\int_1^{1+t} \frac{\widetilde{L}_0(s)}{s}ds\Big)^{\frac{1}{1-\sigma}} \approx \Big(\int_1^{1+t} \frac{\widetilde{L}_0(s)}{s}ds\Big)^{\frac{1}{1-\sigma}}$ on $[1, \infty)$.

We recall that for $i \in \{1, 2\}$,

$$\nu_i = \min(1, \frac{2-\lambda_i + (\alpha-2)\sigma_i}{1-\sigma_i}) \quad \text{and} \quad \zeta_i = \max(0, \frac{2-\mu_i + (\alpha-2)\sigma_i}{1-\sigma_i}).$$

Since $\nu_1 < \nu_2$ is equivalent to $\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1} < \frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2}$ and $1 + (\alpha-1)\sigma_1 < \lambda_1$, we deduce from (1.7) and Remark 3.1(i) that for $t \in (0, 1]$,

$$\theta(t) \approx \begin{cases} \Big(\int_0^t \frac{L_1(s)}{s}ds\Big)^{\frac{1}{1-\sigma_1}} & \text{if } \lambda_1 = 2 + (\alpha-2)\sigma_1 \text{ and } \lambda_2 < 2 + (\alpha-2)\sigma_2, \\ t^{\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1}} (L_1(t))^{\frac{1}{1-\sigma_1}} & \text{if } \frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1} < \frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1 + (\alpha-1)\sigma_1 < \lambda_1 < 2 + (\alpha-2)\sigma_1, \\ t^{\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1}} L(t) & \text{if } \frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1} = \frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1 + (\alpha-1)\sigma_1 < \lambda_1 < 2 + (\alpha-2)\sigma_1, \\ tM(t), & \text{if } \lambda_1 = 1 + (\alpha-1)\sigma_1 \text{ and } \lambda_2 = 1 + (\alpha-1)\sigma_2, \\ t\Big(\int_t^{\eta} \frac{L_1(s)}{s}ds\Big)^{\frac{1}{1-\sigma_1}} & \text{if } \lambda_1 = 1 + (\alpha-1)\sigma_1 \text{ and } \lambda_2 < 1 + (\alpha-1)\sigma_2, \\ t & \text{if } \lambda_1 < 1 + (\alpha-1)\sigma_1, \\ N(t) & \text{if } \lambda_1 = 2 + (\alpha-2)\sigma_1 \text{ and } \lambda_2 = 2 + (\alpha-2)\sigma_2. \end{cases} \tag{3.1}$$

On the other hand, since $\zeta_1 < \zeta_2$ is equivalent to

$$\frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1} < \frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2} \quad \text{and} \quad 1 + (\alpha-1)\sigma_2 \leq \mu_2 < 2 + (\alpha-2)\sigma_2,$$

we deduce from from (1.8) and Remark 3.1(ii) that for $t \geq 1$

$$\theta(t) \approx \begin{cases} (\int_1^{t+1} \frac{\widetilde{L}_2(s)}{s}ds)^{\frac{1}{1-\sigma_2}} & \text{if } \mu_2 = 2 + (\alpha-2)\sigma_2 \text{ and } \mu_1 > 2 + (\alpha-2)\sigma_1, \\ t^{\frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}} \Big(\widetilde{L}_2(t)\Big)^{\frac{1}{1-\sigma_2}} & \text{if } \frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1} < \frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1 + (\alpha-1)\sigma_2 < \mu_2 < 2 + (\alpha-2)\sigma_2, \\ t^{\frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}} \widetilde{L}(t) & \text{if } \frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1} = \frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1 + (\alpha-1)\sigma_2 < \mu_2 < 2 + (\alpha-2)\sigma_2, \\ t\widetilde{N}(t), & \text{if } \mu_2 = 1 + (\alpha-1)\sigma_2 \text{ and } \mu_1 = 1 + (\alpha-1)\sigma_1, \\ t\Big(\int_t^{\infty} \frac{\widetilde{L}_2(s)}{s}ds\Big)^{\frac{1}{1-\sigma_2}} & \text{if } \mu_2 = 1 + (\alpha-1)\sigma_2 \text{ and } \mu_1 > 1 + (\alpha-1)\sigma_1, \\ 1 & \text{if } \mu_2 > 2 + (\alpha-2)\sigma_2, \\ \widetilde{M}(t) & \text{if } \mu_1 = 2 + (\alpha-2)\sigma_1 \text{ and } \mu_2 = 2 + (\alpha-2)\sigma_2. \end{cases} \tag{3.2}$$

3.1. Proof of Theorem 1.4

Lemma 3.2. *For $r, s > 0$, we have*

$$2^{-\max(1-\sigma_1,1-\sigma_2)}(r+s) \le r^{1-\sigma_1}(r+s)^{\sigma_1} + s^{1-\sigma_2}(r+s)^{\sigma_2} \le 2(r+s).$$

Proof. Let $r, s > 0$ and put

$$t = \frac{r}{r+s}.$$

Since for $0 \le t \le 1$, we have

$$2^{-\max(1-\sigma_1,1-\sigma_2)} \le t^{1-\sigma_1} + (1-t)^{1-\sigma_2} \le 2,$$

this implies the result. □

The following lemmas will be useful in the proof of Theorem 1.4.

Lemma 3.3 ([11])**.**

(*i*) *On $(0, \eta)$, we have*

$$\int_t^\eta \frac{(M^{\sigma_1}L_1 + M^{\sigma_2}L_2)(s)}{s} ds \approx M(t).$$

(*ii*) *Assume that for $i \in \{1,2\}$, $\int_0^\eta \frac{L_i(s)}{s} ds < \infty$. Then, for $t \in (0, \eta)$,*

$$\int_0^t \frac{(N^{\sigma_1}L_1 + N^{\sigma_2}L_2)(s)}{s} ds \approx N(t).$$

Lemma 3.4 ([9])**.**

(*i*) *On $(1, \infty)$, we have*

$$\int_1^{t+1} \frac{(\widetilde{M}^{\sigma_1}\widetilde{L}_1 + \widetilde{M}^{\sigma_2}\widetilde{L}_2)(s)}{s} ds \approx \widetilde{M}(t).$$

(*ii*) *Assume that for $i \in \{1,2\}$, $\int_1^\infty \frac{\widetilde{L}_i(s)}{s} ds < \infty$. Then, for $t > 1$,*

$$\int_t^\infty \frac{(\widetilde{N}^{\sigma_1}\widetilde{L}_1 + \widetilde{N}^{\sigma_2}\widetilde{L}_2)(s)}{s} ds \approx \widetilde{N}(t).$$

We are ready to prove Theorem 1.4.

For $t > 0$, we let

$$\omega(t) = a_1(t)t^{(\alpha-2)\sigma_1}\theta^{\sigma_1}(t) + a_2(t)t^{(\alpha-2)\sigma_2}\theta^{\sigma_2}(t)$$

and we will prove that

$$t^{2-\alpha}V(\omega)(t) \approx \theta(t).$$

We recall that for $i \in \{1,2\}$, $\nu_i = \min(1, \frac{2-\lambda_i+(\alpha-2)\sigma_i}{1-\sigma_i})$ and $\zeta_i = \max(0, \frac{2-\mu_i+(\alpha-2)\sigma_i}{1-\sigma_i})$ and

$$a_i(t) \approx t^{-\lambda_i}(1+t)^{\lambda_i-\mu_i} L_i(1 \wedge t)\widetilde{L}_i(1 \vee t),\ t > 0.$$

Throughout the proof, we use Proposition 2.4, Lemma 2.6 and Lemma 2.7 to verify that some functions are in $\mathcal{K}$ and in $\mathcal{K}^\infty$.

We distinguish the following cases:

Case 1: If $\nu_1 < \nu_2$ and $\zeta_1 < \zeta_2$, then

$$\theta(t) = t^{\nu_1}(1+t)^{\zeta_2-\nu_1}\Psi_{L_1,\lambda_1,\sigma_1}(1 \wedge t)\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}(1 \vee t).$$

Therefore,

$$\begin{aligned}\omega(t) \approx & t^{-\lambda_1+\nu_1\sigma_1+(\alpha-2)\sigma_1}(1+t)^{\lambda_1-\mu_1+(\zeta_2-\nu_1)\sigma_1} \times (L_1\Psi^{\sigma_1}_{L_1,\lambda_1,\sigma_1})(1\wedge t)(\widetilde{L}_1\widetilde{\Psi}^{\sigma_1}_{\widetilde{L}_2,\mu_2,\sigma_2})(1\vee t)\\ &+ t^{-\lambda_2+\nu_1\sigma_2+(\alpha-2)\sigma_2}(1+t)^{\lambda_2-\mu_2+(\zeta_2-\nu_1)\sigma_2} \times (L_2\Psi^{\sigma_2}_{L_1,\lambda_1,\sigma_1})(1\wedge t)(\widetilde{L}_2\widetilde{\Psi}^{\sigma_2}_{\widetilde{L}_2,\mu_2,\sigma_2})(1\vee t).\end{aligned}$$

Since $\nu_1 < \nu_2$ and $\zeta_1 < \zeta_2$, we deduce by Proposition 2.4 that

$$\omega(t) \approx t^{-\lambda_1+\nu_1\sigma_1+(\alpha-2)\sigma_1}(L_1\Psi^{\sigma_1}_{L_1,\lambda_1,\sigma_1})(1\wedge t)\times(1+t)^{\lambda_1-\nu_1\sigma_1-(\alpha-2)\sigma_1-(\mu_2-\zeta_2\sigma_2-(\alpha-2)\sigma_2)}(\widetilde{L}_2\widetilde{\Psi}^{\sigma_2}_{\widetilde{L}_2,\mu_2,\sigma_2})(1\vee t).$$

Furthermore, by using (1.5) and Proposition 2.5, we have

$$\int_0^\eta s^{1-\lambda_1+\nu_1\sigma_1+(\alpha-2)\sigma_1}(L_1\Psi^{\sigma_1}_{L_1,\lambda_1,\sigma_1})(s)ds < \infty,$$
$$\int_1^\infty s^{-\mu_2+\zeta_2\sigma_2+(\alpha-2)\sigma_2}(\widetilde{L}_2\widetilde{\Psi}^{\sigma_2}_{\widetilde{L}_2,\mu_2,\sigma_2})(s)ds < \infty.$$

So, by applying Proposition 2.8 with $\beta = \lambda_1-\nu_1\sigma_1-(\alpha-2)\sigma_1$, $\gamma = \mu_2-\zeta_2\sigma_2-(\alpha-2)\sigma_2$, $L_0 = L_1\Psi^{\sigma_1}_{L_1,\lambda_1,\sigma_1} \in \mathcal{K}$ and $\widetilde{L}_0 = \widetilde{L}_2\widetilde{\Psi}^{\sigma_2}_{\widetilde{L}_2,\mu_2,\sigma_2} \in \mathcal{K}^\infty$, we obtain for $t > 0$

$$t^{2-\alpha}V\omega(t) \approx \psi_\beta(1\wedge t)\phi_\gamma(1\vee t),$$

where, for $r \in (0,1]$,

$$\psi_\beta(r) = \begin{cases} \int_0^r \frac{L(s)}{s}\left(\int_0^s \frac{L(t)}{t}dt\right)^{\frac{\sigma_1}{1-\sigma_1}} ds & \text{if } \lambda_1 = 2+(\alpha-2)\sigma_1 \text{ and} \lambda_2 < 2+(\alpha-2)\sigma_2,\\ r^{2-\beta}L_1(r)(L_1(r))^{\frac{\sigma_1}{1-\sigma_1}} & \text{if } \frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1} < \frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1+(\alpha-1)\sigma_1 < \lambda_1 < 2+(\alpha-2)\sigma_2,\end{cases}$$

$$= \begin{cases} \left(\int_0^r \frac{L(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}} & \text{if } \lambda_1 = 2+(\alpha-2)\sigma_1 \text{ and } \lambda_2 < 2+(\alpha-2)\sigma_2,\\ r^{\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1}}(L_1(r))^{\frac{1}{1-\sigma_1}} & \text{if } \frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1} < \frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1+(\alpha-1)\sigma_1 < \lambda_1 < 2+(\alpha-2)\sigma_2, \end{cases},$$

and for $r \geq 1$,

$$\phi_\gamma(r) = \begin{cases} r\int_r^\infty \frac{\widetilde{L}_2(s)}{s}\left(\int_s^\infty \frac{\widetilde{L}_2(t)}{t}dt\right)^{\frac{\sigma_2}{1-\sigma_2}} ds & \text{if } \mu_2 = 1+(\alpha-1)\sigma_2 \\ & \text{and } \mu_1 > 1+(\alpha-1)\sigma_1,\\ r^{2-\gamma}\widetilde{L}_2(r)(\widetilde{L}_2(r))^{\frac{\sigma_2}{1-\sigma_2}} & \text{if } \frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1} < \frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1+(\alpha-1)\sigma_2 < \mu_2 < 2+(\alpha-2)\sigma_2,\end{cases}$$

$$= \begin{cases} r\left(\int_r^\infty \frac{\widetilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma_2}} & \text{if } \mu_2 = 1+(\alpha-1)\sigma_2 \\ & \text{and } \mu_1 > 1+(\alpha-1)\sigma_1,\\ r^{\frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}}(\widetilde{L}_2(r))^{\frac{1}{1-\sigma_2}} & \text{if } \frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1} < \frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1+(\alpha-1)\sigma_2 < \mu_2 < 2+(\alpha-2)\sigma_2.\end{cases}$$

Hence, Proposition 1.3(iii) and (3.1)–(3.2) give that

$$t^{2-\alpha}V\omega(t) \approx \theta(t).$$

Case 2: If $\nu_1 < \nu_2$ and $\zeta_1 = \zeta_2$, then for $t > 0$,

$$\theta(t) = t^{\nu_1}(1+t)^{\zeta_2-\nu_1}\Psi_{L_1,\lambda_1,\sigma_1}(1\wedge t)\left(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1} + \widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}\right)(1\vee t).$$

In this case,

$$\omega(t) \approx t^{-\lambda_1+\nu_1\sigma_1+(\alpha-2)\sigma_1}(1+t)^{\lambda_1-\mu_1+(\zeta_2-\nu_1)\sigma_1} \times (L_1\Psi^{\sigma_1}_{L_1,\lambda_1,\sigma_1})(1\wedge t)\left[\widetilde{L}_1\left(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}\right)^{\sigma_1}\right](1\vee t)$$
$$+ t^{-\lambda_2+\nu_1\sigma_2+(\alpha-2)\sigma_2}(1+t)^{\lambda_2-\mu_2+(\zeta_2-\nu_1)\sigma_2} \times (L_2\Psi^{\sigma_2}_{L_1,\lambda_1,\sigma_1})(1\wedge t)\left[\widetilde{L}_2\left(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}\right)^{\sigma_2}\right](1\vee t).$$

Since $\nu_1<\nu_2$ and $\zeta_1=\zeta_2$, we deduce that

$$\begin{aligned}\omega(t) &\approx t^{-\lambda_1+\nu_1\sigma_1+(\alpha-2)\sigma_1}(1+t)^{\lambda_1-\mu_1+(\zeta_1-\nu_1)\sigma_1} \times (L_1\Psi^{\sigma_1}_{L_1,\lambda_1,\sigma_1})(1\wedge t)\left[\widetilde{L}_1\left(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}\right)^{\sigma_1}\right](1\vee t)\\ &\quad + t^{-\lambda_1+\nu_1\sigma_1+(\alpha-2)\sigma_1}(1+t)^{\lambda_1-\nu_1\sigma_1-(\alpha-2)\sigma_1-(\mu_2-\zeta_2\sigma_2-(\alpha-2)\sigma_2)}\\ &\quad \times (L_1\Psi^{\sigma_1}_{L_1,\lambda_1,\sigma_1})(1\wedge t)\left[\widetilde{L}_2\left(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}\right)^{\sigma_2}\right](1\vee t)\\ &= \omega_1(t)+\omega_2(t).\end{aligned}$$

So, by applying Proposition 2.8 with $\beta=\lambda_1-\nu_1\sigma_1-(\alpha-2)\sigma_1$ and $\gamma_i=\mu_i-\zeta_i\sigma_i-(\alpha-2)\sigma_i$, $i\in\{1,2\}$ to estimate $t^{2-\alpha}V\omega_1(t)$ and $t^{2-\alpha}V\omega_2(t)$, we obtain

$$t^{2-\alpha}V\omega(t)\approx t^{2-\alpha}V\omega_1(t)+t^{2-\alpha}V\omega_2(t)\approx \psi_\beta(1\wedge t)\left[\phi_{\gamma_1}(1\vee t)+\phi_{\gamma_2}(1\vee t)\right],$$

where, for $r\in(0,1]$,

$$\psi_\beta(r)=\begin{cases}\left(\int_0^r \dfrac{L(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}} & \text{if } \lambda_1=2+(\alpha-2)\sigma_1 \text{ and } \lambda_2<2+(\alpha-2)\sigma_2,\\ r^{\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1}}(L_1(r))^{\frac{1}{1-\sigma_1}} & \text{if } \frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1}<\frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2}\\ & \text{and } 1+(\alpha-1)\sigma_1<\lambda_1<2+(\alpha-2)\sigma_2,\end{cases},$$

and for $r\geq 1$,

$$\begin{cases}\text{if } \mu_2=2+(\alpha-2)\sigma_2 \text{ and } \mu_1>2+(\alpha-2)\sigma_1,\\ \phi_{\gamma_1}(r)+\phi_{\gamma_2}(r)=1+\int_1^{r+1}\frac{\widetilde{L}_2(s)}{s}(1+\left(\int_1^{s+1}\frac{\widetilde{L}_2(t)}{t}dt\right)^{\frac{1}{1-\sigma_2}})^{\sigma_2}ds\\ \text{if } \frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1}=\frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2} \text{ and } 1+(\alpha-1)\sigma_2<\mu_2<2+(\alpha-2)\sigma_2,\\ \phi_{\gamma_1}(r)+\phi_{\gamma_2}(r)=r^{\frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}}\left[\widetilde{L}_1(r)(\widetilde{L}(r))^{\sigma_1}+\widetilde{L}_2(r)(\widetilde{L}(r))^{\sigma_2}\right]\\ \text{if } \mu_2=1+(\alpha-1)\sigma_2 \text{ and } \mu_1=1+(\alpha-1)\sigma_1,\\ \phi_{\gamma_1}(r)+\phi_{\gamma_2}(r)=r\int_r^\infty\frac{(\widetilde{L}_1\widetilde{N}^{\sigma_1}(s)+\widetilde{L}_2\widetilde{N}^{\sigma_2})(s)}{s}ds\\ \text{if } \mu_2>2+(\alpha-2)\sigma_2,\\ \phi_{\gamma_1}(r)+\phi_{\gamma_2}(r)=2\\ \text{if } \mu_1=2+(\alpha-2)\sigma_1\\ \phi_{\gamma_1}(r)+\phi_{\gamma_2}(r)=\int_1^{r+1}\frac{(\widetilde{M}^{\sigma_1}\widetilde{L}_1+\widetilde{M}^{\sigma_2}\widetilde{L}_2)(s)}{s}ds\end{cases} \tag{3.3}$$

So by using Remark 3.1, Lemma 3.2 and Lemma 3.4, we deduce that

$$\phi_{\gamma_1}(r)+\phi_{\gamma_2}(r)\approx\begin{cases}\left(\int_1^{r+1}\frac{\widetilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma_2}} & \text{if } \mu_2=2+(\alpha-2)\sigma_2 \text{ and } \mu_1>2+(\alpha-2)\sigma_1,\\ r^{\frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}}\widetilde{L}(r) & \text{if } \frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1}=\frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}\\ & \text{and } 1+(\alpha-1)\sigma_2<\mu_2<2+(\alpha-2)\sigma_2,\\ r\widetilde{N}(r) & \text{if } \mu_2=1+(\alpha-1)\sigma_2 \text{ and } \mu_1=1+(\alpha-1)\sigma_1,\\ 1 & \text{if } \mu_2>2+(\alpha-2)\sigma_2,\\ \widetilde{M}(r) & \text{if } \mu_1=2+(\alpha-2)\sigma_1 \text{ and} \mu_2=2+(\alpha-2)\sigma_2.\end{cases}$$

Hence by using again Proposition 1.3(iii) and (3.1)–(3.2), we deduce that

$$t^{2-\alpha}V\omega(t) \approx \theta(t).$$

Case 3: If $\nu_1 = \nu_2$ and $\zeta_1 < \zeta_2$, then for $t > 0$,

$$\theta(t) = t^{\nu_1}(1+t)^{\zeta_2-\nu_1}\left(\Psi_{L_1,\lambda_1,\sigma_1} + \Psi_{L_2,\lambda_2,\sigma_2}\right)(1\wedge t)\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}(1\vee t).$$

So we obtain

$$\begin{aligned}\omega(t) \approx{}& t^{-\lambda_1+\nu_1\sigma_1+(\alpha-2)\sigma_1}(1+t)^{\lambda_1-\mu_1+(\zeta_2-\nu_1)\sigma_1} \times (L_1\left(\Psi_{L_1,\lambda_1,\sigma_1} + \Psi_{L_2,\lambda_2,\sigma_2}\right)^{\sigma_1})(1\wedge t)(\widetilde{L}_1\widetilde{\Psi}^{\sigma_1}_{\widetilde{L}_2,\mu_2,\sigma_2})(1\vee t)\\ &+ t^{-\lambda_2+\nu_1\sigma_2+(\alpha-2)\sigma_2}(1+t)^{\lambda_2-\mu_2+(\zeta_2-\nu_1)\sigma_2} \times (L_2\left(\Psi_{L_1,\lambda_1,\sigma_1} + \Psi_{L_2,\lambda_2,\sigma_2}\right)^{\sigma_2})(1\wedge t)(\widetilde{L}_2\widetilde{\Psi}^{\sigma_2}_{\widetilde{L}_2,\mu_2,\sigma_2})(1\vee t).\end{aligned}$$

Since $\nu_1 = \nu_2$ and $\zeta_1 < \zeta_2$, we deduce that

$$\begin{aligned}\omega(t) \approx{}& t^{-\lambda_1+\nu_1\sigma_1+(\alpha-2)\sigma_1}(1+t)^{\lambda_1-\nu_1\sigma_1-(\alpha-2)\sigma_1-(\mu_2-\zeta_2\sigma_2-(\alpha-2)\sigma_2)}\\ &\times (L_1\left(\Psi_{L_1,\lambda_1,\sigma_1} + \Psi_{L_2,\lambda_2,\sigma_2}\right)^{\sigma_1})(1\wedge t)(\widetilde{L}_2\widetilde{\Psi}^{\sigma_2}_{\widetilde{L}_2,\mu_2,\sigma_2})(1\vee t)\\ &+ t^{-\lambda_2+\nu_2\sigma_2+(\alpha-2)\sigma_2}(1+t)^{\lambda_2-\mu_2+(\zeta_2-\nu_2)\sigma_2}\\ &\times (L_2\left(\Psi_{L_1,\lambda_1,\sigma_1} + \Psi_{L_2,\lambda_2,\sigma_2}\right)^{\sigma_2})(1\wedge t)(\widetilde{L}_2\widetilde{\Psi}^{\sigma_2}_{\widetilde{L}_2,\mu_2,\sigma_2})(1\vee t)\\ ={}&\omega_1(t) + \omega_2(t).\end{aligned}$$

So, using again Proposition 2.8 with $\beta_i = \lambda_i - \nu_i\sigma_i - (\alpha-2)\sigma_i$, $i \in \{1,2\}$ and $\gamma = \mu_2 - \zeta_2\sigma_2 - (\alpha-2)\sigma_2$, to estimate $t^{2-\alpha}V\omega_1(t)$ and $t^{2-\alpha}V\omega_2(t)$, we obtain

$$t^{2-\alpha}V\omega(t) \approx t^{2-\alpha}V\omega_1(t) + t^{2-\alpha}V\omega_2(t) \approx \left[\psi_{\beta_1}(1\wedge t) + \psi_{\beta_2}(1\wedge t)\right]\phi_\gamma(1\vee t),$$

where, for $r \geq 1$ (see Case 1),

$$\phi_\gamma(r) = \begin{cases} r\left(\int_r^\infty \frac{\widetilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma_2}} & \text{if } \mu_2 = 1+(\alpha-1)\sigma_2 \text{ and } \mu_1 > 1+(\alpha-1)\sigma_1,\\ r^{\frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}}(\widetilde{L}_2(r))^{\frac{1}{1-\sigma_2}} & \text{if } \frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1} < \frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}\\ & \text{and } 1+(\alpha-1)\sigma_2 < \mu_2 < 2+(\alpha-2)\sigma_2\end{cases}$$

and for $r \in (0,1]$,

$$\begin{aligned}&\psi_{\beta_1}(r) + \psi_{\beta_2}(r)\\ &= \begin{cases} r^{\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1}}\left[L_1(r)(L(r))^{\sigma_1} + L_2(r)(L(r))^{\sigma_2}\right] & \text{if } \frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1} = \frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2}\\ & \text{and } 1+(\alpha-1)\sigma_1 < \lambda_1 < 2+(\alpha-2)\sigma_1,\\ r\int_r^\eta \frac{(L_1M^{\sigma_1}+L_2M^{\sigma_2})(s)}{s}ds & \text{if } \lambda_1 = 1+(\alpha-1)\sigma_1\\ & \text{and } \lambda_2 = 1+(\alpha-1)\sigma_2,\\ r\int_r^\eta \frac{L_1(s)}{s}\left(1+\left(\int_s^\eta \frac{L_1(t)}{t}dt\right)^{\frac{1}{1-\sigma_1}}\right)^{\sigma_1} + r & \text{if } \lambda_1 = 1+(\alpha-1)\sigma_1\\ & \text{and } \lambda_2 < 1+(\alpha-1)\sigma_2,\\ 2r & \text{if } \lambda_1 < 1+(\alpha-1)\sigma_1,\\ \int_0^r \frac{(L_1N^{\sigma_1}+L_2N^{\sigma_2})(s)}{s}ds & \text{if } \lambda_1 = 2+(\alpha-2)\sigma_1\\ & \text{and } \lambda_2 = 2+(\alpha-2)\sigma_2.\end{cases}\end{aligned}$$

Hence, by applying Lemma 3.2, Remark 3.1 and Lemma 3.3, we deduce that

$$\psi_{\beta_1}(r)+\psi_{\beta_2}(r)\approx\begin{cases} r^{\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1}}L(r) & \text{if } \frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1}=\frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1+(\alpha-1)\sigma_1<\lambda_1<2+(\alpha-2)\sigma_1, \\ rM(r) & \text{if } \lambda_1=1+(\alpha-1)\sigma_1 \text{ and } \lambda_2=1+(\alpha-1)\sigma_2, \\ r\left(\int_r^\eta \frac{L_1(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}} & \text{if } \lambda_1=1+(\alpha-1)\sigma_1 \text{ and } \lambda_2<1+(\alpha-1)\sigma_2, \\ 2r & \text{if } \lambda_1<1+(\alpha-1)\sigma_1, \\ N(r) & \text{if } \lambda_1=2+(\alpha-2)\sigma_1 \text{ and } \lambda_2=2+(\alpha-2)\sigma_2. \end{cases} \tag{3.4}$$

Using (3.1)–(3.2), we deduce that

$$t^{2-\alpha}V\omega(t)\approx\theta(t).$$

Case 4: If $\nu_1=\nu_2$ and $\zeta_1=\zeta_2$, then for $t>0$,

$$\theta(t)=t^{\nu_1}(1+t)^{\zeta_2-\nu_1}\left(\Psi_{L_1,\lambda_1,\sigma_1}+\Psi_{L_2,\lambda_2,\sigma_2}\right)(1\wedge t)\times\left(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2}\right)(1\vee t).$$

In this case, we have

$$\begin{aligned}\omega(t)\approx&t^{-\lambda_1+\nu_1\sigma_1+(\alpha-2)\sigma_1}(1+t)^{\lambda_1-\mu_1+(\zeta_2-\nu_1)\sigma_1}\\ &\times(L_1(\Psi_{L_1,\lambda_1,\sigma_1}+\Psi_{L_2,\lambda_2,\sigma_2})^{\sigma_1})(1\wedge t)(\widetilde{L}_1(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2})^{\sigma_1})(1\vee t)\\ &+t^{-\lambda_2+\nu_1\sigma_2+(\alpha-2)\sigma_2}(1+t)^{\lambda_2-\mu_2+(\zeta_2-\nu_1)\sigma_2}\\ &\times(L_2(\Psi_{L_1,\lambda_1,\sigma_1}+\Psi_{L_2,\lambda_2,\sigma_2})^{\sigma_2})(1\wedge t)(\widetilde{L}_2(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2})^{\sigma_2})(1\vee t).\end{aligned}$$

Since $\nu_1=\nu_2$ and $\zeta_1=\zeta_2$, we deduce that

$$\begin{aligned}\omega(t)\approx&t^{-\lambda_1+\nu_1\sigma_1+(\alpha-2)\sigma_1}(1+t)^{\lambda_1-\mu_1+(\zeta_1-\nu_1)\sigma_1}\\ &\times(L_1(\Psi_{L_1,\lambda_1,\sigma_1}+\Psi_{L_2,\lambda_2,\sigma_2})^{\sigma_1})(1\wedge t)(\widetilde{L}_1(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2})^{\sigma_1})(1\vee t)\\ &+t^{-\lambda_2+\nu_2\sigma_2+(\alpha-2)\sigma_2}(1+t)^{\lambda_2-\mu_2+(\zeta_2-\nu_2)\sigma_2}\\ &\times(L_2(\Psi_{L_1,\lambda_1,\sigma_1}+\Psi_{L_2,\lambda_2,\sigma_2})^{\sigma_2})(1\wedge t)(\widetilde{L}_2(\widetilde{\Psi}_{\widetilde{L}_1,\mu_1,\sigma_1}+\widetilde{\Psi}_{\widetilde{L}_2,\mu_2,\sigma_2})^{\sigma_2})(1\vee t)\\ =&\omega_1(t)+\omega_2(t).\end{aligned}$$

Applying Proposition 2.8 with $\beta_i=\lambda_i-\nu_i\sigma_i-(\alpha-2)\sigma_i$ and $\gamma_i=\mu_i-\zeta_i\sigma_i-(\alpha-2)\sigma_i$, $i\in\{1,2\}$ to estimate $t^{2-\alpha}V\omega_1(t)$ and $t^{2-\alpha}V\omega_2(t)$, we obtain

$$t^{2-\alpha}V\omega(t)\approx t^{2-\alpha}V\omega_1(t)+t^{2-\alpha}V\omega_2(t)\approx\left[\psi_{\beta_1}(1\wedge t)+\psi_{\beta_2}(1\wedge t)\right]\times\left[\phi_{\gamma_1}(1\vee t)+\phi_{\gamma_2}(1\vee t)\right],$$

where the expression for $\psi_{\beta_1}(1\wedge t)+\psi_{\beta_2}(1\wedge t)$ (resp. $\phi_{\gamma_1}(1\vee t)+\phi_{\gamma_2}(1\vee t)$) is given in (3.4) (resp. (3.3)). The required result follows by similar arguments as in the previous cases.

3.2. Proof of Theorem 1.5

Let $1<\alpha<2$, $\sigma_1,\sigma_2\in(-1,1)$ and assume that hypothesis (H) is satisfied. By Theorem 1.4, there exists an $M>1$ such that for each $t>0$

$$\frac{1}{M}\theta(t)\leq t^{2-\alpha}V\omega(t)\leq M\theta(t), \tag{3.5}$$

where $\omega(t)=a_1(t)t^{(\alpha-2)\sigma_1}\theta^{\sigma_1}(t)+a_2(t)t^{(\alpha-2)\sigma_2}\theta^{\sigma_2}(t)$.

Put $\sigma=\max(|\sigma_1|,|\sigma_2|)$ and $c_0=M^{\frac{1+\sigma}{1-\sigma}}$. In order to use a fixed point theorem, we let

$$\Lambda=\left\{v\in C_0([0,\infty)):\frac{\theta(t)}{c_0(1+t)}\leq v(t)\leq\frac{c_0\theta(t)}{1+t},\ t>0\right\}.$$

and we define the operator T on Λ by

$$Tv(t) = \frac{t^{2-\alpha}}{1+t}\int_0^\infty G_\alpha(t,s)\left[a_1(s)s^{(\alpha-2)\sigma_1}(1+s)^{\sigma_1}v^{\sigma_1}(s) + a_2(s)s^{(\alpha-2)\sigma_2}(1+s)^{\sigma_2}v^{\sigma_2}(s)\right]ds. \tag{3.6}$$

Since $t \to \dfrac{\theta(t)}{1+t} \in C_0([0,\infty))$, it follows that Λ is not empty.

On the other hand, by using (3.5) and a simple computation, one has

$$Tv(t) \le \frac{c_0\theta(t)}{1+t} \ \text{ and } \ Tv(t) \ge \frac{\theta(t)}{c_0\,(1+t)} \text{ for all } v\in\Lambda \text{ and } t>0. \tag{3.7}$$

By Lemma 2.3 (i), there exists a $c>0$ such that for all $t,s>0$,

$$\frac{t^{2-\alpha}G_\alpha(t,s)}{1+t} \le c\frac{\max(1,t)\min(1,s)}{1+t} \le c\min(1,s), \tag{3.8}$$

This implies that there exists a $\widetilde{c}>0$ such that, for each $v\in\Lambda$ and $t>0$,

$$|Tv(t)| \le \widetilde{c}\int_0^\infty \min(1,s)\omega(s)ds.$$

Now by hypothesis (H) and Proposition 2.5, the function $t\mapsto \min(1,t)\omega(t)$ is in $L^1(0,\infty)$, which implies that the family $\{Tv(t),\ v\in\Lambda\}$ is uniformly bounded.

Using (3.8) and the fact that for each $s>0$, the function $t\mapsto \dfrac{t^{2-\alpha}G_\alpha(t,s)}{1+t}$ is in $C_0([0,\infty))$, we deduce that the family $\{Tv(t),\ v\in\Lambda\}$ is equicontinuous in $[0,\infty]$.

Hence, it follows by Ascoli's theorem that $T(\Lambda)$ is relatively compact in $C_0([0,\infty))$. Therefore $T(\Lambda)\subset\Lambda$.

Next, we shall prove the continuity of T in the supremum norm. Let $(v_k)_k$ be a sequence in Λ which converges to v in Λ. Using again (3.8) and Lebesgue's theorem, we deduce that $Tv_k(t)\to Tv(t)$ as $k\to\infty$, for $t>0$.

Since $T(\Lambda)$ is relatively compact in $C_0([0,\infty))$, then the point-wise convergence implies the uniform convergence. Thus we proved that T is a compact mapping from Λ to itself.

So, by the Schauder fixed point theorem, there exists a function $v\in\Lambda$ such that

$$v(t) = \frac{t^{2-\alpha}}{1+t}\int_0^\infty G_\alpha(t,s)\left[a_1(s)s^{(\alpha-2)\sigma_1}(1+s)^{\sigma_1}v^{\sigma_1}(s) + a_2(s)s^{(\alpha-2)\sigma_2}(1+s)^{\sigma_2}v^{\sigma_2}(s)\right]ds.$$

Put $u(t)=t^{\alpha-2}(1+t)v(t)$. Then $u\in C_{2-\alpha}([0,\infty))$ and u satisfies on $(0,\infty)$ the equation

$$u(t) = V(a_1u^{\sigma_1}+a_2u^{\sigma_2})(t).$$

Since the function $s\mapsto \min(1,s)(a_1u^{\sigma_1}+a_2u^{\sigma_2})(s)$ is continuous and integrable on $(0,\infty)$, then by Lemma 2.3 (ii), we deduce that the function u is a positive solution to problem (1.1) satisfying (1.10).

Finally, it remains to prove that u is the unique positive solution in $C_{2-\alpha}([0,\infty))$ satisfying (1.10). Let $u,v\in C_{2-\alpha}([0,\infty))$ be two positive solutions to problem (1.1) satisfying (1.10). Then, there exists a constant $m>1$ such that

$$\frac{1}{m} \le \frac{u}{v} \le m.$$

This implies that the set

$$J=\{m\ge 1 : \frac{1}{m}\le\frac{u}{v}\le m\}$$

is not empty. Let $\overline{c}=\inf J$. Then $\overline{c}\ge 1$ and we have $\frac{1}{\overline{c}}v\le u\le \overline{c}v$. It follows that $u^{\sigma_i}\le \overline{c}^{\sigma}v^{\sigma_i}$ for $i\in\{1,2\}$.

Consequently

$$\begin{cases} -D^\alpha(\overline{c}^\sigma v-u) = a_1(\overline{c}^\sigma v^{\sigma_1}-u^{\sigma_1}) + a_2(\overline{c}^\sigma v^{\sigma_2}-u^{\sigma_2}) \ge 0, \\ \lim_{t\to 0} t^{2-\alpha}(\overline{c}^\sigma v-u)(t)=0, \quad \lim_{t\to\infty} t^{1-\alpha}(\overline{c}^\sigma v-u)(t)=0. \end{cases}$$

This implies by Lemma 2.2 that $\overline{c}^\sigma v-u = V(a_1(\overline{c}^\sigma v^{\sigma_1}-u^{\sigma_1})+a_2(\overline{c}^\sigma v^{\sigma_2}-u^{\sigma_2}))\ge 0$. By symmetry, we also have $v\le\overline{c}^\sigma u$. Hence $\overline{c}^\sigma\in J$ and $\overline{c}\le\overline{c}^\sigma$. Since $|\sigma|<1$, then $\overline{c}=1$ and therefore $u=v$.

Example 3.5. Let $1 < \alpha < 2$ and $-1 < \sigma_1 < 0 < \sigma_2 < 1$.

For $i \in \{1,2\}$, let $\lambda_i < 2 + (\alpha-2)\sigma_i$, $\mu_i > 1 + (\alpha-1)\sigma_i$ such that

$$\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1} \le \frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2} \text{ and } \frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1} \le \frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}.$$

Let $a_i \in C^+((0,\infty))$ such that

$$a_i(t) \approx t^{-\lambda_i}(1+t)^{\lambda_i-\mu_i}\log(\frac{2}{t\wedge 1}), \quad t > 0.$$

Then, by Theorem 1.5, problem (1.1) has a unique positive solution $u \in C_{2-\alpha}([0,\infty))$ satisfying for $t > 0$,

$$u(t) \approx t^{\alpha-2}\theta(t),$$

where for $t \in (0,1]$,

$$\theta(t) \approx \begin{cases} t^{\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1}}\left(\log(\frac{2}{t})\right)^{\frac{1}{1-\sigma_1}} & \text{if } \frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1} < \frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1+(\alpha-1)\sigma_1 < \lambda_1 < 2+(\alpha-2)\sigma_1, \\ t^{\frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1}}\left(\log(\frac{2}{t})\right)^{\frac{1}{1-\sigma_2}} & \text{if } \frac{2-\lambda_1+(\alpha-2)\sigma_1}{1-\sigma_1} = \frac{2-\lambda_2+(\alpha-2)\sigma_2}{1-\sigma_2} \\ & \text{and } 1+(\alpha-1)\sigma_1 < \lambda_1 < 2+(\alpha-2)\sigma_1, \\ t\left(\log(\frac{2}{t})\right)^{\frac{2}{1-\sigma_2}}, & \text{if } \lambda_1 = 1+(\alpha-1)\sigma_1 \text{ and } \lambda_2 = 1+(\alpha-1)\sigma_2, \\ t\left(\log(\frac{2}{t})\right)^{\frac{2}{1-\sigma_1}} & \text{if } \lambda_1 = 1+(\alpha-1)\sigma_1 \text{ and } \lambda_2 < 1+(\alpha-1)\sigma_2, \\ t & \text{if } \lambda_1 < 1+(\alpha-1)\sigma_1, \end{cases}$$

and for $t \ge 1$,

$$\theta(t) \approx \begin{cases} t^{\frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2}} & \text{if } \frac{2-\mu_1+(\alpha-2)\sigma_1}{1-\sigma_1} \le \frac{2-\mu_2+(\alpha-2)\sigma_2}{1-\sigma_2} \text{ and } 1+(\alpha-1)\sigma_2 < \mu_2 < 2+(\alpha-2)\sigma_2, \\ 1 & \text{if } \mu_2 > 2+(\alpha-2)\sigma_2, \\ (\log(1+t))^{\frac{1}{1-\sigma_2}} & \text{if } \mu_1 = 2+(\alpha-2)\sigma_1 \text{ and } \mu_2 = 2+(\alpha-2)\sigma_2. \end{cases}$$

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