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# Existence of solutions for nonlinear impulsive $q_k$-difference equations with first-order $q_k$-derivatives

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**Abstract**

In this paper, we study the nonlinear second-order impulsive $q_k$-difference equations with Sturm-Liouville type, in which nonlinear team and impulsive teams are dependent on first-order $q_k$-derivatives. We obtain the existence and uniqueness results of solutions for the problem by Banach's contraction mapping principle and Schaefer's fixed point theorems. Finally, we give two examples to demonstrate the use of the main results. 

*Keywords:* $q_k$-derivative, $q_k$-integral, impulsive $q_k$-difference equation, boundary value problem, fixed point theorem.
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## 1. Introduction

The $q$-difference equations initiated in the beginning of the 20th century [1, 10, 13, 17], is a very interesting field in difference equations. In the last few decades, it has evolved into a multidisciplinary subject and plays an important role in several fields of physics, such as cosmic strings and black holes [18], conformal quantum mechanics [23], nuclear and high energy physics [16]. However, the theory of boundary value problems (BVPs) for nonlinear $q$-difference equations is still in the initial stages and many aspects of this theory need to be explored. The book by Kac and Cheung [14] covers many of the fundamental aspects of quantum

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calculus. A variety of new results can be found in the papers [2, 3, 4, 5, 6, 7, 8, 11, 12, 24, 25, 26, 27] and the references therein.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Henceforth, impulsive differential equations have gained considerable importance due to their application in various sciences, such as control theory, population dynamics and medicine and so on. For some recent results on the theory of impulsive differential equations, see [9, 15, 19, 21] and the references therein. To the best of our knowledge, the study of BVPs for nonlinear impulsive $q_k$-difference equation with Sturm-Liouville type is yet to be initiated.

Recently, in [22], C. Thaiprayoon, J. Tariboon and S.K. Ntouyas studied the separated boundary value problem for impulsive $q_k$-integro-difference equation:

$$\begin{cases} D_{q_k}^2 x(t) = f(t, x(t), (S_{q_k}x)(t)), & t \in J, \quad t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \cdots, m, \\ D_{q_k}x(t_k^+) - D_{q_{k-1}}x(t_k) = I_k^*(x(t_k)), & k = 1, 2, \cdots, m, \\ x(0) + D_{q_0}x(0) = 0, \quad x(1) + D_{q_m}x(1) = 0, \end{cases}$$

where $J = [0, T], 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $f : J \times R^2 \to R$, $(S_{q_k}x)(t) = \int_{t_k}^t \phi(t, s)x(s)d_{q_k}s$, $t \in (t_k, t_{k+1}]$, $k = 1, 2, \cdots, m$, $\phi : J \times J \to [0, +\infty)$ is a continuous function, $I_k, I_k^* \in C(R, R), \Delta x(t_k) = x(t_k^+) - x(t_k), x(t_k^+) = \lim_{h\to 0^+} x(t_k + h), 0 < q_k < 1$ for $k = 1, 2, \cdots, m$.

Motivated by the work above, in this paper, we study the existence of solutions for a boundary value problem with nonlinear second-order impulsive $q_k$-difference equations

$$\begin{cases} D_{q_k}^2 x(t) = f(t, x(t), D_{q_k}x(t)), & t \in J, \quad t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k), D_{q_k}x(t_k)), & k = 1, 2, \cdots, m, \\ \Delta D_{q_k}x(t_k) = I_k^*(x(t_k), D_{q_k}x(t_k)), & k = 1, 2, \cdots, m, \\ \alpha x(0) - \beta D_{q_0}x(0) = 0, \quad \delta x(1) + \gamma D_{q_m}x(1) = 0, \end{cases} \tag{1.1}$$

where $J = [0, 1], 0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1, J_0 = [t_0, t_1], J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \cdots, m$, $f : J \times R^2 \to R$ is a continuous function, $I_k, I_k^* \in C(R^2, R)$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $\Delta D_{q_k}x(t_k) = D_{q_k}x(t_k^+) - D_{q_{k-1}}x(t_k)$, $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $D_{q_k}x(t_k^+) = \lim_{h\to 0^+} D_{q_k}x(t_k + h)$, $0 < q_k < 1$ for $k = 1, 2, \cdots, m$, $\alpha, \beta, \delta, \gamma$ are given nonnegative constants and $\delta\beta + \delta\alpha + \gamma\alpha \neq 0$.

We deal with the existence and uniqueness of solutions for BVP (1.1) by using the Schauder's fixed point theorem and Banach's contraction mapping principle and obtain multiplicity results which extend and improve the known results.

## 2. Preliminary results

In this section, firstly, let us recall some basic concepts of $q_k$-derivative and $q_k$-integral ([21]). For a fixed $k \in N \cup \{0\}$, let $J_k := [t_k, t_{k+1}] \subset R$ be an interval and $0 < q_k < 1$ be a constant.

**Definition 2.1.** Assume $f : J_k \to R$ is a continuous function and let $t \in J_k$. Then the expression

$$D_{q_k}f(t) = \frac{f(t) - f(q_k t + (1 - q_k)t_k)}{(1 - q_k)(t - t_k)}, \qquad t \neq t_k, \quad D_{q_k}f(t_k) = \lim_{t\to t_k} D_{q_k}f(t).$$

is called the $q_k$-derivative of function $f$ at $t$.

Note that if $t_k = 0$ and $q_k = q$, then $D_{q_k}f = D_q f$, here $D_q f$ is the well-known $q$-derivative of function $f(t)$ defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}.$$

**Definition 2.2.** Let $f : J_k \to R$ is a continuous function, we call the second-order $q_k$-derivative $D^2_{q_k} f$ provided $D_{q_k} f$ is $q_k$-differentiable on $J_k$ with $D^2_{q_k} f = D_{q_k}(D_{q_k} f) : J_k \to R$. Similarly, we define higher order $q_k$-derivative $D^n_{q_k} f : J_k \to R$.

**Definition 2.3.** Assume $f : J_k \to R$ is a continuous function. Then the $q_k$-integral is defined by

$$\int_{t_k}^{t} f(s) d_{q_k} s = (1 - q_k)(t - t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1 - q_k^n) t_k),$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$ then the definite $q_k$-integral is defined by

$$\int_{a}^{t} f(s) d_{q_k} s = \int_{t_k}^{t} f(s) d_{q_k} s - \int_{t_k}^{a} f(s) d_{q_k} s.$$

Note that if $t_k = 0, q_k = q$, then the $q_k$-integral reduces to $q$-integral

$$\int_{0}^{t} f(s) d_q s = (1 - q) \sum_{n=0}^{\infty} q^n f(q^n t)$$

for $t \in [0, \infty)$.

**Lemma 2.4.** *If $x(t)$ is a solution of (1.1), then for any $t \in J_k, k = 0, 1, 2, \cdots, m$,*

$$\begin{aligned} x(t) = &-(\eta_1 + \eta_2 t)\Bigg[\delta\Bigg(\sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f(\tau, x(\tau), D_{q_{k-1}} x(\tau)) d_{q_{k-1}} \tau d_{q_{k-1}} s + \sum_{k=1}^{m} I_k(x(t_k), D_{q_k} x(t_k))\Bigg) \\ &\qquad + \sum_{k=1}^{m+1} (\delta(1 - t_k) + \gamma) \int_{t_{k-1}}^{t_k} f(s, x(s), D_{q_{k-1}} x(s)) d_{q_{k-1}} s \\ &\qquad + \sum_{k=1}^{m} (\delta(1 - t_k) + \gamma) I_k^*(x(t_k), D_{q_k} x(t_k))\Bigg] \\ &+ \sum_{0<t_k<t} \Bigg(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f(\tau, x(\tau), D_{q_{k-1}} x(\tau)) d_{q_{k-1}} \tau d_{q_{k-1}} s + I_k(x(t_k), D_{q_k} x(t_k))\Bigg) \\ &+ \sum_{0<t_k<t} \Bigg(\int_{t_{k-1}}^{t_k} f(s, x(s), D_{q_{k-1}} x(s)) d_{q_{k-1}} s + I_k^*(x(t_k), D_{q_k} x(t_k))\Bigg)(t - t_k) \\ &+ \int_{t_k}^{t} \int_{t_k}^{s} f(\tau, x(\tau), D_{q_{k-1}} x(\tau)) d_{q_k} \tau d_{q_k} s, \end{aligned} \tag{2.1}$$

*with $\Sigma_{0<0}(\cdot) = 0$, where $\eta_1 = \frac{\beta}{\delta\beta + \delta\alpha + \gamma\alpha}$, $\eta_2 = \frac{\alpha}{\delta\beta + \delta\alpha + \gamma\alpha}$.*

*Proof.* For $t \in J_0$, using $q_0$-integral for the first equation of (1.1), we get

$$D_{q_0} x(t) = D_{q_0} x(0) + \int_0^t f(s, x(s) D_{q_0} x(s)) d_{q_0} s, \tag{2.2}$$

which leads to

$$D_{q_0} x(t_1) = D_{q_0} x(0) + \int_0^{t_1} f(s, x(s), D_{q_0} x(s)) d_{q_0} s. \tag{2.3}$$

For $t \in J_0$, we obtain by $q_0$-integrating (2.2),

$$x(t) = x(0) + D_{q_0} x(0) t + \int_0^t \int_0^s f(\tau, x(\tau), D_{q_0} x(\tau)) d_{q_0} \tau d_{q_0} s.$$

Using $\alpha x(0) - \beta D_{q_0} x(0) = 0$, we have

$$x(t) = \frac{\beta + \alpha t}{\alpha} D_{q_0} x(0) + \int_0^t \int_0^s f(\tau, x(\tau), D_{q_0} x(\tau)) d_{q_0}\tau d_{q_0} s.$$

In particular, for $t = t_1$,

$$x(t_1) = \frac{\beta + \alpha t_1}{\alpha} D_{q_0} x(0) + \int_0^{t_1} \int_0^s f(\tau, x(\tau), D_{q_0} x(\tau)) d_{q_0}\tau d_{q_0} s. \tag{2.4}$$

For $t \in J_1 = (t_1, t_2]$, $q_1$-integrating (1.1), we have

$$D_{q_1} x(t) = D_{q_1} x(t_1^+) + \int_{t_1}^t f(s, x(s), D_{q_1} x(s)) d_{q_1} s.$$

From the second impulsive equation of (1.1), we get

$$\begin{aligned} D_{q_1} x(t) =& D_{q_0} x(0) + \int_0^{t_1} f(s, x(s), D_{q_0} x(s)) d_{q_0} s + I_1^*(x(t_1), D_{q_1} x(t_1)) \\ &+ \int_{t_1}^t f(s, x(s), D_{q_1} x(s)) d_{q_1} s. \end{aligned} \tag{2.5}$$

Applying $q_1$-integral to (2.5) for $t \in J_1$, we have

$$\begin{aligned} x(t) =& x(t_1^+) + [D_{q_0} x(0) + \int_0^{t_1} f(s, x(s), D_{q_0} x(s)) d_{q_0} s + I_1^*(x(t_1), D_{q_1} x(t_1))](t - t_1) \\ &+ \int_{t_1}^t \int_{t_1}^s f(\tau, x(\tau), D_{q_1} x(\tau)) d_{q_1}\tau d_{q_1} s. \end{aligned} \tag{2.6}$$

Using the second impulsive equation of (1.1) with (2.4) and (2.6), we have

$$\begin{aligned} x(t) =& \frac{\beta + \alpha t}{\alpha} D_{q_0} x(0) + \int_0^{t_1} \int_0^s f(\tau, x(\tau), D_{q_0} x(\tau)) d_{q_0}\tau d_{q_0} s + I_1(x(t_1), D_{q_1} x(t_1)) \\ &+ \left[ \int_0^{t_1} f(s, x(s), D_{q_0} x(s)) d_{q_0} s + I_1^*(x(t_1), D_{q_1} x(t_1)) \right] (t - t_1) \\ &+ \int_{t_1}^t \int_{t_1}^s f(\tau, x(\tau), D_{q_1} x(\tau)) d_{q_1}\tau d_{q_1} s. \end{aligned}$$

Repeating the above process, for $t \in J$ we have

$$\begin{aligned} x(t) =& \frac{\beta + \alpha t}{\alpha} D_{q_0} x(0) + \sum_{0<t_k<t} (\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s f(\tau, x(\tau), D_{q_{k-1}} x(\tau)) d_{q_{k-1}}\tau d_{q_{k-1}} s + I_k(x(t_k), D_{q_k} x(t_k))) \\ &+ \sum_{0<t_k<t} (\int_{t_{k-1}}^{t_k} f(s, x(s), D_{q_{k-1}} x(s)) d_{q_{k-1}} s + I_k^*(x(t_k), D_{q_k} x(t_k)))(t - t_k) \\ &+ \int_{t_k}^t \int_{t_k}^s f(\tau, x(\tau), D_{q_k} x(\tau)) d_{q_k}\tau d_{q_k} s. \end{aligned} \tag{2.7}$$

For $t = 1$, we obtain

$$x(1) = \frac{\beta + \alpha}{\alpha} D_{q_0} x(0) + \sum_{k=1}^m (\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s f(\tau, x(\tau), D_{q_{k-1}} x(\tau)) d_{q_{k-1}}\tau d_{q_{k-1}} s + I_k(x(t_k), D_{q_k} x(t_k)))$$

$$+\sum_{k=1}^{m}(\int_{t_{k-1}}^{t_k} f(s,x(s),D_{q_{k-1}}x(s))d_{q_{k-1}}s+I_k^*(x(t_k),D_{q_k}x(t_k)))(1-t_k)$$
$$+\int_{t_m}^{1}\int_{t_m}^{s} f(\tau,x(\tau),D_{q_m}x(\tau))d_{q_m}\tau d_{q_m}s.$$

It is easy to see that

$$\begin{aligned} D_{q_k}x(t) =&D_{q_0}x(0)+\sum_{0<t_k<t}(\int_{t_{k-1}}^{t_k} f(s,x(s),D_{q_{k-1}}x(s))d_{q_{k-1}}s+I_k^*(x(t_k),D_{q_k}x(t_k))) \\ &+\int_{t_k}^{t} f(s,x(s),D_{q_k}x(s))d_{q_k}s. \end{aligned} \tag{2.8}$$

For $t=1$, we obtain

$$D_{q_m}x(1)=D_{q_0}x(0)+\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f(s,x(s),D_{q_{k-1}}x(\tau))d_{q_{k-1}}s+\sum_{k=1}^{m}I_k^*(x(t_k),D_{q_k}x(t_k)).$$

Applying the boundary condition $\delta x(1)+\gamma D_{q_m}x(1)=0$, we obtain

$$\begin{aligned} D_{q_0}x(0) =&\frac{-\alpha}{\delta\beta+\delta\alpha+\gamma\alpha}\Bigg[\delta\Bigg(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau),D_{q_{k-1}}x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+\sum_{k=1}^{m}I_k(x(t_k),D_{q_k}x(t_k))\Bigg) \\ &+\sum_{k=1}^{m+1}(\delta(1-t_k)+\gamma)\int_{t_{k-1}}^{t_k} f(s,x(s),D_{q_{k-1}}x(s))d_{q_{k-1}}s \\ &+\sum_{k=1}^{m}(\delta(1-t_k)+\gamma)I_k^*(x(t_k),D_{q_k}x(t_k))\Bigg]. \end{aligned}$$

Setting $\eta_1=\frac{\beta}{\delta\beta+\delta\alpha+\gamma\alpha}$, $\eta_2=\frac{\alpha}{\delta\beta+\delta\alpha+\gamma\alpha}$ and substituting the value $D_{q_0}x(0)$ into (2.7), we get (2.1) as requested. This completes the proof. □

We consider a Banach space $PC(J,R)$ with the norm $\|x\|=\max\{\|x\|_\infty,\|D_{q_k}x\|_\infty\}$, and $\|\cdot\|_\infty=\sup\{|\cdot|,t\in J,t\neq t_k\},x\in PC(J,R)$, where $PC(J,R)=\{x:J\to R:x(t)$ is continuous everywhere except for some $t_k$ at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-)=x(t_k),\ k=1,2,\cdots,m\}$.

Define an integral operator $T:PC(J,R)\to PC(J,R)$ by

$$\begin{aligned} Tx(t)=&-(\eta_1+\eta_2 t)\Bigg[\delta\Bigg(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau),D_{q_{k-1}}x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+\sum_{k=1}^{m}I_k(x(t_k),D_{q_k}x(t_k))\Bigg) \\ &+\sum_{k=1}^{m+1}(\delta(1-t_k)+\gamma)\int_{t_{k-1}}^{t_k} f(s,x(s),D_{q_{k-1}}x(s))d_{q_{k-1}}s \\ &+\sum_{k=1}^{m}(\delta(1-t_k)+\gamma)I_k^*(x(t_k),D_{q_k}x(t_k))\Bigg] \\ &+\sum_{0<t_k<t}\Bigg(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau,x(\tau),D_{q_{k-1}}x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k(x(t_k),D_{q_k}x(t_k))\Bigg) \\ &+\sum_{0<t_k<t}\Bigg(\int_{t_{k-1}}^{t_k} f(s,x(s),D_{q_{k-1}}x(s))d_{q_{k-1}}s+I_k^*(x(t_k),D_{q_k}x(t_k))\Bigg)(t-t_k) \\ &+\int_{t_k}^{t}\int_{t_k}^{s} f(\tau,x(\tau),D_{q_k}x(\tau))d_{q_k}\tau d_{q_k}s, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}D_{q_k}Tx(t) =&(-\eta_2)\Big[\delta\Big(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f(\tau, x(\tau), D_{q_{k-1}}x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s + \sum_{k=1}^{m} I_k(x(t_k), D_{q_k}x(t_k))\Big)\\ &+\sum_{k=1}^{m+1}(\delta(1-t_k)+\gamma)\int_{t_{k-1}}^{t_k} f(s, x(s), D_{q_{k-1}}x(s))d_{q_{k-1}}s\\ &+\sum_{k=1}^{m}(\delta(1-t_k)+\gamma)I_k^*(x(t_k), D_{q_k}x(t_k))\Big]\\ &+\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k} f(s, x(s), D_{q_{k-1}}x(s))d_{q_{k-1}}s + I_k^*(x(t_k), D_{q_k}x(t_k))\Big)\\ &+\int_{t_k}^{t} f(s, x(s), D_{q_k}x(s))d_{q_k}s.\end{aligned} \tag{2.10}$$

Obviously, $T$ is well defined and $x \in PC(J, R)$ is solution of BVP (1.1) if and only if $x$ is a fixed point of $T$.

**Theorem 2.5** ([20])**.** *(Schaefer's fixed point theorem) Let $T$ be a continuous and compact mapping of a Banach space $X$ into itself, such that the set $E = \{x \in X : x = \lambda Tx$ for some $0 \le \lambda \le 1\}$ is bounded. Then $T$ has a fixed point.*

## 3. Existence and uniqueness results

Throughout this paper, we adopt the following assumptions:

($H_1$) $f(t, x, y) \in C(J \times R^2, R)$, and there exist $L_1(t), L_2(t) \in C(J, R^+)$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le L_1(t)|x_1 - x_2| + L_2(t)|y_1 - y_2|,$$

for each $t \in J$ and $(x_1, y_1), (x_2, y_2) \in R^2$.

($H_2$) $I_k, I_k^* \in C(R^2, R)$ and there exist four positive functions $L_3(t), L_4(t), L_5(t), L_6(t) \in C(J, R^+)$ such that

$$|I_k(x_1, y_1) - I_k(x_2, y_2)| \le L_3(t)|x_1 - x_2| + L_4(t)|y_1 - y_2|,$$
$$|I_k^*(x_1, y_1) - I_k^*(x_2, y_2)| \le L_5(t)|x_1 - x_2| + L_6(t)|y_1 - y_2|,$$

for each $(x_1, y_1), (x_2, y_2) \in R^2$, $k = 1, 2, \cdots, m$. And Let $L = \max_{t\in J}(L_3(t) + L_4(t))$, $L^* = \max_{t\in J}(L_5(t) + L_6(t))$.

($H_3$) $f(t, x, y) \in C(J \times R^2, R)$, and there exist two positive constants $L_1, L_2$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le L_1|x_1 - x_2| + L_2|y_1 - y_2|,$$

for each $t \in J$ and $(x_1, y_1), (x_2, y_2) \in R^2$.

($H_4$) $I_k, I_k^* \in C(R^2, R)$ and there exist four positive constants $L_3, L_4, L_5, L_6$ such that

$$|I_k(x_1, y_1) - I_k(x_2, y_2)| \le L_3|x_1 - x_2| + L_4|y_1 - y_2|,$$
$$|I_k^*(x_1, y_1) - I_k^*(x_2, y_2)| \le L_5|x_1 - x_2| + L_6|y_1 - y_2|,$$

for each $(x_1, y_1), (x_2, y_2) \in R^2$, $k = 1, 2, \cdots, m$.

($H_5$) $f(t, x, y) \in C(J \times R^2, R)$, and there exist three functions $p(t), q(t), h(t) \in C(J, R^+)$ such that

$$|f(t, x, y)| \le p(t)|x| + q(t)|y| + h(t),$$

for each $t \in J$ and $(x, y) \in R^2$.

(H$_6$) $I_k, I_k^* \in C(R^2, R)$ and there exist positive constants $a_k, b_k, c_k, d_k, e_k, f_k$ such that

$$|I_k(x,y)| \leq a_k|x| + b_k|y| + e_k \text{ and } |I_k^*(x,y)| \leq c_k|x| + d_k|y| + f_k,$$

for each $(x, y) \in R^2, k = 1, 2, \cdots, m$, and note

$$a = \sum_{k=1}^{m} a_k, \quad b = \sum_{k=1}^{m} b_k, \quad c = \sum_{k=1}^{m} c_k, \quad d = \sum_{k=1}^{m} d_k, \quad e = \sum_{k=1}^{m} e_k, \quad f = \sum_{k=1}^{m} f_k.$$

(H$_7$) $f(t, x, y) \in C(J \times R^2, R)$, and there exists a constant $N > 0$ such that

$$|f(t,x,y)| \leq N|x|,$$

for each $t \in J$ and $(x, y) \in R^2$.

(H$_8$) $I_k, I_k^* \in C(R^2, R)$ and there exist two positive constants $M_1, M_2$ such that

$$|I_k(x,y)| \leq M_1 \text{ and } |I_k^*(x,y)| \leq M_2,$$

for each $(x, y) \in R^2$ and $k = 1, 2, \cdots, m$.

(H$_9$) $f(t, x, y) \in C(J \times R^2, R)$, and there exists a constant $M_0 > 0$ such that

$$|f(t,x,y)| \leq M_0,$$

for each $t \in J$ and $(x, y) \in R^2$.

At the same time, we set

$$N_1 = \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (L_1(\tau) + L_2(\tau)) d_{q_{k-1}}\tau d_{q_{k-1}}s, \quad N_2 = \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (L_1(s) + L_2(s)) d_{q_{k-1}}s,$$

$$p_1 = \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (p(\tau) + q(\tau)) d_{q_{k-1}}\tau d_{q_{k-1}}s, \qquad p_2 = \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (p(s) + q(s)) d_{q_{k-1}}s,$$

$$h_1 = \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} h(\tau) d_{q_{k-1}}\tau d_{q_{k-1}}s, \quad h_2 = \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} h(s) d_{q_{k-1}}s, \quad v = \sum_{k=1}^{m+1} \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}}.$$

In this section, we will apply various fixed point theorems to BVP (1.1). First, we give the uniqueness result based on Banach's contraction principle.

**Theorem 3.1.** *Assume that* (H$_1$) *and* (H$_2$) *hold. In addition, if* $\Lambda < 1$ *holds, then the impulsive* $q_k$-*difference equation BVP* (1.1) *has a unique solution, where* $\Lambda = (\eta_1 + \eta_2)[\delta Q + \gamma(N_2 + mL^*)] + Q, Q = N_1 + N_2 + m(L + L^*)$.

*Proof.* let us set $K_1 = \sup_{t\in J} |f(t, 0, 0)|, K_2 = \sup\{|I_k(0, 0)| : k = 1, 2, \cdots, m\}$ and $K_3 = \sup\{|I_k^*(0, 0)| : k = 1, 2, \cdots, m\}$. We choose a suitable constant $r$ by $r \geq \frac{\Lambda^*}{1-\varepsilon}$, where $\Lambda \leq \varepsilon < 1$ and $\Lambda^* = (\eta_1 + \eta_2)(\delta Q^* + \gamma(K_1 + mK_3)) + Q^*,\ Q^* = K_1 v + mK_2 + K_1 + mK_3$.

Now, we show that $TB_r \subset B_r$, where $B_r = \{x \in PC(J, R) : \| x \| \leq r\}$. For each $x \in B_r$, we have

$$|Tx(t)| \leq \sup_{t\in J} \Bigg\{ \Bigg| -(\eta_1 + \eta_2 t)\Bigg[\delta\Bigg(\sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f(\tau, x(\tau), D_{q_{k-1}}x(\tau)) d_{q_{k-1}}\tau d_{q_{k-1}}s + \sum_{k=1}^{m} I_k(x(t_k), D_{q_k}x(t_k))\Bigg)$$
$$+ \sum_{k=1}^{m+1} (\delta(1 - t_k) + \gamma) \int_{t_{k-1}}^{t_k} f(s, x(s), D_{q_{k-1}}x(s)) d_{q_{k-1}}s$$

$$
\begin{aligned}
&+\sum_{k=1}^{m}(\delta(1-t_k)+\gamma)I_k^*(x(t_k),D_{q_k}x(t_k))\Big]\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}f(\tau,x(\tau),D_{q_{k-1}}x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k(x(t_k),D_{q_k}x(t_k))\right)\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}f(s,x(s),D_{q_{k-1}}x(s))d_{q_{k-1}}s+I_k^*(x(t_k),D_{q_k}x(t_k))\right)(t-t_k)\\
&+\int_{t_k}^{t}\int_{t_k}^{s}f(\tau,x(\tau),D_{q_k}x(\tau))d_{q_k}\tau d_{q_k}s\Bigg|\Bigg\}\\
\le&(\eta_1+\eta_2)\Bigg[\delta\Bigg(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(|f(\tau,x(\tau),D_{q_{k-1}}x(\tau))-f(\tau,0,0)|+|f(\tau,0,0)|)d_{q_{k-1}}\tau d_{q_{k-1}}s\\
&+\sum_{k=1}^{m}(|I_k(x(t_k),D_{q_k}x(t_k))-I_k(0,0)|+|I_k(0,0)|)\Bigg)\\
&+\sum_{k=1}^{m+1}(\delta+\gamma)\int_{t_{k-1}}^{t_k}(|f(s,x(s),D_{q_{k-1}}x(s))-f(s,0,0)|+|f(s,0,0)|)d_{q_{k-1}}s\\
&+\sum_{k=1}^{m}(\delta+\gamma)(|I_k^*(x(t_k),D_{q_k}x(t_k))-I_k^*(0,0)|+|I_k^*(0,0)|)\Bigg]\\
&+\sum_{k=1}^{m}\Bigg(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(|f(\tau,x(\tau),D_{q_{k-1}}x(\tau))-f(\tau,0,0)|+|f(\tau,0,0)|)d_{q_{k-1}}\tau d_{q_{k-1}}s\\
&+|I_k(x(t_k),D_{q_k}x(t_k))-I_k(0,0)|+|I_k(0,0)|\Bigg)\\
&+\sum_{k=1}^{m}\Bigg(\int_{t_{k-1}}^{t_k}(|f(s,x(s),D_{q_{k-1}}x(s))-f(s,0,0)|+|f(s,0,0)|)d_{q_{k-1}}s\\
&+(|I_k^*(x(t_k),D_{q_k}x(t_k))-I_k^*(0,0)|+|I_k^*(0,0)|)\Bigg)(1-t_k)\\
&+\int_{t_m}^{t}\int_{t_m}^{s}(|f(\tau,x(\tau),D_{q_{k-1}}x(\tau))-f(\tau,0,0)|+|f(\tau,0,0)|)d_{q_m}\tau d_{q_m}s\\
\le&(\eta_1+\eta_2)\Bigg[\delta\Bigg(r\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(L_1(\tau)+L_2(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+K_1\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\\
&+m((L_3(t)+L_4(t))r+K_2)\Bigg)\\
&+(\delta+\gamma)\Bigg(r\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(L_1(s)+L_2(s))d_{q_{k-1}}s+K_1+m(L_5(t)+L_6(t))r+K_3\Bigg)\Bigg]\\
&+r\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(L_1(\tau)+L_2(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+K_1\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\\
&+m((L_3(t)+L_4(t))r+K_2)\\
&+r\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(L_1(s)+L_2(s))d_{q_{k-1}}s+K_1+m((L_5(t)+L_6(t))r+K_3)\\
\le&[(\eta_1+\eta_2)(\delta(N_1+N_2+m(L+L^*))+\gamma(N_2+mL^*))+N_1+N_2+m(L+L^*)]r
\end{aligned}
$$

$$
\begin{aligned}
&+ [(\eta_1+\eta_2)(\delta(K_1 v + mK_2 + K_1 + mK_3) + \gamma(K_1 + mK_3)) + K_1 v + mK_2 + K_1 + mK_3] \\
&= \Lambda r + \Lambda^* \le (\Lambda + 1 - \varepsilon) r \le r,
\end{aligned}
$$

and

$$
\begin{aligned}
|D_{q_k} Tx(t)| \le & \sup_{t\in J} \Bigg\{ \Bigg| (-\eta_2) \Bigg[ \delta \Bigg( \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f(\tau, x(\tau), D_{q_{k-1}} x(\tau)) d_{q_{k-1}}\tau d_{q_{k-1}} s + \sum_{k=1}^{m} I_k(x(t_k), D_{q_k} x(t_k)) \Bigg) \\
& + \sum_{k=1}^{m+1} (\delta(1-t_k)+\gamma) \int_{t_{k-1}}^{t_k} f(s, x(s), D_{q_{k-1}} x(s)) d_{q_{k-1}} s \\
& + \sum_{k=1}^{m} (\delta(1-t_k)+\gamma) I_k^*(x(t_k), D_{q_k} x(t_k)) \Bigg] \\
& + \sum_{0<t_k<t} \Bigg( \int_{t_{k-1}}^{t_k} f(s, x(s), D_{q_{k-1}} x(s)) d_{q_{k-1}} s + I_k^*(x(t_k), D_{q_k} x(t_k)) \Bigg) \\
& + \int_{t_k}^{t} f(s, x(s), D_{q_k} x(s)) d_{q_k} s \Bigg| \Bigg\} \\
\le & \eta_2 \Bigg[ \delta \Bigg( \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (|f(\tau, x(\tau), D_{q_{k-1}} x(\tau)) - f(\tau,0,0)| + |f(\tau,0,0)|) d_{q_{k-1}}\tau d_{q_{k-1}} s \\
& + \sum_{k=1}^{m} (|I_k(x(t_k), D_{q_k} x(t_k)) - I_k(0,0)| + |I_k(0,0)|) \Bigg) \\
& + \sum_{k=1}^{m+1} (\delta+\gamma) \int_{t_{k-1}}^{t_k} (|f(s, x(s), D_{q_{k-1}} x(s)) - f(s,0,0)| + |f(s,0,0)|) d_{q_{k-1}} s \\
& + \sum_{k=1}^{m} (\delta+\gamma)(|I_k^*(x(t_k), D_{q_k} x(t_k)) - I_k^*(0,0)| + |I_k^*(0,0)|) \Bigg] \\
& + \sum_{k=1}^{m} \Bigg( \int_{t_{k-1}}^{t_k} (|f(s, x(s), D_{q_{k-1}} x(s)) - f(s,0,0)| + |f(s,0,0)|) d_{q_{k-1}} s \\
& + (|I_k^*(x(t_k), D_{q_k} x(t_k)) - I_k^*(0,0)| + |I_k^*(0,0)|) \Bigg) (1-t_k) \\
& + \int_{t_m}^{t} (|f(s, x(s), D_{q_{k-1}} x(s)) - f(s,0,0)| + |f(s,0,0)|) d_{q_m} s \\
\le & \eta_2 \Bigg[ \delta \Bigg( r \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (L_1(\tau) + L_2(\tau)) d_{q_{k-1}}\tau d_{q_{k-1}} s + K_1 \sum_{k=1}^{m+1} \frac{(t_k - t_{k-1})^2}{1+q_{k-1}} \\
& + m((L_3(t) + L_4(t)) r + K_2) \Bigg) \\
& + (\delta+\gamma)(r \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (L_1(s) + L_2(s)) d_{q_{k-1}} s + K_1 + m(L_5(t) + L_6(t)) r + K_3) \Bigg] \\
& + r \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (L_1(s) + L_2(s)) d_{q_{k-1}} s + K_1 + m((L_5(t) + L_6(t)) r + K_3) \\
\le & [\eta_2(\delta(N_1 + N_2 + m(L + L^*)) + \gamma(N_2 + mL^*)) + N_2 + mL^*] r \\
& + [\eta_2(\delta(K_1 v + mK_2 + K_1 + mK_3) + \gamma(K_1 + mK_3)) + K_1 + mK_3] \\
\le & \Lambda r + \Lambda^*
\end{aligned}
$$

$$
\begin{aligned}
&\le(\Lambda+1-\varepsilon)r\\
&\le r.
\end{aligned}
$$

Hence, we obtain that $TB_r \subset B_r$.

Now, for $x, y \in PC(J, R)$ and for each $t \in J$, we have

$$
\begin{aligned}
|Tx(t)-Ty(t)| \le&(\eta_1+\eta_2 t)\Bigg[\delta\Bigg(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau),D_{q_{k-1}}x(\tau))-f(\tau,y(\tau),D_{q_{k-1}}y(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\\
&+\sum_{k=1}^{m}|I_k(x(t_k),D_{q_k}x(t_k))-I_k(y(t_k),D_{q_k}y(t_k))|\Bigg)\\
&+\sum_{k=1}^{m+1}(\delta(1-t_k)+\gamma)\int_{t_{k-1}}^{t_k}|f(s,x(s),D_{q_{k-1}}x(s))-f(s,y(s),D_{q_{k-1}}y(s))|d_{q_{k-1}}s\\
&+\sum_{k=1}^{m}(\delta(1-t_k)+\gamma)|I_k^*(x(t_k),D_{q_k}x(t_k))-I_k^*(y(t_k),D_{q_k}y(t_k))|\Bigg]\\
&+\sum_{0<t_k<t}\Bigg(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau),D_{q_{k-1}}x(\tau))-f(\tau,y(\tau),D_{q_{k-1}}y(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\\
&+|I_k(x(t_k),D_{q_k}x(t_k))-I_k(y(t_k),D_{q_k}y(t_k))|\Bigg)\\
&+\sum_{0<t_k<t}\Bigg(\int_{t_{k-1}}^{t_k}|f(s,x(s),D_{q_{k-1}}x(s))-f(s,y(s),D_{q_{k-1}}y(s))|d_{q_{k-1}}s\\
&+|I_k^*(x(t_k),D_{q_k}x(t_k))-I_k^*(y(t_k),D_{q_k}y(t_k))|\Bigg)(1-t_k)\\
&+\int_{t_k}^{t}\int_{t_k}^{s}|f(\tau,x(\tau),D_{q_k}x(\tau))-f(\tau,y(\tau),D_{q_k}y(\tau))|d_{q_k}\tau d_{q_k}s\\
\le&(\eta_1+\eta_2)\Bigg[\delta\Bigg(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(L_1(\tau)+L_2(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+m(L_3(t)+L_4(t))\Bigg)\|x-y\|\\
&+(\delta+\gamma)\Bigg(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(L_1(s)+L_2(s))d_{q_{k-1}}s+m(L_5(t)+L_6(t))\Bigg)\|x-y\|\Bigg]\\
&+\Bigg(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(L_1(\tau)+L_2(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+m(L_3(t)+L_4(t))\Bigg)\|x-y\|\\
&+\Bigg(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(L_1(s)+L_2(s))d_{q_{k-1}}s+m(L_5(t)+L_6(t))\Bigg)\|x-y\|\\
\le&[(\eta_1+\eta_2)(\delta Q+\gamma(N_2+mL^*))+Q]\|x-y\|\le\Lambda\|x-y\|<\|x-y\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|D_{q_k}Tx(t)-D_{q_k}Ty(t)|\\
&\quad\le|-\eta_2|\Bigg[\delta\Bigg(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau),D_{q_{k-1}}x(\tau))-f(\tau,y(\tau),D_{q_{k-1}}y(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\\
&\qquad+\sum_{k=1}^{m}|I_k(x(t_k),D_{q_k}x(t_k))-I_k(y(t_k),D_{q_k}y(t_k))|\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{m+1}(\delta(1-t_k)+\gamma)\int_{t_{k-1}}^{t_k}|f(s,x(s),D_{q_{k-1}}x(s))-f(s,y(s),D_{q_{k-1}}y(s))|d_{q_{k-1}}s\\
&+\sum_{k=1}^{m}(\delta(1-t_k)+\gamma)|I_k^*(x(t_k),D_{q_k}x(t_k))-I_k^*(y(t_k),D_{q_k}y(t_k))|\Big]\\
&+\sum_{0<t_k<t}(\int_{t_{k-1}}^{t_k}|f(s,x(s),D_{q_{k-1}}x(s))-f(s,y(s),D_{q_{k-1}}y(s))|d_{q_{k-1}}s\\
&+|I_k^*(x(t_k),D_{q_k}x(t_k))-I_k^*(y(t_k),D_{q_k}y(t_k))|)\\
&+\int_{t_k}^{t}|f(s,x(s),D_{q_k}x(s))-f(s,y(s),D_{q_k}y(s))|d_{q_k}s\\
\le&\eta_2\Big[\delta(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(L_1(\tau)+L_2(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+m(L_3(t)+L_4(t)))\|x-y\|\\
&\qquad(\delta+\gamma)(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(L_1(s)+L_2(s))d_{q_{k-1}}s+m(L_5(t)+L_6(t)))\|x-y\|\Big]\\
&+(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(L_1(s)+L_1(s))d_{q_{k-1}}s+m(L_5(t)+L_6(t)))\|x-y\|\\
&=[\eta_2(\delta Q+\gamma(N_2+mL^*))+N_2+mL^*]\|x-y\|\\
\le&\Lambda\|x-y\|<\|x-y\|.
\end{aligned}
$$

Therefore, we obtain that $\|Tx-Ty\|<\| x-y \|$, so $T$ is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle. This completes the proof of Theorem 3.1. □

**Corollary 3.2.** *Assume that* (H$_1$) *and* (H$_4$) *hold. In addition, if* $\Lambda_1<1$ *holds, then the impulsive* $q_k$*-difference equation BVP* (1.1) *has a unique solution, where* $\Lambda_1=(\eta_1+\eta_2)[\delta Q_1+\gamma(N_2+m(L_5+L_6))]+Q_1, Q_1=N_1+N_2+m(L_3+L_4+L_5+L_6)$.

**Corollary 3.3.** *Assume that* (H$_3$) *and* (H$_4$) *hold. In addition, if* $\Lambda_2<1$ *hold, then the BVP* (1.1) *has a unique solution, where* $\Lambda_2=(\eta_1+\eta_2)[\delta Q_2+\gamma(L_1+L_2+m(L_5+L_6))]+Q_2, Q_2=(L_1+L_2)(v+1)+m(L_3+L_4+L_5+L_6)$.

The next existence result is based on the Schaefer's fixed point theorem.

**Lemma 3.4.** *Assume that* (H$_5$) *and* (H$_6$) *hold. Then* $T$ *is completely continuous.*

*Proof.* The proof consists of several steps.

(i) By the continuity of $f$, $I_k$ and $I_k^*$, it is easy to get $T$ is continuous.

(ii) $T$ maps bounded sets into bounded sets in $PC(J,R)$. Let $B_r=\{x\in PC(J,R):\|x\|\le r\}$ be a bounded set in $PC(J,R)$ and $x\in B_r$. Then we have

$$
\begin{aligned}
|Tx(t)|\le&(\eta_1+\eta_2)\Big[\delta\Big(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau),D_{q_{k-1}}x(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s+\sum_{k=1}^{m}|I_k(x(t_k),D_{q_k}x(t_k))|\Big)\\
&+\sum_{k=1}^{m+1}(\delta(1-t_k)+\gamma)\int_{t_{k-1}}^{t_k}|f(s,x(s),D_{q_{k-1}}x(s))|d_{q_{k-1}}s\\
&+\sum_{k=1}^{m}(\delta(1-t_k)+\gamma)|I_k^*(x(t_k),D_{q_k}x(t_k))|\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau),D_{q_{k-1}}x(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s+|I_k(x(t_k),D_{q_k}x(t_k))|\right)\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,x(s),D_{q_{k-1}}x(s))|d_{q_{k-1}}s+|I_k^*(x(t_k),D_{q_k}x(t_k))|\right)(1-t_k)\\
&+\int_{t_m}^{1}\int_{t_m}^{s}|f(\tau,x(\tau),D_{q_m}x(\tau))|d_{q_m}\tau d_{q_m}s\\
\le&(\eta_1+\eta_2)\Bigg[\delta\Bigg(r\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(p(\tau)+q(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s\\
&+\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}h(\tau)d_{q_{k-1}}\tau d_{q_{k-1}}s+r\sum_{k=1}^{m}(a_k+b_k)+\sum_{k=1}^{m}e_k\Bigg)\\
&+(\delta+\gamma)(r\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(p(s)+q(s))d_{q_{k-1}}s+\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}h(s)d_{q_{k-1}}s)\\
&+(\delta+\gamma)(r\sum_{k=1}^{m}(c_k+d_k)+\sum_{k=1}^{m}f_k)\Bigg]\\
&+r\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(p(\tau)+q(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}h(\tau)d_{q_{k-1}}\tau d_{q_{k-1}}s\\
&+r\sum_{k=1}^{m}(a_k+b_k)+\sum_{k=1}^{m}e_k\\
&+r\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(p(s)+q(s))d_{q_{k-1}}s+\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}h(s)d_{q_{k-1}}s+r\sum_{k=1}^{m}(c_k+d_k)+\sum_{k=1}^{m}f_k\\
\le&[(\eta_1+\eta_2)(\delta(p_1+p_2+a+b+c+d)+\gamma(p_2+c+d))+p_1+p_2+a+b+c+d]r\\
&+[(\eta_1+\eta_2)(\delta(h_1+h_2+e+f)+\gamma(h_2+f))+h_1+h_2+e+f]=\Gamma r+\Gamma^*:=M,
\end{aligned}
$$

and

$$
\begin{aligned}
|D_{q_k}Tx(t)|\le&\eta_2\Bigg[\delta\Bigg(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau),D_{q_{k-1}}x(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s+\sum_{k=1}^{m}|I_k(x(t_k),D_{q_k}x(t_k))|\Bigg)\\
&+\sum_{k=1}^{m+1}(\delta(1-t_k)+\gamma)\int_{t_{k-1}}^{t_k}|f(s,x(s),D_{q_{k-1}}x(s))|d_{q_{k-1}}s\\
&+\sum_{k=1}^{m}(\delta(1-t_k)+\gamma)|I_k^*(x(t_k),D_{q_k}x(t_k))|\Bigg]\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,x(s),D_{q_{k-1}}x(s))|d_{q_{k-1}}s+|I_k^*(x(t_k),D_{q_k}x(t_k))|\right)\\
&+\int_{t_m}^{1}|f(s,x(s),D_{q_m}x(s))|d_{q_m}s\\
\le&\eta_2\Bigg[\delta\Bigg(r\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(p(\tau)+q(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s\\
&+\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}h(\tau)d_{q_{k-1}}\tau d_{q_{k-1}}s+r\sum_{k=1}^{m}(a_k+b_k)+\sum_{k=1}^{m}e_k\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ (\delta+\gamma)\Big(r\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(p(s)+q(s))d_{q_{k-1}}s+\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}h(s)d_{q_{k-1}}s\Big)\\
&+ (\delta+\gamma)\Big(r\sum_{k=1}^{m}(c_k+d_k)+\sum_{k=1}^{m}f_k\Big)\Big]\\
&+ r\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(p(s)+q(s))d_{q_{k-1}}s+\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}h(s)d_{q_{k-1}}s+r\sum_{k=1}^{m}(c_k+d_k)+\sum_{k=1}^{m}f_k\\
\le&[\eta_2(\delta(p_1+p_2+a+b+c+d)+\gamma(p_2+c+d))+p_2+c+d]r\\
&+[\eta_2(\delta(h_1+h_2+e+f)+\gamma(h_2+f))+h_2+f]=\Gamma_0 r+\Gamma_0^*=M_0,
\end{aligned}
$$

where $\Gamma=(\eta_1+\eta_2)[\delta Q_3+\gamma(p_2+c+d)]+Q_3,\ Q_3=p_1+p_2+a+b+c+d,\ \Gamma^*=(\eta_1+\eta_2)[\delta Q_3^*+\gamma(h_2+f)]+Q_3^*,\ Q_3^*=h_1+h_2+e+f$. And $\Gamma_0=\eta_2[\delta Q_3+\gamma(p_2+c+d)+p_2+c+d,\Gamma_0^*=\eta_2[\delta Q_3^*+\gamma(h_2+f)]+h_2+f$. Obviously, $M\ge M_0$. Thus $\|Tx\|\le M$.

(iii) $T$ maps bounded sets into equicontinuous sets of $PC(J,R)$.

Let $\tau_1,\tau_2\in J_i\in(t_i,t_{i+1})$ for some $i\in\{0,1,2,\cdots,m\}$ and $B_r$ be bound set of $PC(J,R)$ as before. Then for $x\in B_r$, we have

$$
\begin{aligned}
&|Tx(\tau_2)-Tx(\tau_1)|\\
&\le\eta_2|\tau_2-\tau_1|\Big[\delta\Big(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,x(\tau),D_{q_{k-1}}x(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s+\sum_{k=1}^{m}|I_k(x(t_k),D_{q_k}x(t_k))|\Big)\\
&\quad+\sum_{k=1}^{m+1}(\delta(1-t_k)+\gamma)\int_{t_{k-1}}^{t_k}|f(s,x(s),D_{q_{k-1}}x(s))|d_{q_{k-1}}s\\
&\quad+\sum_{k=1}^{m}(\delta(1-t_k)+\gamma)|I_k^*(x(t_k),D_{q_k}x(t_k))|\Big]\\
&\quad+|\tau_2-\tau_1|\sum_{k=1}^{i}\Big(\int_{t_{k-1}}^{t_k}|f(s,x(s),D_{q_{k-1}}x(s))|d_{q_{k-1}}s+|I_k^*(x(t_k),D_{q_k}x(t_k))|\Big)\\
&\quad+\Big|\int_{t_i}^{\tau_2}\int_{t_i}^{s}|f(\tau,x(\tau),D_{q_i}x(\tau))|d_{q_i}\tau d_{q_i}s-\int_{t_i}^{\tau_1}\int_{t_i}^{s}|f(\tau,x(\tau),D_{q_i}x(\tau))|d_{q_i}\tau d_{q_i}s\Big|\\
&\le\eta_2|\tau_2-\tau_1|\Big[\delta(rp_1+h_1+r(a+b)+e)+(\delta+\gamma)(rp_2+h_2+r(c+d)+f)\Big]\\
&\quad+|\tau_2-\tau_1|(rp_2+h_2+r(c+d)+f)+|\tau_2-\tau_1|\int_{t_i}^{\xi}|f(\tau,x(\tau),D_{q_i}x(\tau))|d_{q_i}\tau\to0,\quad(\tau_2-\tau_1\to0),
\end{aligned}
$$

where $\xi\in[\min\{\tau_1,\tau_2\},\max\{\tau_1,\tau_2\}]$, and

$$
\begin{aligned}
|D_{q_i}Tx(\tau_2)-D_{q_i}Tx(\tau_1)|\le&\Big|\sum_{\tau_1\le t_k<\tau_2}\Big(\int_{t_{k-1}}^{t_k}f(s,x(s),D_{q_{k-1}}x(s))d_{q_{k-1}}s+I_k^*(x(t_k),D_{q_k}x(t_k))\Big)\Big|\\
&+\Big|\int_{t_i}^{\tau_2}f(s,x(s),D_{q_i}x(s))d_{q_i}s-\int_{t_i}^{\tau_1}f(s,x(s),D_{q_i}x(s))d_{q_i}s\Big|\\
\le&\Big|\int_{\tau_1}^{\tau_2}f(s,x(s),D_{q_i}x(s))d_{q_i}s\Big|\\
\le&\Big|\int_{\tau_1}^{\tau_2}(p(s)|x(s)|+q(s)|D_{q_i}x(s)|+h(s))d_{q_i}s\Big|\\
\le&\Big|r\int_{\tau_1}^{\tau_2}(p(s)+q(s))d_{q_i}s+\int_{\tau_1}^{\tau_2}h(s))d_{q_i}s\Big|\to0,\qquad(\tau_2-\tau_1\to0).
\end{aligned}
$$

As a consequence of the Arzelá-Ascoli theorem, we can conclude that $T : PC(J,R) \to PC(J,R)$ is completely continuous. This proof is completed. □

**Theorem 3.5.** *Assume that* ($H_5$) *and* ($H_6$) *hold. Suppose further* $\Gamma < 1$ *holds, then BVP* (1.1) *has at least one solution, where* $\Gamma = (\eta_1 + \eta_2)[\delta Q_3 + \gamma(p_2 + c + d)] + Q_3, Q_3 = p_1 + p_2 + a + b + c + d$.

*Proof.* In view of Lemma 3.4, it is easy to know that $T$ is completely continuous. It is clear that $x \in PC(J,R)$ is a solution of BVP (1.1) if only if $x$ is a fixed point of $T$.

Next, we show that the set

$$E = \{x \in PC(J,R) : x = \lambda Tx \text{ for some } 0 \le \lambda \le 1\}$$

is bounded, which is independent of $\lambda$. Let $x \in E$, then $x(t) = \lambda Tx(t)$ for some $0 \le \lambda \le 1$.

By ($H_5$) and ($H_6$), that for each $t \in J$, by (ii) of prove in Lemma 3.4, we have

$$|x(t)| = |\lambda Tx(t)| \le |Tx(t)| \le \Gamma\|x\| + \Gamma^*,$$

and

$$|D_{q_k}x(t)| = |\lambda D_{q_k}Tx(t)| \le |D_{q_k}Tx(t)| \le \Gamma_0\|x\| + \Gamma_0^*,$$

therefore,

$$\|x(t)\| \le \Gamma\|x\| + \Gamma^*.$$

Hence,

$$\|x\| \le \frac{\Gamma^*}{1-\Gamma} := M_0.$$

This show that the set $E$ is bounded. By Theorem 2.5, we obtain that BVP (1.1) has at least one solution. This proof is completed. □

**Corollary 3.6.** *Assume that* ($H_6$) *and* ($H_7$) *hold. In addition, if* $\Gamma_1 < 1$ *holds, where* $\Gamma_1 = (\eta_1 + \eta_2)[\delta Q_4 + \gamma(N + c + d)] + Q_4, Q_4 = N(v + 1) + a + b + c + d$. *Then the BVP* (1.1) *has at least one solution.*

**Corollary 3.7.** *Assume that* ($H_5$) *and* ($H_8$) *hold. In addition, if* $\Gamma_2 < 1$ *holds, where* $\Gamma_2 = (\eta_1 + \eta_2)[\delta(p_1 + p_2) + \gamma p_2] + p_1 + p_2$. *Then the BVP* (1.1) *has at least one solution.*

**Corollary 3.8.** *Assume that* ($H_7$) *and* ($H_8$) *hold. In addition, if* $\Gamma_3 < 1$ *holds, where* $\Gamma_3 = N[(\eta_1+\eta_2)(\delta(v+1) + \gamma) + v + 1]$. *Then the BVP* (1.1) *has at least one solution.*

**Corollary 3.9.** *Assume that* ($H_8$) *and* ($H_9$) *hold. Then the BVP* (1.1) *has at least one solution.*

## 4. Example

**Example 4.1.** Consider the following BVP for second-order impulsive $q$-difference equation:

$$\begin{cases} D^2_{q_k}x(t) = \frac{\sin(|x(t)|)}{3(10+t)^3} + \frac{2}{3\cdot 10^3}\ln(1 + |D_{q_k}x(t)|), & t \in J = [0,1], t \ne t_k, \\ \Delta x(t_k) = \frac{|x(t_k)|}{9(10+|x(t_k)|)} + \frac{1}{10^2}e^{-|D_{q_k}x(t_k)|}, & t_k = \frac{k}{10}, k = 1,2,\cdots,9, \\ \Delta D_{q_k}x(t_k) = \frac{1}{9}\tan^{-1}(\frac{1}{10}|x(t_k)|) + \frac{1}{10^2}|D_{q_k}x(t_k)|, & t_k = \frac{k}{10} \quad k = 1,2,\cdots,9, \\ x(0) - D_{\frac{2}{3}}(x(0)) = 0, \quad x(1) + D_{\frac{1}{6}}(x(1)) = 0. & \end{cases} \tag{4.1}$$

Here, $q_k = \frac{2}{3+k}$ $(k = 0,1,2,\cdots,m)$, $m = 9$ and $f(t,x,D_{q_k}x) = \frac{\sin(|x|)}{3(10+t)^3} + \frac{2}{3\cdot 10^3}\ln(1 + |D_{q_k}x|)$, $I_k(x, D_{q_k}x) = \frac{|x|}{9(10+|x|)} + \frac{1}{10^2}e^{-|D_{q_k}x|}$, $I^*_k(x, D_{q_k}x) = \frac{1}{9}\tan^{-1}(\frac{1}{10}|x|) + \frac{1}{10^2}|D_{q_k}x|$. Obviously, $\forall t \in J$, $(x_1,y_1),(x_2,y_2) \in R^2$, we have

$$|f(t,(x_1,y_1)) - f(t,(x_2,y_2))| \le \frac{1}{3\cdot 10^3}|x_1 - x_2| + \frac{2}{3\cdot 10^3}|y_1 - y_2|,$$

$$|I_k(x_1,y_1)-I_k(x_2,y_2)| \le \frac{1}{90}|x_1-x_2|+\frac{1}{10^2}|y_1-y_2|,$$
$$|I_k^*(x_1,y_1)-I_k^*(x_2,y_2)| \le \frac{1}{90}|x_1-x_2|+\frac{1}{10^2}|y_1-y_2|.$$

Then, $L_1=\frac{1}{3\cdot 10^3}$, $L_2=\frac{2}{3\cdot 10^3}$, $L_3=\frac{1}{90}$, $L_4=\frac{1}{10^2}$, $L_5=\frac{1}{90}$, $L_6=\frac{1}{10^2}$ and $v=\frac{1380817}{180180}$. Clearly, $\Lambda_2\approx 0.89006<1$. By Corollary 3.3, we obtain that BVP (4.1) has a unique solution.

**Example 4.2.** Consider the following BVP for second-order impulsive $q$-difference equation:

$$\begin{cases} D^2_{q_k}x(t)=\frac{e^{-2t}|D_{q_k}x(t)|}{10(1+|D_{q_k}x(t)|)}\ln(1+|x(t)|), & t\in J, t\neq t_k,\\ \Delta x(t_k)=\frac{1}{10^k}\sin(|x(t_k)|)+\frac{1}{5^k}(1+|D_{q_k}x(t_k)|)^{\frac{1}{5}}+\frac{1}{2^k}, & t_k=\frac{k}{5}, k=1,2,3,4,\\ \Delta D_{q_k}x(t_k)=\frac{1}{10^k}\arctan(|x(t_k)|)+\frac{1}{5^{k+1}}|D_{q_k}x(t_k)|+\frac{1}{4^k}, & t_k=\frac{k}{5}, k=1,2,3,4,\\ \frac{1}{8}x(0)-\frac{1}{8}D_{\frac{1}{2}}x(0)=0, \quad x(1)+D_{\frac{1}{4}}x(1)=0. \end{cases} \tag{4.2}$$

Here, $q_k=\frac{2}{4+k}$ $(k=0,1,2,3,4)$, $m=4, \alpha=\beta=\frac{1}{8}, \delta=\gamma=1$, and $f(t,x,D_{q_k})=\frac{e^{-2t}|D_{q_k}x|}{10(1+|D_{q_k}x|)}\ln(1+|x|)$, $I_k(x,D_{q_k})=\frac{1}{10^k}\sin(|x|)+\frac{1}{5^k}(1+|D_{q_k}x|)^{\frac{1}{5}}+\frac{1}{2^k}$, $I_k^*(x,D_{q_k})=\frac{1}{10^k}\arctan(|x|)+\frac{1}{5^{k+1}}|D_{q_k}x|+\frac{1}{4^k}$. Obviously, we have

$$|f(t,x,D_{q_k})|\le \frac{1}{10}|x|,$$
$$|I_k(x,D_{q_k})|\le \frac{1}{10^k}|x(t_k)|+\frac{1}{5^{k+1}}|D_{q_k}x(t_k)|+\frac{1}{2^k},$$
$$|I_k^*(x,D_{q_k})|\le \frac{1}{10^k}|x|+\frac{1}{5^{k+1}}|D_{q_k}x|+\frac{1}{4^k}.$$

Therefore, $N=\frac{1}{10}$, $a_k=\frac{1}{10^k}$, $b_k=\frac{1}{5^{k+1}}$, $c_k=\frac{1}{4^k}$, $d_k=\frac{1}{5^{k+1}}$, $e_k=\frac{1}{2^k}$, $f_k=\frac{1}{4^k}$. We can find that $\eta_1+\eta_2=\frac{2}{3}$, $a=c=\frac{1111}{10000}$, $b=d=\frac{156}{3125}$, $v=\frac{4421}{31500}$. Clearly, $\Gamma_1\approx 0.9008<1$. By Corollary 3.6, we obtain that BVP (4.2) has at least one solution.

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