Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2680–2687
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

A noncompactness measure for tvs-metric cone spaces and some applications

Raúl Fierro

Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Casilla 4059, Valparaíso, Chile.

Instituto de Matemáticas, Universidad de Valparaíso, Casilla 123-V, Valparaíso, Chile.

Communicated by C. Vetro

Abstract

We provide a natural topology for a cone metric space and a noncompactness measure is defined for this space, which enables us to extend existing results for mappings and set-valued mappings defined on classical metric spaces. Moreover it is proved that the topology of any uniform topological space is generated by a cone metric.

Keywords: Approximate and fixed points, noncompactness measure, uniform spaces, set-valued mapping, tvs-cone metric space.
2010 MSC: 46A50, 47H10, 47H04.

1. Introduction

It has been proven by Huang and Zhang in [19] that, the fixed point theory based on cone metric spaces is not a banal extension of the classical theory based on metric spaces. Indeed, these authors introduced an example of contraction, on a cone metric space, which is not contraction in a standard metric space. As in [1, 3, 4, 6, 10, 22, 24, 29, 31], this has led some authors to publish numerous papers both on fixed point theory as on properties of cone metric spaces. However, it has not been paid attention enough to the topology of these spaces. Indeed, by making use of sequences, some concepts such as closed set and the completeness of these spaces are defined. In general, results of these works are obtained by means of sequential properties of the cone metric, but without reference to the topology of the corresponding space, which is often not explicitly defined.

Email address: `raul.fierro@pucv.cl`, `raul.fierro@uv.cl` (Raúl Fierro)

Received 2015-12-27

The main aim of this paper is providing a natural topology for a cone metric space. We prove that this topology is generated by a uniformity. Reciprocally, it is proved that the topology of any uniform space is generated by a cone metric. Of course, the topology of a cone metric space depends strongly on the topological vector spaces where the cone metric takes its values. For instance, when this latter topology is locally convex, the topology of the cone metric space so is. One of the advantages of having a topology for the spaces under our study, is that the precompact sets can be characterized by means of their cone metrics. Indeed, in this paper a noncompactness measure is defined for these spaces and precompactness of sets is characterized by this measure, which also enables us to define condensing mappings and condensing set-valued mappings. Originally, this type of measure was introduced by Kuratowski in [20] allowing to characterize relatively compact sets in complete (classical) metric spaces. Other authors such as [18, 25] have characterized relatively compact sets in locally convex spaces. In [13], a noncompactness measure is defined for cone metric spaces with normal cone in a Banach space. However, these authors do not prove that the noncompactness measure of a compact set equals zero.

The Banach contraction theorem allows to prove that a non-expansive self-mapping defined on a bounded, closed and convex subset of a Banach space has an approximate fixed point. Furthermore, if this subset is compact, then the non-expansive self-mapping has a fixed point. A proof of these results is given, for instance, in Chapter 3 of [17]. By making use of a general result, introduced in this work, and a version of the Nadler theorem in the context of cone metric spaces, we prove that non-expansive set-valued mappings with respect to cone metric, have also approximate fixed points. Moreover, when a set-valued mapping has compact domain, it has a fixed point.

Another application, of the main results of this paper, is an extension of a fixed point theorem by Darbo in [11], which is proved in the context of cone metric spaces.

The paper is organized as follows. The topology of a cone metric space is introduced in Section 2. Also in this section, it is proved that cone metric spaces are uniform topological spaces and the noncompactness measure on these spaces is presented. In Section 3 the main results are stated and proved. Finally, Section 4 is devoted to some applications to approximate and fixed points.

2. Preliminaries

Let E be a complete topological real vector space with θ as zero element and usual notations for addition and scalar product. A cone is a nonempty closed subset P of E such that $P \cap (-P) = \{\theta\}$ and for each $\alpha \geq 0$, $\alpha P + P \subseteq P$. Given a cone P of E, a partial order is defined on E as $x \preceq y$, if and only if, $y - x \in P$. We denote by $x \prec y$ whenever $x \preceq y$ and $x \neq y$. Moreover, the notations $x \ll y$ means that $y - x$ belongs to $\operatorname{int}(P)$, the interior of P. As natural, the notations $x \succeq y$, $x \succ y$ and $x \gg y$ mean $y \preceq x$, $y \prec x$ and $y \ll x$, respectively. In what follows, it is assumed that P is a cone of E. We refer to [2] for notations and facts, about ordered vector spaces.

Remark 2.1. For each $a, b, c \in E$ such that $a \preceq b \ll c$, we have $a \ll c$.

A cone metric space is a pair (X, d), where X is a nonempty set and $d : X \times X \to E$ is a function satisfying the following two conditions: i) for all $x, y \in X$, $d(x, y) = \theta$, if and only if, $x = y$, and ii) for all $x, y, z \in X$, $d(x, y) \preceq d(x, z) + d(y, z)$.

In what follows, (X, d) and P stand for a cone metric space and the cone defining the order on E, respectively.

Remark 2.2. Note that for all $x, y \in X$, $d(x, y) \succeq \theta$, and $d(x, y) = d(y, x)$ (see also [9]).

Next, we define a topology on X based on its cone metric d. Let $\mathcal{C}$ be a base of neighborhoods of θ and for each $C \in \mathcal{C}$, we denote

$$U_C = \{(x, y) \in X \times X : d(x, y) \in C\},$$

$\mathcal{B}_X = \{U_C : C \in \mathcal{C}\}$ and $\mathcal{U}_X = \{U \subset X \times X : \exists V \in \mathcal{B}_X, V \subset U\}$. Without loss of generality, in the sequel, we assume every $C \in \mathcal{C}$ is balanced, i.e. for each $\alpha \in \mathbb{R}$ such that $|\alpha| \leq 1$, we have $\alpha C \subseteq C$. Notice that $\mathcal{U}_X$

is a uniformity on X, if and only if, $\mathcal{B}_X$ is a fundamental system of entourages. Theorem 2.3 below shows that indeed $(X,\mathcal{U}_X)$ is a uniform space.

As usual, the diagonal of X is defined as $\Delta = \{(x, y) \in X \times X : x = y\}$.

Theorem 2.3. *The family $\mathcal{B}_X$ is a fundamental system of entourages for the uniformity $\mathcal{U}_X$ on X. I.e., $\mathcal{B}_X$ is a filter base satisfying the following three conditions:*

(i) $\Delta = \bigcap_{C\in\mathcal{C}} U_C$;

(ii) for each $U \in \mathcal{B}_X$, there exists $V \in \mathcal{B}_X$, such that $V \subset U^{-1}$;

(iii) for each $U \in \mathcal{B}_X$, there exists $W \in \mathcal{B}_X$, such that $W \circ W \subset U$.

Proof. Conditions (i) and (ii) follow directly from the definition of cone metric. Let us prove condition (iii). Let $U \in \mathcal{B}_X$ and $C \in \mathcal{C}$ satisfy $U = U_C$. Hence, there exists $B \in \mathcal{C}$ such that $B + B \subset C$. Let $W = U_B$ and $(x, y) \in W \circ W$. There exists $z \in X$ such that $(x, z) \in W$ and $(z, y) \in W$ and consequently $d(x, z) \in B$ and $d(z, y) \in B$. Thus $d(x, y) \preceq d(x, z) + d(z, y) \in B + B \subset C$ and accordingly $W \circ W \subset U$.

From (i), $\emptyset \notin \mathcal{B}_X$. Let $A, B \in \mathcal{C}$ and $C \in \mathcal{C}$ such that $C \subset A \cap B$. Hence $U_C \subset U_A \cap U_B$. Consequently, $\mathcal{B}_X$ is a filter base on $X \times X$, which concludes the proof. □

In what follows, we consider the space X endowed with the topology generated by $\mathcal{U}_X$. Hence, a base for the topology of X is given by means of the family $\{U_C[a]; a \in X, C \in \mathcal{C}\}$, where $U_C[a] = \{x \in X : d(x, a) \in C\}$. Moreover, it is worth noting that X satisfies the first countability axiom, whenever E does it. In particular, when E is metrizable, X is 1° countable.

Remark 2.4. Notice that for each $\epsilon \gg \theta$ and $a \in X$, the ball $B(a, \epsilon) = \{x \in X : d(x, a) \ll \epsilon\}$ is an open set. However, in general, it is easy to give examples showing that the family of all balls is not a base for the topology of X. Consequently, the topology generated by $\{B(a, \epsilon) : \epsilon \gg \theta\}$, is, in general, weaker than the topology of X generated by $\mathcal{U}_X$. In [9], this weaker topology has been considered to prove some results.

For each $A \subseteq X$ and $C \in \mathcal{C}$, we denote $U_C[A] = \bigcup_{x\in A} U_C[x]$. A subset B of X is said to be bounded, if there exists an (order) upper bounded $C \in \mathcal{C}$ such that $B \times B \subseteq U_C$. We denote by 2^X the family of all nonempty subsets of X, by $\mathcal{B}(X)$ the family of all bounded subsets of X, by $\mathcal{C}(X)$ the family of all closed and nonempty subsets of X and $\mathcal{CB}(X) = \mathcal{C}(X) \cap \mathcal{B}(X)$. A subfamily $\mathcal{H} = \{H_C\}_{C\in\mathcal{C}}$ of $\mathcal{CB}(X) \times \mathcal{CB}(X)$ is defined as follows:

$$H_C = \{(A, B) \in \mathcal{CB}(X) \times \mathcal{CB}(X) : A \subseteq U_C[B] \text{ and } B \subseteq U_C[A]\}.$$

It is well known, see [8] for instance, that $\mathcal{H}$ is a fundamental system of entourages for a uniformity on $\mathcal{CB}(X)$. The topology on $\mathcal{CB}(X)$ induced by $\mathcal{H}$ is referred to as the $\mathcal{H}$-topology. Let $\mathcal{K}(X)$ denote the family of all nonempty compact subsets of X. Since $\mathcal{K}(X) \subseteq \mathcal{CB}(X)$, we consider $\mathcal{K}(X)$ endowed with the induced $\mathcal{H}$-topology. In [27] is proved that $\mathcal{K}(X)$ is $\mathcal{H}$-complete, if and only if, $(X,\mathcal{U}_X)$ so is.

In accordance with the uniformity generating the topology of X, a subset D of X is precompact, if for each $C \in \mathcal{C}$, there exist $x_1, \ldots, x_r \in X$ such that $D \subseteq U_C[x_1] \cup \cdots \cup U_C[x_r]$. Moreover, a filter $\mathcal{F}$ on X is a Cauchy filter (c.f. [7]), if for each $C \in \mathcal{C}$, there exist $A \in \mathcal{F}$ such that $A \times A \subseteq U_C$. We define a noncompactness measure as follows: for $D \in \mathcal{B}(X)$, $\mathcal{M}_X(D)$ denotes the family of all $C \in \mathcal{C}$ such that there exist $x_1, \ldots, x_r \in X$ satisfying $D \subseteq U_C[x_1] \cup \cdots \cup U_C[x_r]$. Accordingly, (c.f. [7]), $D \in \mathcal{B}(X)$ is precompact, if and only if, $\mathcal{M}_X(D) = \mathcal{C}$. Thus, $\mathcal{M}_X(\cdot)$ is a noncompactness measure.

It is easy to see the following three properties hold:

a) $\mathcal{M}_X(B) \subseteq \mathcal{M}_X(A)$ whenever $A \subseteq B$, $\quad (B \in \mathcal{B}(X))$.

b) $\mathcal{M}_X(A \cup B) = \mathcal{M}_X(A) \cap \mathcal{M}_X(B)$, $\quad (A, B \in \mathcal{B}(X))$.

c) $\mathcal{M}_X(\overline{A}) = \mathcal{M}_X(A)$, $\quad (A \in \mathcal{B}(X))$.

d) $\mathcal{M}_X(A) \subseteq C + \mathcal{M}_X(U_C[A])^1, \quad (A \in \mathcal{B}(X), C \in \mathcal{C})$.

Let $D \subseteq X$ and $\mathcal{B}$ be a filterbase in X. We say that $\mathcal{B}$ converges to D, in the $\mathcal{H}$-topology, if and only if, for each $C \in \mathcal{C}$, there exists $B \in \mathcal{B}$ such that $B \subseteq U_C[D]$ and $D \subseteq U_C[B]$.

3. Main results

We have seen that a cone space enjoys of a Hausdorff uniform structure. The next result states that any Hausdorff uniform space has the topology induced by a cone metric.

Theorem 3.1. *Let $(X,\mathcal{U})$ be a Hausdorff uniform space, i.e. $\bigcap_{U\in\mathcal{U}} U = \Delta$. Then, there exists a cone metric $d : X \times X \to E$ generating the topology of X.*

Proof. It is well-known that the uniformity $\mathcal{U}$ is generated by a family of separating pseudo metrics $\{d_\lambda\}_{\lambda\in\Lambda}$ on X. Let $E = \mathbb{R}^\Lambda$ be the real vector space of all functions from Λ to $\mathbb{R}$, endowed with the usual operations of addition and scalar multiplication. By considering E with the product topology and defining $P = \{x \in E : \forall\lambda \in \Lambda, x_\lambda \geq 0\}$, we have P is a cone of E and a cone metric $d : X \times X \to E$ is defined as $d(x, y) = \{d_\lambda(x, y)\}_{\lambda\in\Lambda}$. A local base for E is given by the family $\mathcal{C}$ of the all sets having the form

$$C_{\lambda_1,\ldots,\lambda_r}(a, \epsilon) = \{x \in E : \max_{1\leq i\leq r} |x_{\lambda_i} - a_{\lambda_i}| < \epsilon\}, \quad \lambda_1,\ldots\lambda_r \in \Lambda, \quad \epsilon > 0, \quad a \in E.$$

It is easy to see that $\{U_C; C \in \mathcal{C}\}$, where $U_C = \{(x, y) \in X \times X : d(x, y) \in C\}$, is a fundamental system of entourages generating the uniformity $\mathcal{U}$. Therefore, d is a cone metric generating the topology of X. □

Theorem 3.2 below extends a classical result (Theorem 1') by Kuratowski in [20].

Theorem 3.2. *Let $\mathcal{B} \subseteq \mathcal{CB}(X)$ be a filter base on X and $D = \bigcap_{B\in\mathcal{B}} B$. Suppose $(X,\mathcal{U}_X)$ is complete and $\bigcup_{B\in\mathcal{B}} \mathcal{M}_X(B) = \mathcal{C}$. Then, the following two conditions hold:*

(i) D is compact and nonempty,

(ii) $\mathcal{B}$ converges to D in the $\mathcal{H}$-topology.

Proof. Since D is closed, X is complete and $\mathcal{M}_X(D) = \mathcal{C}$, we have D is compact. Let $\mathcal{F}$ be the filter generated by $\mathcal{B}$, i.e., $\mathcal{F} = \{A \subseteq X : \exists B \in \mathcal{B}, B \subseteq A\}$. In order to prove D is nonempty, let $\mathcal{F}^*$ be a ultrafilter such that $\mathcal{F} \subseteq \mathcal{F}^*$. Let $C \in \mathcal{C}$ and $C' \in \mathcal{C}$ such that $C' + C' \subseteq C$. By assumption, there exist $B \in \mathcal{B}$ and $x_1, \ldots, x_r \in X$ such that $B \subseteq U_{C'}[x_1] \cup \cdots \cup U_{C'}[x_r]$. Since $B \in \mathcal{F}^*$ and $\mathcal{F}^*$ is a ultrafilter, there exists $i \in \{1, \ldots, r\}$ such that $U_{C'}[x_i] \in \mathcal{F}^*$ (see Corollary in Section §6.4, Chapter I in [7], for instance). Thus, $U_{C'}[x_i] \times U_{C'}[x_i] \subseteq U_C$ and consequently, $\mathcal{F}^*$ is a Cauchy filter, which converges to some point $x^* \in X$. But, $\{x^*\} = \bigcap_{F\in\mathcal{F}^*} F \subseteq \bigcap_{F\in\mathcal{F}} F = \bigcap_{B\in\mathcal{B}} B$ and therefore D is nonempty.

Next, we prove the convergence of $\mathcal{B}$ to E in the $\mathcal{H}$-topology. Suppose there exists $C \in \mathcal{C}$ such that for any $B \in \mathcal{B}$, $B \cap U_C[D]^c \neq \emptyset$. Consequently, $\widetilde{\mathcal{B}} = \{B \cap L_C : B \cap U_C[D]^c\}$ is a filter base on $U_C[D]^c$. Moreover $\bigcup_{B\in\widetilde{\mathcal{B}}} \mathcal{M}_X(B) \supseteq \bigcup_{B\in\mathcal{B}} \mathcal{M}_X(B) = \mathcal{C}$ and $U_C[D]^c$ is complete. These facts imply from (i) that $\emptyset \neq \bigcap_{B\in\widetilde{\mathcal{B}}} B = \bigcap_{B\in\mathcal{B}} B \cap U_C[D]^c = D \cap U_C[D]^c$, which is a contradiction. Therefore, there exists $B \in \mathcal{B}$ such that $B \subseteq U_C[D]$ and the proof is complete. □

Example 3.3. Let $f : X \to X$ be a continuous mapping and for each $C \in \mathcal{C}$ denote $B_C = \{x \in X : f(x) \in \overline{U_C[x]}\}$. Suppose $(X,\mathcal{U}_X)$ is complete and $\mathcal{B} = \{B_C; C \in \mathcal{C}\}$ is a filter base satisfying $\bigcup_{C\in\mathcal{C}} \mathcal{M}_X(B_C) = \mathcal{C}$. Then, from Theorem 3.2, the set of the all fixed points of f is compact and nonempty. Moreover, $\mathcal{B}$ converges to this set in the $\mathcal{H}$-topology.

[1] $C + \mathcal{M}_X(U_C[A]) = \{C + C' : C' \in \mathcal{M}_X(U_C[A])\}$

Corollary 3.4 (Kuratowski [20]). *Let (X, d) be a complete metric space, α_X be the noncompactness measure on X, $\{B_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of nonempty, closed and bounded subsets of X and $D = \bigcap_{n\in\mathbb{N}} B_n$. Suppose* $\lim_{n\to\infty} \alpha_X(B_n) = 0$. *Then, the following two conditions hold:*

(i) D is compact and nonempty,
(ii) $\mathcal{B}$ converges to D in the Hausdorff metric.

Proof. Let $\mathcal{C} = \{(-\epsilon, \epsilon) : \epsilon > 0\}$ and $C \in \mathcal{C}$. Hence, there exists $\epsilon > 0$ such that $C = (-\epsilon, \epsilon)$. Let $n_0 \in \mathbb{N}$ such that $\alpha_X(B_{n_0}) < \epsilon$. Hence, $C \in \mathcal{M}_X(B_{n_0})$ and consequently $\bigcup_{n\in\mathbb{N}} \mathcal{M}_X(B_n) = \mathcal{C}$. Therefore, conditions (i) and (ii) follows from Theorem 3.2, which completes the proof. □

Remark 3.5. Note that whether $\mathcal{B}$ is countable, in Theorem 3.2, in order to $\bigcap_{B\in\mathcal{B}} B$ is compact and nonempty, it suffices that X is sequentially complete, with respect to the uniformity $\mathcal{U}_X$.

Let $T : D \subseteq X \to 2^X$ be a set-valued mapping. We say that T is condensing, if for each $A \subseteq D$ such that $A \in \mathcal{B}(X)$ and $\mathcal{M}_X(A) \neq \mathcal{C}$, we have $T(A) \in \mathcal{B}(X)$, $\mathcal{M}_X(A) \subseteq \mathcal{M}_X(T(A))$ and $\mathcal{M}_X(A) \neq \mathcal{M}_X(T(A))$.

A subfamily $\mathcal{D}$ of 2^X is said to be stable under intersections, if for any $\mathcal{D}' \subseteq \mathcal{D}$, we have $\bigcap_{D\in\mathcal{D}'} D \in \mathcal{D}$.

Theorem 3.6. *Suppose (X, U_X) is complete. Let $\mathcal{D}$ be stable under intersections subfamily of $\mathcal{CB}(X)$, $D \in \mathcal{D}$ and $T : D \to 2^D$ be a condensing set-valued mapping. Then, there exists $C \in \mathcal{D}$, a compact subset of D, such that $T(C) \subseteq C$.*

Proof. Let $x_0 \in D$ and $\Sigma = \{K \in \mathcal{D} : x_0 \in K \subseteq D \text{ and } T(K) \subseteq K\}$. Due to $D \in \Sigma$, we have $\Sigma \neq \emptyset$. Let $B = \bigcap_{K\in\Sigma} K$ and $C = \overline{\{x_0\} \cup T(B)}$. We have $T(B) \subseteq \bigcap_{K\in\Sigma} T(K) \subseteq B$ and $x_0 \in B$. Moreover, since B is closed, $C \subseteq B$. Thus $T(C) \subseteq T(B) \subseteq C$, $C \in \Sigma$ and $B = C$. Since $\mathcal{D}$ is stable under intersections, we have $C \in \mathcal{D}$. From properties b) and c), we obtain $\mathcal{M}_X(C) = \mathcal{M}_X(T(B)) = \mathcal{M}_X(T(C))$ and due to T is condensing, we have $\mathcal{M}_X(C) = \mathcal{C}$. Since C is closed, we have C is compact and the proof is complete. □

Example 3.7. Suppose X is a complete topological vector space. Hence, under conditions and notations stated in Theorem 3.6, $\mathcal{D}$ can be chosen as the family of all nonempty convex sets and consequently, there exists a compact convex subset C of X such that $T(C) \subseteq C$. This fact, in case the Schauder conjecture were correct (see *The Scottish Book* [21], Problem 54), would imply the existence of a fixed point of T, when T is a single-valued mapping.

4. Applications to approximate and fixed points

In this section, the cone P is assumed to be normal (c.f. [2]), i.e. for each $C \in \mathcal{C}$, and $x, y \in C$, one has $\{z \in E : x \preceq z \preceq y\} \subseteq C$. Let $T : D \subseteq X \to 2^X$ be a set-valued mapping. A point $x^* \in D$ is said to be a fixed point of T, if $x^* \in Tx^*$ and T is said to have an approximate fixed point (see [4, 12, 14, 26, 30] for related concepts), if for any $C \in \mathcal{C}$, there exists $x \in X$ such that $Tx \cap U_C[x] \neq \emptyset$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

A concept extending the classical Hausdorff metric is stated as follows. For $x \in X$ and $A, B \in \mathcal{CB}(X)$, we define $s(x, B)$ and $s(A, B)$ as follows:

$$s(x, B) = \bigcup_{b\in B} \{\epsilon \succ \theta : d(x, b) \preceq \epsilon\}$$

and

$$s(A, B) = \bigcap_{a\in A} s(a, B) \cap \bigcap_{b\in B} s(b, A).$$

Let $k \geq 0$ and $T : X \to \mathcal{CB}(X)$ be a set-valued mapping satisfying

$$kd(x, y) \in s(Tx, Ty), \qquad \text{for all } x, y \in X. \tag{4.1}$$

As in [5, 6, 10, 22, 28], we say T is a contraction whenever $k < 1$. In this work, the set-valued mapping T is said to be non-expansive, whenever (4.1) is satisfied with $k = 1$. In [16, 24, 31], other definitions of contraction for set-valued mappings are given in the context of cone metric spaces. Also, these definitions are based on extensions of the classical Hausdorff metric.

The following lemma is stated and proved in [28] (see also [6]) whether E is a locally convex space and in [31] whether E is a Banach space. Since we are not assuming these conditions and the completeness condition stated here is different, we give an explicit proof, which is simple and similar to that given by Nadler in [23], for set-valued contraction with respect to the classical Hausdorff metric.

Lemma 4.1. *Suppose $(X,\mathcal{U}_X)$ is complete and let $T : X \to \mathcal{CB}(X)$ be a contraction. Then, T has a fixed point.*

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. Since $kd(x_0, x_1) \in \bigcap_{x\in Tx_0} s(x, Tx_1) \subseteq s(x_1, Tx_1)$, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \preceq kd(x_0, x_1)$. It follows by induction that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that, for each $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$ and $d(x_{n+1}, x_{n+2}) \preceq kd(x_n, x_{n+1})$. Hence, for each $n, p \in \mathbb{N}$, $d(x_n, x_{n+p}) \preceq \{k^n/(1-k)\}d(x_0, x_1)$ and since E is normal, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\mathcal{U}_X$. Thus, there exists $x^* \in X$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. On the other hand, $kd(x_{n+1}, x^*) \in s(Tx_n, Tx^*)$ and hence there exists $y_n \in Tx^*$ such that $d(x_{n+1}, y_n) \preceq kd(x_{n+1}, x^*)$. Accordingly, $d(x^*, y_n) \preceq d(x^*, x_{n+1}) + kd(x_{n+1}, x^*)$. Let $C, C' \in \mathcal{C}$ and $N \in \mathbb{N}$ such that $C' + C' \subseteq C$ and $d(x_n, x^*) \in C'$ for all $n \geq N$. We have, $d(x^*, x_{n+1}) + kd(x_{n+1}, x^*) \in C' + kC' \subseteq C$ and since E is normal, $d(x^*, y_n) \in C$, for all $n \geq N$. This proves that $\{y_n\}_{n\in\mathbb{N}}$ converges to x^* and due to Tx^* is closed, we have $x^* \in Tx^*$, which completes the proof. □

Next, we assume X is a vector space endowed with a cone norm on E. That is, there exists a function $\|\cdot\| : X \to E$ satisfying the following three conditions: (a) for all $x \in X$, $\|x\| = 0$ implies $x = 0$, (b) for all $x \in X$ and $\lambda \in \mathbb{R}$, $\|\lambda x\| = |\lambda|\|x\|$, and (c) for all $x, y \in X$, $\|x + y\| \preceq \|x\| + \|y\|$. We denote by d the cone metric induced by $\|\cdot\|$ and, in what follows, for $A, B \in \mathcal{CB}(X)$, $s(A, B)$ is defined in terms of this cone metric.

Since E is normal, it is easy to see that the topology of X, generated by $\mathcal{U}_X$, makes continuous its vector space operations. Consequently, with respect to this topology, X turns out a topological vector space.

Theorem 4.2. *Suppose $(X,\mathcal{U}_X)$ is complete, D is a nonempty closed convex and bounded subset of X and $T : D \to 2^D$ is a non-expansive set-valued mapping. Then, T has an approximate fixed point.*

Proof. For a fixed $z_0 \in D$ and $\epsilon > 0$, we define $T_\epsilon : D \to 2^D$ by $T_\epsilon x = \epsilon z_0 + (1-\epsilon)Tx$. Notice that T_ϵ is well-defined and, for any $x, y \in X$, $s(T_\epsilon x, T_\epsilon y) = (1-\epsilon)s(Tx, Ty)$. Consequently, $(1-\epsilon)d(x, y) \in s(T_\epsilon x, T_\epsilon y)$, i.e. T_ϵ is a contraction. It follows from Lemma 4.1 that there exists $x_\epsilon \in D$ such that $x_\epsilon \in T_\epsilon x_\epsilon$. Let $y_\epsilon \in Tx_\epsilon$ such that $x_\epsilon = \epsilon z_0 + (1-\epsilon)y_\epsilon$. We have $\|x_\epsilon - y_\epsilon\| = \epsilon\|z_0 - y_\epsilon\| \preceq \epsilon c$, where c is an upper bound of $\{d(u, v) : u, v \in D\}$. Since E is normal, for each $C \in \mathcal{C}$ we can choose $\epsilon > 0$ such that $\|x_\epsilon - y_\epsilon\| \in C$. Therefore, $Tx_\epsilon \cap U_C[x_\epsilon] \neq \emptyset$ and the proof is complete. □

Given $T : D \subseteq X \to 2^D$ and $C \in \mathcal{C}$, we denote $B_C[T] = \{x \in D : Tx \cap U_C[x] \neq \emptyset\}$.

Theorem 4.3. *Suppose $(X,\mathcal{U}_X)$ is complete, D is a nonempty closed convex and bounded subset of X, $T : D \to \mathcal{K}(D)$ is a non-expansive set-valued mapping and $\bigcup_{C\in\mathcal{C}} \mathcal{M}_X(B_C[T]) = \mathcal{C}$. Then,* Fix$(T)$ *is compact and nonempty.*

Proof. From Theorem 4.2, $B_C[T]$ is nonempty and hence, $\mathcal{B}_T = \{\overline{B_C[T]}; C \in \mathcal{C}\}$ is a filter base of closed sets. On the other hand, $\bigcup_{C\in\mathcal{C}} \mathcal{M}_X(\overline{B_C[T]}) = \mathcal{C}$ holds due to the assumption and c) in Section 2. Consequently, Theorem 3.2 implies that $\bigcap_{C\in\mathcal{C}} \overline{B_C[T]} \neq \emptyset$. Since $\mathrm{Fix}(T) = \bigcap_{C\in\mathcal{C}} B_C[T]$, it suffices to prove that

$$\bigcap_{C\in\mathcal{C}} \overline{B_C[T]} = \bigcap_{C\in\mathcal{C}} B_C[T]. \tag{4.2}$$

In order to obtain (4.2), we prove that for each $C \in \mathcal{C}$, there exists $C' \in \mathcal{C}$ such that $\overline{B_{C'}(T)} \subseteq B_C[T]$. Let $C \in \mathcal{C}$ and choose $C' \in \mathcal{C}$ satisfying $C' + C' \subseteq C$. Let $y \in \overline{B_{C'}(T)}$ and suppose that $Ty \cap \overline{U_C[y]} = \emptyset$. Since $Ty = \bigcap_{C \in \mathcal{C}} \overline{U_C[Ty]}$ and Ty is compact, there exists $C'' \in \mathcal{C}$ such that $U_{C''}[Ty] \cap U_C[y] = \emptyset$. Notice that for each $x \in U_{C'}[y]$, $U_{C'}[x] \subseteq U_C[y]$. Let $x \in U_{C''}[y]$. Hence $d(x, y) \in s(Tx, Ty)$ and thus, for each $z \in Tx$ there exists $b \in Ty$ such that $d(z, b) \preceq d(x, y)$. Since $d(x, y) \in C''$ and E is normal, we have $d(z, b) \in C''$. Consequently, for each $x \in U_{C''}[y]$, we have $Tx \subseteq U_{C''}[Ty]$ and by defining $V_y = U_{C'}[y] \cap U_{C''}[y]$ we obtain that, for each $x \in V_y$, $Tx \cap U_{C'}[x] \subseteq U_{C''}[Ty] \cap U_C[y] = \emptyset$, i.e. $V_y \subseteq B_{C'}(T)^{\mathrm{c}}$, which is a contradiction. Therefore, $\overline{B_{C'}(T)} \subseteq B_C(T)$ and the proof is complete. □

Corollary 4.4. *Suppose D is a nonempty compact and convex subset of X and $T : D \to \mathcal{K}(D)$ is a non-expansive set-valued mapping. Then,* $\mathrm{Fix}(T)$ *is (compact and) nonempty.*

Remark 4.5. Suppose $\mathcal{C}$ is a base of neighborhoods of θ consisting of convex sets. Hence, $\{U_C[0]; C \in \mathcal{C}\}$ is a base of convex neighborhoods of $0 \in X$. Consequently, X is a locally convex space whether E so is.

For each $D \subseteq X$, let $\mathcal{Q}(D)$ be the family of all nonempty compact convex subsets of D.

Lemma 4.6. *Suppose E is a locally convex space, $C \in \mathcal{Q}(X)$ and let $T : C \to \mathcal{Q}(C)$ be a continuous set-valued mapping. Then, T has a fixed point.*

Proof. It directly follows from Theorem 2 by Fan in [15]. □

The following result extends, to locally convex spaces and for continuous multi-functions, a known theorem by Darbo in [11].

Theorem 4.7. *Suppose E is a locally convex space and $(X, \mathcal{U}_X)$ is complete. Let D be a nonempty bounded closed and convex subset of X and $T : D \to \mathcal{C}(D)$ be a condensing multi-function with convex images. Then, T has a fixed point.*

Proof. Let $\mathcal{D}$ be the subfamily of $\mathcal{CB}(X)$ consisting of all convex subsets of X. Since $\mathcal{D}$ is stable under intersections, Theorem 3.6 implies that there exists a compact and convex subset C of X such that $T(C) \subseteq C$. Since $C \in \mathcal{Q}(X)$ and for each $x \in C$, $Tx \in \mathcal{Q}(C)$, it follows from Lemma 4.6 that T has a fixed point, which completes the proof. □

Acknowledgements

This work was partially supported by FONDECYT grant 1120879 from the Chilean government.

References

[1] R. P. Agarwal, M. A. Khamsi, *Extension of Caristi's fixed point theorem to vector valued metric spaces*, Nonlinear Anal., **74** (2011), 141–145. 1

[2] C. D. Aliprantis, R. Tourky, *Cones and Duality*, American Mathematical Society, Providence, Rhode Island, (2007). 2, 4

[3] I. Altun, V. Rakočević, *Ordered cone metric spaces and fixed point results*, Comput. Math. Appl., **60** (2010), 1145–1151. 1

[4] A. Amini, J. Fakhar, J. Zafarani, *Fixed point theorems for the class S-KKM mappings in abstract convex spaces*, Nonlinear Anal., **66** (2007), 14–21. 1, 4

[5] A. Azam, N. Mehmood, *Multivalued fixed point theorems in cone tvs-cone metric spaces*, Fixed Point Theory Appl., **2013** (2013), 13 pages. 4

[6] A. Azam, N. Mehmood, J. Ahmad, S. Radenović, *Multivalued fixed point theorems in cone b-metric spaces*, J. Inequal. Appl., **2013** (2013), 9 pages. 1, 4

[7] N. Bourbaki, *Elements of Mathematics, General Topology. Part 1*, Hermann, Paris, (1966). 2, 3

[8] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Springer-Verlag, Berlin, (1977). 2

[9] C. Çevik, I. Altun, *Vector metric spaces and some properties*, Topol. Methods Nonlinear Anal., **34** (2009), 375–382. 2.2, 2.4

[10] S. H. Cho, J. S. Bae, *Fixed point theorems for multivalued maps in cone metric spaces*, Fixed Point Theory Appl., **2011** (2011), 7 pages. 1, 4

[11] G. Darbo, *Punti uniti in transformazioni a codominio non compatto*, Rend. Semin. Mat. Univ. Padova, **24** (1955), 84–92. 1, 4

[12] S. Dhompongsa, W. Inthakon, A. Kaewkhao, *Edelstein's method and fixed point theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **350** (2009), 12–17. 4

[13] J. Eisenfeld, V. Lakshmikantham, *Fixed points theorems through abstract cones*, J. Math. Anal. Appl., **52** (1975), 25–35. 1

[14] R. Espínola, W. A. Kirk, *Fixed points and approximate fixed points in product spaces*, Taiwanese J. Math., **5** (2001), 405–416. 4

[15] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann., **142** (1961), 305–310. 4

[16] R. Fierro, *Fixed point theorems for set-valued mappings on TVS-cone metric spaces*, Fixed Point Theory Appl., **2015** (2015), 7 pages. 4

[17] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, United Kingdom, (1990). 1

[18] C. J. Himmelberg, J. R. Porter, F. S. Van Vleck, *Fixed point theorems for condensing multifunctions*, Proc. Amer. Math. Soc., **23** (1969), 635–641. 1

[19] L. G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468–1476. 1

[20] C. Kuratowski, *Sur les espaces completes*, Fund. Math., **15** (1939), 301–309. 1, 3, 3.4

[21] R. D. Mauldin, *The Scottish Book: mathematics from the Scottish Café*, Birkhäuser, Basel, (1981). 3.7

[22] N. Mehmood, A. Azam, L. D. R. Kočinac, *Multivalued fixed point results in cone metric spaces*, Topology Appl., **179** (2015), 156–170. 1, 4

[23] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math., **30** (1969), 475–488. 4

[24] S. Radenović, S. Simić, N. Cakić, Z. Golubović, *A note on tvs-cone metric fixed point theory*, Math. Comput. Model., **54** (2011), 2418–2422. 1, 4

[25] S. Reich, *Fixed points in locally convex spaces*, Math. Z., **125** (1972), 17–31. 1

[26] S. Reich, A. J. Zaslavski, *Approximate fixed points of nonexpansive mappings in unbounded sets*, J. Fixed Point Theory Appl., **13** (2013), 627–632. 4

[27] J. Saint-Raymond, *Topologie sur l'ensemble des compacts non vides d'un espace topologique séparé*, Séminaire Choquet, **9** (1969/70), 6 pages. 2

[28] W. Shatanawi, V. Ćojbašić, S. Radenović, A. Al-Rawashdeh, *Mizoguchi-takahashi-type theorems in tvs-cone metric spaces*, Fixed Point Theory Appl., **2012** (2012), 7 pages. 4

[29] G. Soleimani Rad, H. Rahimi, S. Radenović, *Algebraic cone b-metric spaces and its equivalence*, Miskolc Math. Notes, (In Press). 1

[30] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **340** (2008), 1088–1095. 4

[31] D. Wardowski, *On set-valued contractions of Nadler type in cone metric spaces*, Appl. Math. Lett., **24** (2011), 275–278. 1, 4