Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2688–2696
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Sharpened versions of Mitrinović-Adamović, Lazarević and Wilker's inequalities for trigonometric and hyperbolic functions

Shan-He Wu[a,*], Shu-Guang Li[a], Mihály Bencze[b]

[a]Department of Mathematics, Longyan University, Longyan, Fujian, 364012, P. R. China.
[b]Department of Mathematics, University of Craiova, Craiova, RO-200585, Romania.

Communicated by Y. Yao

Abstract

In this paper, we establish new sharpened versions of Mitrinović-Adamović and Lazarević's inequalities. Further, we provide an application of our results to the improvements of Wilker's inequality for trigonometric and hyperbolic functions. We show that the coefficient assigned to each of these sharpened inequalities is best possible.

Keywords: Mitrinović-Adamović's inequality, Lazarević's inequality, Wilker's inequality, sharpening, best possible coefficient, trigonometric functions, hyperbolic functions.
2010 MSC: 26D05, 26D15, 33B10.

1. Introduction

Mitrinović and Adamović [6] proved that the inequality

$$\cos x < \left(\frac{\sin x}{x}\right)^3 \tag{1.1}$$

holds for all $x \in (0, \pi/2)$, and showed that the exponent 3 is the largest possible.

A hyperbolic analogue of inequality (1.1) was presented by Lazarević [5], which is stated as follows:

*Corresponding author
Email addresses: shanhewu@163.com (Shan-He Wu), shuguanglily@sina.com (Shu-Guang Li), benczemihaly@gmail.com (Mihály Bencze)

Received 2016-02-05

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3, \tag{1.2}$$

where $x \neq 0$, and the exponent 3 is the least possible.

A number of generalizations, improvements and applications relating to Mitrinović-Adamović's inequality (1.1) and Lazarević's inequality (1.2) can be found in the literature [4, 7, 9, 12, 15, 19, 20, 21, 22, 26]. Among these investigations, we remark here that Wu and Baricz [15] dealt with the generalizations of inequalities (1.1) and (1.2) and obtained two excellent results, as follows:

Theorem 1.1. *If* $0 < x < \frac{\pi}{2}$, *then the inequality*

$$1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cos x < \left(\frac{\sin x}{x}\right)^\lambda \tag{1.3}$$

holds if and only if $\lambda < 0$ *or* $\lambda \geq \lambda_0$, *where* $\lambda_0 \approx 1.420330769$ *is the root of the equation* $\lambda/3 + (2/\pi)^\lambda - 1 = 0$.

Theorem 1.2. *If* $x \neq 0$, *then the inequality*

$$1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cosh x < \left(\frac{\sinh x}{x}\right)^\lambda \tag{1.4}$$

holds if and only if $\lambda < 0$ *or* $\lambda \geq 7/5$.

The main purpose of this paper is to establish new sharpened versions of Mitrinović-Adamović's inequality (1.1) and Lazarević's inequality (1.2). Moreover, we provide an application of our results to the improvements of Wilker's inequality for trigonometric and hyperbolic functions.

2. Sharpening of Mitrinović-Adamović and Lazarević's Inequalities

Theorem 2.1. *If* $0 < x < \frac{\pi}{2}$, *then the inequality*

$$\cos x < \left(\frac{\sin x}{x}\right)^3 \left(1 - \frac{12}{5}\left(1 - \frac{x}{\sin x}\right)^2\right) \tag{2.1}$$

holds, where the coefficient $\frac{12}{5}$ *is the best possible.*

Proof. By the Taylor expansions of $\sin x$ and $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + (-1)^{k+1}\frac{(\cos\theta x)}{(2k+2)!}x^{2k+2},$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!} + (-1)^k \frac{(\cos\theta x)}{(2k+1)!}x^{2k+1},$$

where $0 < \theta < 1$, it is easy to observe that

$$\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \tag{2.2}$$

and

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} < \frac{\sin x}{x} < 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \tag{2.3}$$

for $0 < x < \frac{\pi}{2}$. Using inequalities (2.2) and (2.3) together with a simple calculation, it follows that

$$\begin{aligned}
\cos x - \left(\frac{\sin x}{x}\right)^3 \left(1 - \frac{12}{5}\left(1 - \frac{x}{\sin x}\right)^2\right) &= \cos x + \frac{7}{5}\left(\frac{\sin x}{x}\right)^3 - \frac{24}{5}\left(\frac{\sin x}{x}\right)^2 + \frac{12}{5}\left(\frac{\sin x}{x}\right) \\
&< 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{7}{5}\Big(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\Big)^3 \\
&\quad - \frac{24}{5}\Big(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\Big)^2 + \frac{12}{5}\Big(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\Big) \\
&= \frac{1}{423360000} x^6 \left(263x^6 - 13860x^4 + 270060x^2 - 1820\,000\right) \\
&< \frac{1}{423360000} x^6 \left(263 \times 2^2 x^4 - 13860x^4 + 270060 \times 2^2 - 1820000\right) \\
&= \frac{1}{423360000} x^6 (-12808x^4 - 739760) \\
&< 0.
\end{aligned}$$

This proves the desired inequality (2.1).

Next, we need to show that the coefficient $\frac{12}{5}$ is the best possible in inequality (2.1) in the strong sense. Consider inequality (2.1) in a general form as

$$\cos x < \left(\frac{\sin x}{x}\right)^3 \left(1 - \alpha\left(1 - \frac{x}{\sin x}\right)^2\right)$$

$$\Longleftrightarrow \alpha < \frac{(\frac{\sin x}{x})^3 - \cos x}{(1 - \frac{x}{\sin x})^2 (\frac{\sin x}{x})^3}. \tag{2.4}$$

Taking the limit in (2.4) as $x \to 0$, we get

$$\alpha \le \lim_{x \to 0} \frac{(\frac{\sin x}{x})^3 - \cos x}{(1 - \frac{x}{\sin x})^2 (\frac{\sin x}{x})^3} = \frac{12}{5}.$$

Consequently, the coefficient $\alpha = \frac{12}{5}$ is the best possible in inequality (2.1). This completes the proof of Theorem 2.1.

$\square$

Theorem 2.2. *For all nonzero real numbers x, the inequality*

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 - \frac{12}{5}\left(1 - \frac{x}{\sinh x}\right)^2 \tag{2.5}$$

holds, where the coefficient $\frac{12}{5}$ is the best possible.

Proof. We begin by recalling the result asserted by Theorem 1.2 in the introduction section, i.e.,

$$1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cosh x < \left(\frac{\sinh x}{x}\right)^\lambda, \quad \lambda \in \big(-\infty, 0\big) \cup \big[\,\frac{7}{5}, +\infty\big).$$

Choosing $\lambda = \frac{7}{5}$ in the above inequality gives

$$\cosh x < \frac{15}{7}\left(\frac{\sinh x}{x}\right)^{\frac{7}{5}} - \frac{8}{7} \quad (x \neq 0). \tag{2.6}$$

Let $\left(\frac{\sinh x}{x}\right)^{\frac{1}{5}} = t$. Clearly, we have $t > 1$ in light of the Lazarević's inequality

$$1 < \cosh x < \left(\frac{\sinh x}{x}\right)^3 \quad (x \neq 0).$$

Using inequality (2.6) together with a straightforward computation, it follows that

$$\begin{aligned}
&\cosh x - \left(\frac{\sinh x}{x}\right)^3 + \frac{12}{5}\left(1 - \frac{x}{\sinh x}\right)^2 \\
&< \frac{15}{7}\left(\frac{\sinh x}{x}\right)^{\frac{7}{5}} - \frac{8}{7} - \left(\frac{\sinh x}{x}\right)^3 + \frac{12}{5}\left(1 - \frac{x}{\sinh x}\right)^2 \\
&= \frac{15}{7}t^7 - \frac{8}{7} - t^{15} + \frac{12}{5}\left(1 - t^{-5}\right)^2 \\
&= -\frac{(t-1)^3}{t^{10}}\Big(t^{22} + 3t^{21} + 6t^{20} + 10t^{19} + 15t^{18} + 21t^{17} + 28t^{16} \\
&\quad + 36t^{15} + \frac{300}{7}t^{14} + \frac{340}{7}t^{13} + \frac{372}{7}t^{12} + \frac{396}{7}t^{11} + \frac{412}{7}t^{10} + 60t^9 \\
&\quad + 60t^8 + \frac{288}{5}t^7 + \frac{264}{5}t^6 + \frac{228}{5}t^5 + 36t^4 + 24t^3 + \frac{72}{5}t^2 + \frac{36}{5}t + \frac{12}{5}\Big) \\
&< 0.
\end{aligned}$$

Hence

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 - \frac{12}{5}\left(1 - \frac{x}{\sinh x}\right)^2.$$

Next, we shall explain why the coefficient $\frac{12}{5}$ is the best possible in inequality (2.5).
Consider inequality (2.5) in a general form as

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 - \beta\left(1 - \frac{x}{\sinh x}\right)^2$$

$$\Longleftrightarrow \beta < \frac{(\frac{\sinh x}{x})^3 - \cosh x}{(1 - (\frac{x}{\sinh x}))^2}, \tag{2.7}$$

we deduce that

$$\beta \leq \lim_{x \to 0} \frac{(\frac{\sinh x}{x})^3 - \cosh x}{(1 - (\frac{x}{\sinh x}))^2} = \frac{12}{5}.$$

Hence, the coefficient $\beta = \frac{12}{5}$ is the best possible in inequality (2.5). The proof of Theorem 2.2 is thus completed.

□

3. Application to the Improvements of Wilker's Inequality

The inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \quad \left(0 < x < \frac{\pi}{2}\right) \tag{3.1}$$

is called in the literature as Wilker's inequality (see [8]). This beautiful inequality has evoked the interest of many authors, and has motivated a lot of research papers involving its proofs, generalizations, variants and improvements (see [1, 2, 3, 10, 11, 13, 14, 16, 17, 23, 24] and the references therein).

In 2007, an inequality of Wilker-type for hyperbolic functions was presented by Zhu [25]

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}>2\quad\big(x\neq 0\big). \tag{3.2}$$

Recently, Wu et al. [18] gave a sharpening of hyperbolic Wilker-type inequality as follows:

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}>2\sqrt{1+\left(\frac{x}{\sinh x}\right)^3\left|\cos x-\left(\frac{\sin x}{x}\right)^3\right|}\quad\big(x\neq 0\big). \tag{3.3}$$

In this section, we establish new sharpened versions of Wilker's inequality for trigonometric and hyperbolic functions.

Theorem 3.1. *If* $0<x<\frac{\pi}{2}$, *then we have the inequality*

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2+\frac{32}{5}\left(1-\frac{x}{\sin x}\right)^2, \tag{3.4}$$

where the coefficient $\frac{32}{5}$ *is the best possible.*

Proof. In order to prove inequality (3.4), it suffices to prove that the following inequality

$$\frac{\left(\frac{\sin x}{x}\right)^2+\frac{\sin x}{x\cos x}-2}{\left(1-\frac{x}{\sin x}\right)^2}>\frac{32}{5}$$

holds for $0<x<\frac{\pi}{2}$.

By using the result of Theorem 2.1

$$\cos x<\left(\frac{\sin x}{x}\right)^3\left(1-\frac{12}{5}\left(1-\frac{x}{\sin x}\right)^2\right),\quad x\in\big(0,\frac{\pi}{2}\big),$$

we obtain

$$\frac{\left(\frac{\sin x}{x}\right)^2+\frac{\sin x}{x\cos x}-2}{\left(1-\frac{x}{\sin x}\right)^2}>\frac{\left(\frac{\sin x}{x}\right)^2+\left(\frac{\sin x}{x}\right)\left(\left(\frac{\sin x}{x}\right)^3\left(1-\frac{12}{5}\left(1-\frac{x}{\sin x}\right)^2\right)\right)^{-1}-2}{\left(1-\frac{x}{\sin x}\right)^2}. \tag{3.5}$$

Let $\frac{\sin x}{x}=t$. We conclude $\frac{2}{\pi}<t<1$ by virtue of the Jordan's inequality (see [7])

$$\frac{2}{\pi}<\frac{\sin x}{x}<1,\quad x\in\big(0,\frac{\pi}{2}\big).$$

A direct calculation gives

$$\begin{aligned}&\frac{\left(\frac{\sin x}{x}\right)^2+\left(\frac{\sin x}{x}\right)\left(\left(\frac{\sin x}{x}\right)^3\left(1-\frac{12}{5}\left(1-\frac{x}{\sin x}\right)^2\right)\right)^{-1}-2}{\left(1-\frac{x}{\sin x}\right)^2}\\&=\frac{t^2+t\left(t^3\left(1-\frac{12}{5}\left(1-\frac{1}{t}\right)^2\right)\right)^{-1}-2}{\left(1-\frac{1}{t}\right)^2}\\&=\frac{7t^4-10t^3-29t^2}{7t^2-24t+12}.\end{aligned}$$

Consider the function

$$f(t)=\frac{7t^4-10t^3-29t^2}{7t^2-24t+12},\quad t\in\big(\frac{2}{\pi},1\big).$$

Differentiating $f(t)$ with respect to t gives

$$\begin{aligned}
f'(t) &= \frac{2t\left(168t + 408t^2 - 287t^3 + 49t^4 - 348\right)}{(7t^2 - 24t + 12)^2} \\
&= \frac{-2t\left(\left(7t\left(27 - 7t\right) + 19\right)\left(t - 1\right)^2 + 319(1 - t) + 10\right)}{(7t^2 - 24t + 12)^2} \\
&< 0,
\end{aligned}$$

where $\frac{2}{\pi} < t < 1$.

It follows that $f(t)$ is decreasing on $(\frac{2}{\pi}, 1)$. Consequently,

$$f(t) > f(1) = \frac{32}{5} \quad \text{for} \quad t \in \left(\frac{2}{\pi}, 1\right),$$

that is,

$$\frac{\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\sin x}{x}\right)\left(\left(\frac{\sin x}{x}\right)^3\left(1 - \frac{12}{5}\left(1 - \frac{x}{\sin x}\right)^2\right)\right)^{-1} - 2}{\left(1 - \frac{x}{\sin x}\right)^2} > \frac{32}{5}.$$

By inequality (3.5), we get

$$\frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\sin x}{x\cos x} - 2}{\left(1 - \frac{x}{\sin x}\right)^2} > \frac{32}{5},$$

which implies the desired inequality (3.4).

Next, we shall prove the assertion that the coefficient $\frac{32}{5}$ is the best possible in inequality (3.4).

Consider inequality (3.4) in a general form as

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \gamma\left(1 - \frac{x}{\sin x}\right)^2$$

$$\iff \gamma < \frac{(\frac{\sin x}{x})^2 + \frac{\tan x}{x} - 2}{(1 - \frac{x}{\sin x})^2}. \tag{3.6}$$

Taking the limit in (3.6) as $x \to 0$, we obtain

$$\gamma \leq \lim_{x\to 0} \frac{(\frac{\sin x}{x})^2 + \frac{\tan x}{x} - 2}{(1 - \frac{x}{\sin x})^2} = \frac{32}{5}.$$

Consequently, the coefficients $\gamma = \frac{32}{5}$ is the best possible in inequality (3.4). This completes the proof of Theorem 3.1. $\square$

Theorem 3.2. *For all nonzero real numbers x, the following inequality holds*

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 + \frac{8}{5}\left(1 - \left(\frac{x}{\sinh x}\right)^2\right)^2, \tag{3.7}$$

where the coefficient $\frac{8}{5}$ is the best possible.

Proof. To prove inequality (3.7), it is enough to prove that the following inequality

$$\frac{(\frac{\sinh x}{x})^2 + \frac{\tanh x}{x} - 2}{(1 - \frac{x^2}{\sinh^2 x})^2} > \frac{8}{5}$$

holds for $x \neq 0$.

By appealing to inequality (2.6) mentioned in Section 2, i.e.,

$$\cosh x < \frac{15}{7}\left(\frac{\sinh x}{x}\right)^{\frac{7}{5}} - \frac{8}{7} \quad (x \neq 0),$$

we obtain

$$\frac{\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2}{\left(1 - \frac{x^2}{\sinh^2 x}\right)^2} - \frac{8}{5} > \frac{\left(\frac{\sinh x}{x}\right)^2 + \left(\frac{\sinh x}{x}\right)\left(\frac{15}{7}\left(\frac{\sinh x}{x}\right)^{\frac{7}{5}} - \frac{8}{7}\right)^{-1} - 2}{\left(1 - \frac{x^2}{\sinh^2 x}\right)^2} - \frac{8}{5}.$$

Let $\left(\frac{\sinh x}{x}\right)^{\frac{1}{5}} = t$. Obviously, one has $t > 1$ in light of Lazarević's inequality

$$1 < \cosh x < \left(\frac{\sinh x}{x}\right)^3 \quad (x \neq 0).$$

Then, we have

$$\begin{aligned}
&\frac{\left(\frac{\sinh x}{x}\right)^2 + \left(\frac{\sinh x}{x}\right)\left(\frac{15}{7}\left(\frac{\sinh x}{x}\right)^{\frac{7}{5}} - \frac{8}{7}\right)^{-1} - 2}{\left(1 - \frac{x^2}{\sinh^2 x}\right)^2} - \frac{8}{5} \\
&\quad = \frac{t^{10} + t^5\left(\frac{15}{7}t^7 - \frac{8}{7}\right)^{-1} - 2}{(1 - t^{-10})^2} - \frac{8}{5} \\
&\quad = \frac{75t^{37} - 40t^{30} - 270t^{27} + 35t^{25} + 144t^{20} + 240t^{17} - 128t^{10} - 120t^7 + 64}{5\left(t^{10} - 1\right)^2(15t^7 - 8)}.
\end{aligned}$$

Define a function $g(t)$ by

$$g(t) = 75t^{37} - 40t^{30} - 270t^{27} + 35t^{25} + 144t^{20} + 240t^{17} - 128t^{10} - 120t^7 + 64, \quad t \in (1, +\infty).$$

Differentiating $g(t)$ with respect to t gives

$$\begin{aligned}
g'(t) &= 5t^6(555t^{30} - 240t^{23} - 1458t^{20} + 175t^{18} + 576t^{13} + 816t^{10} - 256t^3 - 168) \\
&= 5t^6 g_1(t).
\end{aligned}$$

where $g_1(t) = 555t^{30} - 240t^{23} - 1458t^{20} + 175t^{18} + 576t^{13} + 816t^{10} - 256t^3 - 168$.
Now, computing the derivative of $g_1(t)$ gives

$$\begin{aligned}
g_1'(t) &= 2t^2(8325t^{27} - 2760t^{20} - 14580t^{17} + 1575t^{15} + 3744t^{10} + 4080t^7 - 384) \\
&= 2t^2 g_2(t).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
g_2'(t) &= 15t^6(14985t^{20} - 3680t^{13} - 16524t^{10} + 1575t^8 + 2496t^3 + 1904) \\
&= 15t^6 g_3(t).
\end{aligned}$$

$$\begin{aligned}
g_3'(t) &= 4t^2\left(74925t^{17} - 11960t^{10} - 41310t^7 + 3150t^5 + 1872\right) \\
&= 4t^2\left(11960t^{10}(t^7 - 1) + 41310t^7(t^{10} - 1) + 21655t^{17} + 3150t^5 + 1872\right) \\
&> 0.
\end{aligned}$$

From $g\prime_3(t) > 0$ for $t \in (1, +\infty)$ and $g_3(1) = 756 > 0$, we conclude that the function g_3 is increasing on $(1, +\infty)$, and deduce that $g_3(t) > g_3(1) > 0$ for $t \in (1, +\infty)$.

Similar to the discussions made above, by using the functional relationships

$$g_2'(t) = 15t^6 g_3(t), \quad g_1'(t) = 2t^2 g_2(t), \quad g'(t) = 5t^6 g_1(t),$$

together with $g_2(1) = g_1(1) = g(1) = 0$, we deduce that each of the functions $g_3,\ g_2,\ g_1,\ g$ is increasing on $(1, +\infty)$, and conclude that, for $t \in (1, +\infty)$,

$$g_3(t) > 0, \quad g_2(t) > 0, \quad g_1(t) > 0, \quad g(t) > 0.$$

Hence, we have

$$\begin{aligned}\frac{\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2}{\left(1 - \frac{x^2}{\sinh^2 x}\right)^2} - \frac{8}{5} &> \frac{\left(\frac{\sinh x}{x}\right)^2 + \left(\frac{\sinh x}{x}\right)\left(\frac{15}{7}\left(\frac{\sinh x}{x}\right)^{\frac{7}{5}} - \frac{8}{7}\right)^{-1} - 2}{\left(1 - \frac{x^2}{\sinh^2 x}\right)^2} - \frac{8}{5} \\ &= \frac{g\left(t\right)}{5\left(t^{10} - 1\right)^2\left(15t^7 - 8\right)} \\ &> 0.\end{aligned}$$

Inequality (3.7) is proved.

Next, we need to verify that the coefficient $\frac{8}{5}$ is the best possible in inequality (3.7). For this, we consider the general form of inequality (3.7), i.e.,

$$\begin{aligned}&\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 + \mu\left(1 - \frac{x^2}{\sinh^2 x}\right)^2 \\ &\Longleftrightarrow \mu < \frac{(\frac{\sinh x}{x})^2 + \frac{\tanh x}{x} - 2}{(1 - \frac{x^2}{\sinh^2 x})^2}.\end{aligned} \tag{3.8}$$

Taking the limit in (3.8) as $x \to 0$, we obtain

$$\mu \le \lim_{x \to 0} \frac{(\frac{\sinh x}{x})^2 + \frac{\tanh x}{x} - 2}{(1 - \frac{x^2}{\sinh^2 x})^2} = \frac{8}{5}.$$

Thus, the coefficients $\frac{8}{5}$ is the best possible in inequality (3.7), that is, it cannot be replaced by a larger constant. The proof of Theorem 3.2 is completed. □

Acknowledgments

The present investigation was supported, in part, by the Natural Science Foundation of Fujian Province of China (No.2016J01023) and, in part, by the Foundation of the National Training Programs of China for Undergraduate Innovation and Entrepreneurship (No.201411312002).

References

[1] C. P. Chen, *Sharp Wilker- and Huygens-type inequalities for inverse trigonometric and inverse hyperbolic functions,* Integral Transforms Spec. Funct., **23** (2012), 865–873. 3

[2] C. P. Chen, W. S. Cheung, *Wilker and Huygens-type inequalities and solution to Oppenheim's problem,* Integral Transforms Spec. Funct., **23** (2012), 325–336. 3

[3] B. N. Guo, B. M. Qiao, F. Qi, W. Li, *On new proofs of Wilker's inequalities involving trigonometric functions,* Math. Inequal. Appl., **6** (2003), 19–22. 3

[4] R. Klén, M. Visuri, M. Vuorinen, *On Jordan type inequalities for hyperbolic functions,* J. Inequal. Appl., **2010** (2010), 14 pages. 1
[5] I. Lazarević, *Certain inequalities with hyperbolic functions,* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., **159-170** (1966), 41–48. 1
[6] D. S. Mitrinović, D. D. Adamović, *Sur une inégalité élémentaire où interviennent des fonctions trigonométriques*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., **143–155** (1965), 23–34. 1
[7] D. S. Mitrinović, P. M. Vasić, *Analytic Inequalities*, Springer-Verlag, New York, (1970). 1, 3
[8] L. H. Montgomery, D. J. Vaaler, J. Delany, E. D. Knuth, D. Vialetto, M. S. Wilker, J. B. Wilker, *Problems and Solutions: Elementary Problems: E3301-E3306,* Amer. Math. Monthly, **96** (1989), 54–55. 3
[9] C. Mortici, *A subtly analysis of Wilker inequality*, Appl. Math. Comput., **231** (2014), 516–520. 1
[10] E. Neuman, *On Wilker and Huygens type inequalities,* Math. Inequal. Appl., **15** (2012), 271–279. 3
[11] E. Neuman, *Wilker and Huygens-type inequalities for Jacobian elliptic and theta functions,* Integral Transforms Spec. Funct., **25** (2014), 240–248. 3
[12] I. Pinelis, *L' Hospital rules for monotonicity and the Wilker-Anglesio inequality,* Amer. Math. Monthly, **111** (2004), 905–909. 1
[13] J. S. Sumner, A. A. Jagers, M. Vowe, J. Anglesio, *Inequalities involving trigonometric functions,* Amer. Math. Monthly, **98** (1991), 264–267. 3
[14] S. Wu, *On extension and refinement of Wilker's inequality,* Rocky Mountains J. Math., **39** (2009), 683–687. 3
[15] S. Wu, Á. Baricz, *Generalizations of Mitrinović, Adamović and Lazarević's inequalities and their applications*, Publ. Math. Debrecen, **75** (2009), 447–458. 1
[16] S. Wu, H. M. Srivastava, *A weighted and exponential generalization of Wilker's inequality and its applications,* Integral Transform. Spec. Funct., **18** (2007), 529–535. 3
[17] S. Wu, H. M. Srivastava, *A further refinement of Wilker's inequality,* Integral Transform. Spec. Funct., **19** (2008), 757–765. 3
[18] S. H. Wu, H. P. Yue, Y. P. Deng, Y. M. Chu, *Several improvements of Mitrinović-Adamović and Lazarević's inequalities with applications to the sharpening of Wilker-type inequalities,* J. Nonlinear Sci. Appl., **9** (2016), 1755–1765. 3
[19] Z. H. Yang, *Renements of a two-sided inequality for trigonometric functions*, J. Math. Inequal., **7** (2013), 601–615. 1
[20] C. Y. Yang, *Inequalities on generalized trigonometric and hyperbolic functions,* J. Math. Anal. Appl., **419** (2014), 775–782. 1
[21] Z. H. Yang, Y. M. Chu, *A note on Jordan, Mitrinović-Adamović, and Cusa inequalities,* Abstr. Appl. Anal., **2014** (2014), 12 pages. 1
[22] L. Yin, L. Huang, F. Qi, *Some inequalities for the generalized trigonometric and hyperbolic functions,* Turkish J. Anal. Number theo., **2** (2014), 96–101. 1
[23] L. Zhang, L. Zhu, *A new elementary proof of Wilker's inequalities,* Math. Inequal. Appl., **11** (2008), 149–151. 3
[24] L. Zhu, *A new simple proof of Wilker's inequality,* Math. Inequal. Appl., **8** (2005), 749–750. 3
[25] L. Zhu, *On Wilker-type inequalities,* Math. Inequal. Appl., **10** (2007), 727–731. 3
[26] L. Zhu, *Generalized Lazarević's inequality and its applications II,* J. Inequal. Appl., **2009** (2009), 4 pages. 1