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Some new identities on the Apostol-Bernoulli polynomials of higher order derived from Bernoulli basis

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Abstract

In the present paper, we introduce a method in order to obtain some new interesting relations and identities of the Apostol-Bernoulli polynomials of higher order, which are derived from Bernoulli polynomial basis. Finally, by utilizing this method, we also get formulas for the convolutions of Bernoulli and Euler polynomials in terms of Apostol-Bernoulli polynomials of higher order.

Keywords: Generating function, Bernoulli polynomials of higher order, Euler polynomials of higher order, Hermite polynomials, Apostol-Bernoulli polynomials of higher order, Apostol-Euler polynomials of higher order, identities.
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1. Introduction

For $t \in \mathbb{C}$, the Euler polynomials have the following Taylor expansion at $t = 0$ (known as generating function):

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = e^{tE(x)} = \frac{2}{e^t+1} e^{xt}, \ \ (|t| < \pi) \tag{1.1}$$

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with the usual convention about replacing of $(E(x))^n := E_n(x)$, (see [1, 4, 9, 10, 11, 14, 16, 20]).

There are also explicit formulas for the Euler polynomials, e.g.,

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k},$$

where E_k means the Euler numbers. Conversely, the Euler numbers are expressed with the Euler polynomials through $E_k = 2^k E_k(1/2)$. These numbers can be computed by:

$$(E+1)^n + (E-1)^n = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0, \end{cases}$$

(see [7] and [17]).

For $|t| < 2\pi$ with $t \in \mathbb{C}$, the Bernoulli polynomials are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = e^{tB(x)} = \frac{t}{e^t - 1} e^{xt},$$

where we have used $(B(x))^n := B_n(x)$, symbolically. In the case $x = 0$, we have $B_n(0) := B_n$ that stands for n-th Bernoulli number. This number can be computed via

$$(B+1)^n - B_n = \delta_{n,1},$$

where $\delta_{n,1}$ stands for Kronecker delta, (see [3, 5, 11, 15]).

The Euler polynomials of order k are defined by the exponential generating function as follows:

$$\left(\frac{2}{e^t + 1}\right)^k e^{xt} = e^{tE^{(k)}(x)} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} \quad (k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}), \tag{1.2}$$

with the usual convention about replacing $(E^{(k)}(x))^n$ by $E_n^{(k)}(x)$. In the special case, $x = 0$, $E_n^{(k)}(0) := E_n^{(k)}$ are called Apostol-Euler numbers of order k, (see [14] and [16]).

In the complex plane, Apostol-Euler polynomials $E_n(x \mid \lambda)$ and Apostol-Bernoulli polynomials $B_n(x \mid \lambda)$ are given by [16]

$$\frac{2}{\lambda e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x \mid \lambda) \frac{t^n}{n!}, \quad (|t| < \log(-\lambda)), \tag{1.3}$$

$$\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x \mid \lambda) \frac{t^n}{n!}, \quad (|t| < \log \lambda). \tag{1.4}$$

In [16], Apostol-Euler polynomials of higher order $E_n^{(k)}(x \mid \lambda)$ and Apostol-Bernoulli polynomials of higher order $B_n^{(k)}(x \mid \lambda)$ are given by the following generating functions:

$$\left(\frac{2}{\lambda e^t + 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x \mid \lambda) \frac{t^n}{n!}, (|t| < \log(-\lambda)), \tag{1.5}$$

$$\frac{t^k}{(\lambda e^t - 1)^k} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x \mid \lambda) \frac{t^n}{n!}, (|t| < \log \lambda). \tag{1.6}$$

In the above expressions, we take the principal value of the logarithm $\log \lambda$, i.e., $\log \lambda = \log |\lambda| + i \arg \lambda \, (-\pi < \arg \lambda \leq \pi)$ when $\lambda \neq 1$; set $\log 1 = 0$ when $\lambda = 1$. Additionally, in the special case, $x = 0$ or $\lambda = 1$ in (1.5) and (1.6), we have $E_n^{(k)}(0 \mid \lambda) := E_n^{(k)}(\lambda)$ and $B_n^{(k)}(0 \mid \lambda) := B_n^{(k)}(\lambda)$, $E_n^{(k)}(x \mid 1) := E_n^{(k)}(x)$

and $B_n^{(k)}(x \mid 1) := B_n^{(k)}(x)$ that stand for Apostol-Euler numbers, Apostol-Bernoulli numbers, the Euler polynomials of order k and the Bernoulli polynomials of order k.

Apostol-Euler polynomials of higher order and Apostol-Bernoulli polynomials of higher order can be expressed in terms of their numbers as follows:

$$E_n^{(k)}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} x^l E_{n-l}^{(k)}(\lambda) \tag{1.7}$$

and

$$B_n^{(k)}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} x^l B_{n-l}^{(k)}(\lambda). \tag{1.8}$$

From (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) we have

$$\begin{aligned} E_n^{(1)}(x \mid \lambda) &:= E_n(x \mid \lambda) \text{ and } E_n^{(1)}(x \mid 1) := E_n(x \mid 1) := E_n(x), \\ B_n^{(1)}(x \mid \lambda) &:= B_n(x \mid \lambda) \text{ and } B_n^{(1)}(x \mid 1) := B_n(x \mid 1) := B_n(x). \end{aligned}$$

By (1.1), we easily get

$$E_n^{(0)}(x \mid \lambda) = B_n^{(0)}(x \mid \lambda) = x^n. \tag{1.9}$$

Applying derivative operator in the both sides of (1.8), we have

$$\frac{d}{dx} B_n^{(k)}(x \mid \lambda) = n B_{n-1}^{(k)}(x \mid \lambda). \tag{1.10}$$

Using (1.6), we arrive to

$$\frac{\lambda B_{n+1}^{(k)}(x+1 \mid \lambda) - B_{n+1}^{(k)}(x \mid \lambda)}{n+1} = B_n^{(k-1)}(x \mid \lambda), \quad (\text{see } [16]). \tag{1.11}$$

The linear operators Λ and D on the space of real-valued differentiable functions are considered as: For $n \in \mathbb{N}$

$$\Lambda f(x) = \lambda f(x+1) - f(x) \text{ and } Df(x) = \frac{df(x)}{dx}. \tag{1.12}$$

Notice that $\Lambda D = D\Lambda$. By (1.12), we have

$$\begin{aligned} \Lambda^2 f(x) = \Lambda(\Lambda f(x)) &= \lambda^2 f(x+2) - 2\lambda f(x+1) + f(x) \\ &= \sum_{l=0}^{2} \binom{2}{l} (-1)^l \lambda^l f(x+l). \end{aligned}$$

By continuing this way, we obtain

$$\Lambda^k f(x) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \lambda^l f(x+l).$$

Consequently, we give the following Lemma.

Lemma 1.1. *Let f be real valued function and $k \in \mathbb{N}$, we have*

$$\Lambda^k f(x) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \lambda^l f(x+l).$$

In particular,

$$\Lambda^k f(0) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \lambda^l f(l). \tag{1.13}$$

Let $P_n = \{q(x) \in \mathbb{Q}[x] \mid \deg q(x) \leq n\}$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}$. Likely, $\{1, x, \cdots, x^n\}$ is the most natural basis for P_n.

Additionally, $\{B_0^{(k)}(x \mid \lambda), B_1^{(k)}(x \mid \lambda), \cdots, B_n^{(k)}(x \mid \lambda)\}$ is also a good basis for the space P_n for our objective of arithmetical applications of Apostol-Bernoulli polynomials of higher order.

If $q(x) \in P_n$, then $q(x)$ can be written as

$$q(x) = \sum_{j=0}^{n} b_j B_j^{(k)}(x \mid \lambda). \tag{1.14}$$

Recently, many mathematicians have studied on the applications of *polynomials* and *q-polynomials* for their finite evaluation schemes, closure under addition, multiplication, differentiation, integration and composition and they are also richly utilized in construction of their generating functions for finding many identities and formulas, (see [1]–[22]).

In this paper, we discover methods for determining b_j from the expression of $q(x)$ in (1.14) and apply those results to arithmetically and combinatorially interesting identities involving $B_0^{(k)}(x \mid \lambda)$, $B_1^{(k)}(x \mid \lambda)$, $\ldots$, $B_n^{(k)}(x \mid \lambda)$.

2. Identities on the Apostol-Bernoulli polynomials of higher order

By (1.11) and (1.12), we see that

$$\Lambda B_n^{(k)}(x \mid \lambda) = \lambda B_n^{(k)}(x+1 \mid \lambda) - B_n^{(k)}(x \mid \lambda) = n B_{n-1}^{(k-1)}(x \mid \lambda), \tag{2.1}$$

and

$$D B_n^{(k)}(x \mid \lambda) = n B_{n-1}^{(k)}(x \mid \lambda). \tag{2.2}$$

Let us assume that $q(x) \in P_n$. Then $q(x)$ can be generated by means of $B_0^{(k)}(x \mid \lambda)$, $B_1^{(k)}(x \mid \lambda)$,$\ldots$, $B_n^{(k)}(x \mid \lambda)$ as follows:

$$q(x) = \sum_{j=0}^{n} b_j B_j^{(k)}(x \mid \lambda). \tag{2.3}$$

Thus, by (2.3) we get

$$\Lambda q(x) = \sum_{j=0}^{n} b_j \Lambda B_j^{(k)}(x \mid \lambda) = \sum_{j=1}^{n} b_l j B_{j-1}^{(k-1)}(x \mid \lambda)$$

and

$$\Lambda^2 q(x) = \Lambda [\Lambda q(x)] = \sum_{j=2}^{n} b_j j (j-1) B_{j-2}^{(k-2)}(x \mid \lambda).$$

By continuing this way, we have

$$\Lambda^k q(x) = \sum_{j=k}^{n} b_j j (j-1) \cdots (j-k+1) B_{j-k}^{(0)}(x \mid \lambda). \tag{2.4}$$

By (1.9) and (2.4), we see that

$$D^s \Lambda^k q(x) = \sum_{j=k+s}^{n} b_j \frac{j!}{(j-k-s)!} x^{j-k-s}. \tag{2.5}$$

Let us take $x = 0$ in (2.5), then we derive the following:

$$\frac{1}{(k+s)!} D^s \Lambda^k q(0) = b_{k+s}. \tag{2.6}$$

From (1.13) and (2.6), we have

$$\begin{aligned} b_{k+s} &= \frac{1}{(k+s)!} D^s \Lambda^k q(0) = \frac{1}{(k+s)!} \Lambda^k D^s q(0) \\ &= \frac{1}{(k+s)!} \sum_{a=0}^{k} (-1)^a \binom{k}{a} \lambda^a D^s q(a). \end{aligned} \tag{2.7}$$

Therefore, by (2.3) and (2.7), we have the following theorem.

Theorem 2.1. *For $k \in \mathbb{Z}_+$ and $q(x) \in P_n$, we have*

$$q(x) = \sum_{j=k}^{n} \left(\frac{1}{j!} \sum_{a=0}^{k} (-1)^a \binom{k}{a} \lambda^a D^{j-k} q(a) \right) B_j^{(k)}(x \mid \lambda).$$

Let us take $q(x) = x^n \in P_n$. Then we derive that $D^{j-k} x^n = \frac{n!}{(n-j+k)!} x^{n-j+k}$.

Thus, by Theorem 2.1, we get

$$x^n = \sum_{j=k}^{n} \left(\frac{1}{j!} \sum_{a=0}^{k} (-1)^a \binom{k}{a} \lambda^a \frac{n!}{(n-j+k)!} a^{n-j+k} \right) B_j^{(k)}(x \mid \lambda). \tag{2.8}$$

Therefore, by (2.8), we arrive at the following corollary.

Corollary 2.2. *For $k, n \in \mathbb{Z}_+$, we have*

$$x^n = \sum_{j=k}^{n} \left(\frac{1}{j!} \sum_{a=0}^{k} (-1)^a \binom{k}{a} \lambda^a \frac{n!}{(n-j+k)!} a^{n-j+k} \right) B_j^{(k)}(x \mid \lambda).$$

Let $q(x) = E_n^{(k)}(x) \in P_n$. Also, it is well known in [11] that

$$D^{j-k} E_n^{(k)}(x) = \frac{n!}{(n-j+k)!} E_{n-j+k}^{(k)}(x). \tag{2.9}$$

By Theorem 2.1 and (2.9), we get the following theorem.

Theorem 2.3. *For $k, n \in \mathbb{Z}_+$, we have*

$$E_n^{(k)}(x) = \sum_{j=k}^{n} \sum_{a=0}^{k} \sum_{l=0}^{n-j+k} \frac{\binom{k}{a}\binom{n-j+k}{l} a^l (-\lambda)^a \, n!}{j! \, (n-j+k)!} E_{n-j+k-l}^{(k)} B_j^{(k)}(x \mid \lambda).$$

Let us consider $q(x) = B_n^{(k)}(x) \in P_n$. Then we see that

$$D^{j-k} B_n^{(k)}(x) = \frac{n!}{(n-j+k)!} B_{n-j+k}^{(k)}(x). \tag{2.10}$$

Thanks to Theorem 2.1 and (2.10), we obtain the following theorem.

Theorem 2.4. *For $k, n \in \mathbb{Z}_+$, we have*

$$B_n^{(k)}(x) = \sum_{j=k}^{n} \sum_{a=0}^{k} \sum_{l=0}^{n-j+k} \frac{(-\lambda)^a \binom{k}{a} \binom{n-j+k}{l} a^l n!}{j!\,(n-j+k)!} B_{n-j+k-l}^{(k)} B_j^{(k)}(x \mid \lambda).$$

Hansen [7] derived the following convolution formula:

$$\sum_{k=0}^{m} \binom{m}{k} B_k(x) B_{m-k}(y) = (1-m) B_m(x+y) + (x+y-1) m B_{m-1}(x+y). \tag{2.11}$$

We note that the special case $x = y = 0$ of the last identity

$$B_m = -\frac{\sum_{k=2}^{m-2} \binom{m}{k} B_k B_{m-k}}{m+1}$$

is originally constructed by Euler and Ramanujan (cf. [5]).

Let us now write the following

$$q(x) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y) \in P_n. \tag{2.12}$$

By using derivative operator D^s in the both sides of (2.11), we derive

$$\begin{aligned} D^{j-k} q(x) = (1-n) \frac{n!}{(n-j+k)!} B_{n-j+k}(x+y) + (x+y-1) \frac{n!}{(n-j+k-1)!} B_{n-j+k-1}(x+y) \\ + (j-k) \frac{n!}{(n-j+k)!} B_{n-j+k}(x+y) \end{aligned} \tag{2.13}$$

By Theorem 2.1, (2.12) and (2.13), we arrive at the following theorem.

Theorem 2.5. *For $k, n \in \mathbb{Z}_+$, we have*

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y) = \sum_{j=k}^{n} \frac{1}{j!} \sum_{a=0}^{k} (-1)^a \left(\begin{array}{c} k \\ a \end{array} \right) \lambda^a \{ (1-n) \frac{n!}{(n-j+k)!} B_{n-j+k}(a+y) \\ + (a+y-1) \frac{n!}{(n-j+k-1)!} B_{n-j+k-1}(a+y) \\ + (j-k) \frac{n!}{(n-j+k)!} B_{n-j+k}(a+y) \} B_j^{(k)}(x \mid \lambda). \end{aligned}$$

Dilcher [5] introduced the following interesting identity:

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x) E_{n-k}(y) = 2(1-x-y) E_n(x+y) + 2E_{n+1}(x+y).$$

Let $\sum_{k=0}^{n} \binom{n}{k} E_k(x) E_{n-k}(y) \in P_n$, then we write that

$$q(x) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) E_{n-k}(y). \tag{2.14}$$

By (2.14), we have

$$\begin{aligned} D^{j-k} q(x) = 2\{ \frac{n!}{(n-j+k)!} (1-x-y) E_{n-j+k}(x+y) - (j-k) \frac{n!}{(n-j+k+1)!} E_{n-j+k+1}(x+y) \\ + \frac{(n+1)!}{(n+1-j+k)!} E_{n+1-j+k}(x+y) \}. \end{aligned}$$

As a result of the last identity and Theorem 2.1, we derive the following.

Theorem 2.6. *The following equality holds:*

$$\sum_{k=0}^{n}\binom{n}{k}E_k(x)\,E_{n-k}(y)=2\sum_{j=k}^{n}\frac{1}{j!}\sum_{a=0}^{k}(-1)^a\binom{k}{a}\lambda^a\{\frac{n!}{(n-j+k)!}(1-x-y)\,E_{n-j+k}(x+y)$$
$$-(j-k)\frac{n!}{(n-j+k+1)!}E_{n-j+k+1}(x+y)$$
$$+\frac{(n+1)!}{(n+1-j+k)!}E_{n+1-j+k}(x+y)\}B_j^{(k)}(x\mid\lambda).$$

Remark 2.7. Throughout this paper when we take $\lambda = 1$, our results can easily be related to Bernoulli polynomials of higher order.

Remark 2.8. Theorem 2.1 seems to be plenty large enough for obtaining interesting identities related to special functions in connection with Apostol-Bernoulli polynomials of higher order.

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