Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2705–2717
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Application of single step with three generalized hybrid points block method for solving third order ordinary differential equations

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Communicated by I. K. Argyros

Abstract

This article considers the implementation of one step hybrid block method, three generalized hybrid points developed in collocation interpolation approach. The basic numerical properties of the hybrid block method was established and found to be convergent. The efficiency of the new method was confirmed on some initial value problems and found to give better approximation than the existing methods in term of error.

Keywords: Hybrid method, block method, third order differential equation, single step, three generalized off step points.
2010 MSC: 65L05, 65L06, 65L20.

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1. Introduction and Preliminaries

In this paper, power series of order 7 of the form

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left(\frac{x - x_n}{h}\right)^i. \quad x \in [x_n, x_{n+1}] \tag{1.1}$$

is proposed as an approximation solution to general third order initial value problem (IVP)

$$y''' = f(x, y, y', y''),\ y(a) = \eta_0,\ y'(a) = \eta_1,\ y''(a) = \eta_2,\ x \in [a, b], \tag{1.2}$$

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Received 2015-11-11

where $n = 0, 1, 2, ..., N-1$, $v = 3$ denotes of the number of interpolation points, $m = 5$ represents the number of collocation points, $h = x_n - x_{n-1}$ is constant step size of partition in the interval $[a, b]$ which is given by $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$.

Conventionally, Equation (1.2). is reduced to systems of first order (IVP) and then, suitable methods for first order equations are adopted to solve them.

Reduction approach has been identified to have some drawbacks which include computer burden and alots of human efforts [1, 7, 10] and [14]. Because of this, many researchers have attempted to solve Equation(1) directly . Worthy of note are those of [4, 5, 6, 12] and [13].

Direct method was implemented in different ways such as predictor-corrector method, block linear multistep method and hybrid block method.

Hybrid block method was introduced to combine the advantages of block method and overcoming the zero stability barrier in linear multistep method [2, 3]. This barrier implies that the highest order of zero stability for linear multistep method of step length k is k+2 when k is even and k+1 when k is odd [9].
In this work, efforts are made to develop one step hybrid block method with three generalized hybrid points for solving (1.1) directly.

2. Derivation of the Method

interpolating Equation (1.1) at points x_n, x_{n+s_1}, x_{n+s_2} and collocating its third derivative at all points i.e x_n, x_{n+s_1}, x_{n+s_2} and x_{n+s_2} gives system of equations below

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & s_1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 & s_1^6 & s_1^7 \\ 1 & s_2 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & s_2^6 & s_2^7 \\ 0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24s_1}{h^3} & \frac{60s_1^2}{h^3} & \frac{120s_1^3}{h^3} & \frac{210s_1^4}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24s_2}{h^3} & \frac{60s_2^2}{h^3} & \frac{120s_2^3}{h^3} & \frac{120s_2^4}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24s_3}{h^3} & \frac{60s_3^2}{h^3} & \frac{120s_3^3}{h^3} & \frac{120s_3^4}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} & \frac{210}{h^3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+s_1} \\ y_{n+s_2} \\ f_n \\ f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}. \tag{2.1}$$

Using Gaussian elimination method in (2.1) to find the values of $a_i\, s$, $i = 0(1)8$ and $a_i s$ are then substituted back into equation (1.1). This gives a continuous linear multistep method of the form:

$$y(x) = \alpha_0 y_n + \sum_{i=1}^{2} \alpha_{s_i} y_{n+s_i} + \sum_{i=0}^{1} \beta_i f_{n+i} + \sum_{i=1}^{3} \beta_{s_i} f_{n+s_i}\ . \tag{2.2}$$

The first and second derivatives of equation (2.2) are given by

$$y'(x) = \frac{\partial}{\partial x}\alpha_0 y_n + \sum_{i=0}^{2} \frac{\partial}{\partial x}\alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{\partial}{\partial x}\beta_i(x) f_{n+i} + \sum_{i=1}^{3} \frac{\partial}{\partial x}\beta_{s_i}(x) f_{n+s_i}\ , \tag{2.3}$$

$$y''(x) = \frac{\partial^2}{\partial x^2}\alpha_0 y_n + \sum_{i=1}^{2} \frac{\partial^2}{\partial x^2}\alpha_{s_i}(x) y_{n+s_i} + \sum_{i=0}^{1} \frac{\partial^2}{\partial x^2}\beta_{s_i}(x) f_{n+i} + \sum_{i=1}^{3} \frac{\partial^2}{\partial x^2}\beta_{s_i}(x) f_{n+s_i}\ , \tag{2.4}$$

where

$$\alpha_0 = \frac{(x_n - x + hs_2)(x_n - x + hs_1)}{(h^2 s_1 s_2)}\ ,$$

$$\alpha_{s_1} = \frac{((x - x_n)(x - x_n - hs_2))}{(h^2 s_1 (s_1 - s_2))}\ ,$$

$$\alpha_{s_2} = \frac{(x-x_n)(x_n-x+hs_1)}{(h^2s_2(s_1-s_2))}\ ,$$

$$\begin{aligned}\beta_0 = &-\frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(840h^4s_1s_2s_3)}(7hx^3+3h^3s_1^3x-24x^2x_n^2+7hs_3x^3\\
&+16x^3x_n-4x^4-4x_n^4-7h^4s_1^3+3h^4s_1^4-7h^4s_2^3+3h^4s_2^4+3hs_1x^3+3hs_2x^3+16xx_n^3\\
&-3hs_1x_n^3-3hs_2x_n^3-7hs_3x_n^3+21hxx_n^2-21hx^2x_n+14h^4s_1s_2^2+14h^4s_1^2s_2-4h^4s_1s_2^3\\
&-4h^4s_1^3s_2+21h^4s_1^2s_3-7h^4s_1^3s_3+21h^4s_2^2s_3-7h^4s_2^3s_3-,\ 7h^2s_1x^2-7h^3s_1^2x\\
&-7hx_n^3-7h^2s_2x^2-7h^3s_2^2x+3h^3s_2^3x-14h^2s_3x^2-7h^2s_1x_n^2+7h^3s_1^2x_n-3h^3s_1^3x_n\\
&+7h^3s_2^2x_n-3h^3s_2^3x_n-14h^2s_3x_n^2,-4h^4s_1^2s_2^2+3h^2s_1^2x^2-14h^3s_1s_2x_n-21h^3s_1s_3x_n\\
&+3h^2s_2^2x^2+3h^2s_1^2x_n^2+3h^2s_2^2x_n^2-84h^4s_1s_2s_3+14h^3s_1s_2x+21h^3s_1s_3x+9hs_1xx_n^2\\
&+21h^3s_2s_3x-21h^3s_2s_3x_n+14h^2s_1xx_n+9hs_2xx_n^2-9hs_2x^2x_n+14h^2s_2xx_n\\
&+21hs_3xx_n^2+28h^2s_3xx_n+14h^4s_1s_2^2s_3+14h^4s_1^2s_2s_3-4h^2s_1s_2x^2-4h^3s_1s_2^2x\\
&-4h^3s_1^2s_2x\ ,-7h^2s_1s_3x^2-7h^3s_1^2s_3x-7h^2s_2s_3x^2-4h^2s_1s_2x_n^2+4h^3s_1s_2^2x_n\\
&+4h^3s_1^2s_2x_n-7h^2s_1s_3x_n^2+7h^3s_1^2s_3x_n-7h^2s_2s_3x_n^2+7h^3s_2^2s_3x_n+14h^2s_2s_3xx_n\\
&-6h^2s_2^2xx_n+14h^3s_1s_2s_3x-14h^3s_1s_2s_3x_n+8h^2s_1s_2xx_n+14h^2s_1s_3xx_n-6h^2s_1^2xx_n\\
&-9hs_1x^2x_n-21hs_3x^2x_n-7h^3s_2^2s_3x-7h^2s_2x_n^2)\ ,\end{aligned}$$

$$\begin{aligned}\beta_{s_1} = &\frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(840h^4s_1(s_1-1)(s_1-s_3)(s_1-s_2))}(24x^2x_n^2-14h^2s_2xx_n+4h^3s_1^3x+4h^4s_1^4\\
&+6h^2s_2^2xx_n-3h^3s_1s_2^2x-16x^3x_n+4x_n^4+7h^4s_2^3-3h^4s_2^4+4hs_1x^3-14h^3s_1s_3x_n\\
&-4hs_1x_n^3+3hs_2x_n^3+7hs_3x_n^3-21hxx_n^2+21hx^2x_n+7h^4s_1s_2^2+7h^4s_1^2s_2+3h^3s_1s_2^2x_n\\
&-3h^4s_1^3s_2+14h^4s_1^2s_3-7h^4s_1^3s_3-21h^4s_2^2s_3+7h^4s_2^3s_3-7h^2s_1x^2-7h^3s_1^2x-16xx_n^3\\
&+7h^2s_2x^2+7h^3s_2^2x-3h^3s_2^3x+14h^2s_3x^2-7h^2s_1x_n^2+7h^3s_1^2x_n-4h^3s_1^3x_n+7h^2s_2x_n^2\\
&-7h^3s_2^2x_n+3h^3s_2^3x_n+14h^2s_3x_n^2-3h^4s_1^2s_2^2+4h^2s_1^2x^2-3h^2s_2^2x^2+4h^2s_1^2x_n^2+7hx_n^3\\
&-21h^4s_1s_2s_3+7h^3s_1s_2x+14h^3s_1s_3x-21h^3s_2s_3x-7h^3s_1s_2x_n-3hs_2x^3-3h^2s_2^2x_n^2\\
&+21h^3s_2s_3x_n+12hs_1xx_n^2-12hs_1x^2x_n+14h^2s_1xx_n-9hs_2xx_n^2+9hs_2x^2x_n-7hs_3x^3\\
&-21hs_3xx_n^2+21hs_3x^2x_n-28h^2s_3xx_n+7h^4s_1s_2^2s_3+7h^4s_1^2s_2s_3-3h^2s_1s_2x^2-7hx^3\\
&-3h^3s_1^2s_2x-7h^2s_1s_3x^2-7h^3s_1^2s_3x+7h^2s_2s_3x^2+7h^3s_2^2s_3x-3h^2s_1s_2x_n^2-3h^4s_1s_2^3\\
&+3h^3s_1^2s_2x_n-7h^2s_1s_3x_n^2+7h^3s_1^2s_3x_n+7h^2s_2s_3x_n^2-7h^3s_2^2s_3x_n-8h^2s_1^2xx_n-7h^4s_1^3\\
&+7h^3s_1s_2s_3x-7h^3s_1s_2s_3x_n+6h^2s_1s_2xx_n+14h^2s_1s_3xx_n-14h^2s_2s_3xx_n+4x^4)\ ,\end{aligned}$$

$$\begin{aligned}\beta_{s_2} = &\frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(840h^4s_2(s_2-1)(s_2-s_3)(s_1-s_2))}(7hx^3-24x^2x_n^2+16xx_n^3+3h^3s_1^3x-4x^4\\
&+16x^3x_n-4x_n^4-7h^4s_1^3+3h^4s_1^4+7h^4s_2^3-4h^4s_2^4+3hs_1x^3-4hs_2x^3+21h^4s_1s_2s_3\\
&-3hs_1x_n^3+4hs_2x_n^3-7hs_3x_n^3+21hxx_n^2-21hx^2x_n-7h^4s_1s_2^2-7h^4s_1^2s_2+3h^4s_1s_2^3\\
&+3h^4s_1^3s_2+21h^4s_1^2s_3-7h^4s_1^3s_3-14h^4s_2^2s_3+7h^4s_2^3s_3-7h^2s_1x^2-7h^3s_1^2x-7hx_n^3\\
&+7h^3s_2^2x-4h^3s_2^3x-14h^2s_3x^2-7h^2s_1x_n^2+7h^3s_1^2x_n-3h^3s_1^3x_n+7h^2s_2x_n^2-7h^3s_2^2x_n\\
&+4h^3s_2^3x_n-14h^2s_3x_n^2+3h^4s_1^2s_2^2+3h^2s_1^2x^2-4h^2s_2^2x^2+3h^2s_1^2x_n^2-7h^3s_2^2s_3x_n\\
&+28h^2s_3xx_n+21h^3s_1s_3x-14h^3s_2s_3x+7h^3s_1s_2x_n-21h^3s_1s_3x_n+14h^3s_2s_3x_n\\
&+14h^2s_1xx_n-12hs_2xx_n^2+12hs_2x^2x_n-14h^2s_2xx_n+21hs_3xx_n^2-21hs_3x^2x_n\end{aligned}$$

$$- 7h^4s_1^2s_2s_3 + 3h^2s_1s_2x^2 + 3h^3s_1s_2^2x + 3h^3s_1^2s_2x - 7h^2s_1s_3x^2 - 7h^3s_1^2s_3x + 7hs_3x^3 + 3h^2s_1s_2x_n^2 - 3h^3s_1s_2^2x_n - 3h^3s_1^2s_2x_n - 7h^2s_1s_3x_n^2 + 7h^3s_1^2s_3x_n + 7h^2s_2s_3x_n^2 + 8h^2s_2^2xx_n - 7h^3s_1s_2s_3x + 7h^3s_1s_2s_3x_n - 6h^2s_1s_2xx_n + 14h^2s_1s_3xx_n - 14h^2s_2s_3xx_n - 6h^2s_1^2xx_n + 7h^3s_2^2s_3x - 7h^4s_1s_2^2s_3 - 9hs_1x^2x_n + 7h^2s_2x^2 + 9hs_1xx_n^2 + 7h^2s_2s_3x^2 - 4h^2s_2^2x_n^2 - 7h^3s_1s_2x) ,$$

$$\beta_{s_3} = -\frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(840h^4s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(3h^4s_1^4 - 4h^4s_1^3s_2 - 7h^4s_1^3 - 4h^4s_1^2s_2^2 + 14h^4s_1^2s_2 - 4h^4s_1s_2^3 + 14h^4s_1s_2^2 + 3h^4s_2^4 - 7h^4s_2^3 + 3h^3s_1^3x - 3h^3s_1^3x_n - 4h^3s_1^2s_2x + 4h^3s_1^2s_2x_n - 7h^3s_1^2x + 7h^3s_1^2x_n - 4h^3s_1s_2^2x + 4h^3s_1s_2^2x_n + 14h^3s_1s_2x - 14h^3s_1s_2x_n - 3h^3s_2^3x_n - 7h^3s_2^2x + 7h^3s_2^2x_n + 3h^2s_1^2x^2 - 6h^2s_1^2xx_n + 3h^2s_1^2x_n^2 - 4h^2s_1s_2x^2 - 4x_n^4 - 4h^2s_1s_2x_n^2 - 7h^2s_1x^2 + 14h^2s_1xx_n - 7h^2s_1x_n^2 + 3h^2s_2^2x^2 - 6h^2s_2^2xx_n + 3h^2s_2^2x_n^2 - 7h^2s_2x^2 + 14h^2s_2xx_n - 7h^2s_2x_n^2 + 3hs_1x^3 + 9hs_1xx_n^2 - 3hs_1x_n^3 + 3hs_2x^3 + 16xx_n^3 - 9hs_2x^2x_n + 9hs_2xx_n^2 - 3hs_2x_n^3 + 7hx^3 - 21hx^2x_n + 21hxx_n^2 - 7hx_n^3 - 4x^4 + 16x^3x_n + 3h^3s_2^3x - 24x^2x_n^2 - 9hs_1x^2x_n + 8h^2s_1s_2xx_n) ,$$

$$\beta_1 = \frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(840h^4(s_3-1)(s_2-1)(s_1-1))}(3h^4s_1^4 - 4h^4s_1^3s_2 - 7s_3h^4s_1^3 - 4h^4s_1^2s_2^2 - 4x^4 + 14s_3h^4s_1^2s_2 - 4h^4s_1s_2^3 + 14s_3h^4s_1s_2^2 + 3h^4s_2^4 - 7s_3h^4s_2^3 + 3h^3s_1^3x + 16x^3x_n - 4h^3s_1^2s_2x + 4h^3s_1^2s_2x_n - 7s_3h^3s_1^2x + 7s_3h^3s_1^2x_n - 4h^3s_1s_2^2x + 4h^3s_1s_2^2x_n + 3hs_2x^3 + 14s_3h^3s_1s_2x - 14s_3h^3s_1s_2x_n + 3h^3s_2^3x - 3h^3s_2^3x_n - 7s_3h^3s_2^2x + 7s_3h^3s_2^2x_n - 4x_n^4 + 3h^2s_1^2x^2 - 6h^2s_1^2xx_n + 3h^2s_1^2x_n^2 - 4h^2s_1s_2x^2 + 8h^2s_1s_2xx_n - 4h^2s_1s_2x_n^2 + 16xx_n^3 - 7s_3h^2s_1x^2 + 14s_3h^2s_1xx_n - 7s_3h^2s_1x_n^2 + 3h^2s_2^2x^2 - 6h^2s_2^2xx_n - 3hs_1x_n^3 - 3hs_2x_n^3 - 7s_3h^2s_2x^2 + 14s_3h^2s_2xx_n - 7s_3h^2s_2x_n^2 + 3hs_1x^3 - 9hs_1x^2x_n + 9hs_1xx_n^2 - 7s_3hx_n^3 - 9hs_2x^2x_n + 9hs_2xx_n^2 + 3h^2s_2^2x_n^2 + 7s_3hx^3 - 21s_3hx^2x_n + 21s_3hxx_n^2 - 24x^2x_n^2 - 3h^3s_1^3x_n) .$$

Equation (2.2) is evaluated at the non-interpolating point i.e x_{n+s_3}, x_{n+1} and equations (2.3) and (2.4) are evaluated at all points to produce the discrete schemes and its derivatives. The discrete scheme and its derivatives at x_n are combined in matrix to form a block

$$A^{[3]_3}Y_m^{[3]_3} = B_1^{[3]_3}R_1^{[3]_3} + B_2^{[3]_3}R_2^{[3]_3} + B_3^{[3]_3}R_3^{[3]_3} + h^3D^{[3]_3}R_4^{[3]_3} + h^3E^{[3]_3}R_5^{[3]_3} , \tag{2.5}$$

where

$$A^{[3]_3} = \begin{pmatrix} \frac{(s_2-1)}{(s_1(s_1-s_2))} & -\frac{(s_1-1)}{(s_2(s_1-s_2))} & 0 & 1 \\ \frac{(s_3(s_2-s_3))}{(s_1(s_1-s_2))} & -\frac{(s_3(s_1-s_3))}{(s_2(s_1-s_2))} & 1 & 0 \\ \frac{s_2}{(hs_1(s_1-s_2))} & -\frac{s_1}{(hs_2(s_1-s_2))} & 0 & 0 \\ -\frac{2}{(h^2s_1^2-h^2s_1s_2} & -\frac{2}{(h^2s_2^2-h^2s_1s_2)} & 0 & 0 \end{pmatrix}, \quad Y_m^{[3]_3} = \begin{pmatrix} y_{n+s_1} \\ y_{n+s_2} \\ y_{n+s_3} \\ y_{n+1} \end{pmatrix},$$

$$B_1^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & \frac{((s_1-1)(s_2-1))}{(s_1s_2)} \\ 0 & 0 & 0 & \frac{((s_1-s_3)(s_2-s_3))}{(s_1s_2)} \\ 0 & 0 & 0 & -\frac{(s_1+s_2)}{(hs_1s_2)} \\ 0 & 0 & 0 & \frac{2}{(h^2s_1s_2)} \end{pmatrix}, \ R_1^{[3]_3} \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}, B_2^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$R_2^{[3]_3} = \begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}, \; B_3^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, R_3^{[3]_3} = \begin{pmatrix} y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix}, \; D^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & D_{14}^{[3]_3} \\ 0 & 0 & 0 & D_{24}^{[3]_3} \\ 0 & 0 & 0 & D_{34}^{[3]_3} \\ 0 & 0 & 0 & D_{44}^{[3]_3} \end{pmatrix},$$

$$R_4^{[3]_3} = \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}, \; E^{[3]_3} = \begin{pmatrix} E_{11}^{[3]_3} & E_{12}^{[3]_3} & E_{13}^{[3]_3} & E_{14}^{[3]_3} \\ E_{21}^{[3]_3} & E_{21}^{[3]_3} & E_{23}^{[3]_3} & E_{24}^{[3]_3} \\ E_{31}^{[3]_3} & E_{32}^{[3]_3} & E_{33}^{[3]_3} & E_{34}^{[3]_3} \\ E_{41}^{[3]_3} & E_{42}^{[3]_3} & E_{43}^{[3]_3} & E_{44}^{[3]_3} \end{pmatrix}, \; R_6^{[3]_3} = \begin{pmatrix} f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}.$$

Non-zero elements in $D^{[3]_3}$ and $E^{[3]_3}$ are given by

$$\begin{aligned} D_{14}^{[3]_3} =& -\frac{(s_2-1)(s_1-1)}{(840s_1s_2s_3)}(10s_1s_2 - 4s_2 - 7s_3 - 4s_1^2s_2^2 - 4s_1 + 14s_1s_3 + 14s_2s_3 - 4s_1^3 \\ &+ 10s_1s_2^2 + 10s_1^2s_2 - 4s_1s_2^3 - 4s_1^3s_2 + 14s_1^2s_3 - 7s_1^3s_3 + 14s_2^2s_3 - 7s_2^3s_3 - 4s_1^2 \\ &+ 3s_1^4 - 4s_2^2 - 4s_2^3 + 3s_2^4 + 14s_1s_2^2s_3 + 14s_1^2s_2s_3 - 70s_1s_2s_3 + 3) \ , \end{aligned}$$

$$\begin{aligned} D_{24}^{[3]_3} =& -\frac{(s_1-s_3)(s_2-s_3)}{(840s_1s_2)}(3s_1^4 - 4s_1^3s_2 - 4s_1^3s_3 - 7s_1^3 - 4s_1^2s_2^2 + 10s_1^2s_2s_3 \\ &+ 14s_1^2s_2 - 4s_1^2s_3^2 + 14s_1^2s_3 - 4s_1s_2^3 + 10s_1s_2^2s_3 + 14s_1s_2^2 + 10s_1s_2s_3^2 \\ &- 70s_1s_2s_3 - 4s_1s_3^3 + 14s_1s_3^2 + 3s_2^4 - 4s_2^3s_3 - 7s_2^3 - 4s_2^2s_3^2 + 14s_2^2s_3 \\ &- 4s_2s_3^3 + 14s_2s_3^2 + 3s_3^4 - 7s_3^3) \ , \end{aligned}$$

$$\begin{aligned} D_{34}^{[3]_3} =& \frac{-1}{(840s_3h^2)}(14s_1s_2^2 - 4s_1^2s_2^2 + 14s_1^2s_2 - 4s_1s_2^3 - 4s_1^3s_2 + 21s_1^2s_3 - 7s_1^3s_3 + 3s_2^4 \\ &+ 21s_2^2s_3 - 7s_2^3s_3 - 7s_1^3 + 3s_1^4 - 7s_2^3 + 14s_1s_2^2s_3 + 14s_1^2s_2s_3 - 84s_1s_2s_3) \ , \end{aligned}$$

$$\begin{aligned} D_{44}^{[3]_3} =& \frac{1}{(420h^3s_1s_2s_3)}(14s_1^2s_2^2 - 4s_1^2s_2^3 - 4s_1^3s_2^2 + 14s_1s_2^3 + 14s_1^3s_2 - 4s_1s_2^4 - 4s_1^4s_2 \\ &+ 21s_1^3s_3 - 7s_1^4s_3 + 21s_2^3s_3 - 7s_2^4s_3 - 7s_1^4 + 3s_1^5 - 7s_2^4 + 3s_2^5 - 84s_1s_2^2s_3 \\ &- 84s_1^2s_2s_3 + 14s_1s_2^3s_3 + 14s_1^3s_2s_3 + 14s_1^2s_2^2s_3) \ , \end{aligned}$$

$$\begin{aligned} E_{11}^{[3]_3} =& \frac{(s_2-1)}{840s_1(s_1-s_2)(s_1-s_3)}(4s_2 - 3s_1 + 7s_3 - 3s_1^2s_2^2 + 4s_1s_2 + 7s_1s_3 - 3s_1^2 \\ &- 14s_2s_3 + 4s_1s_2^2 + 4s_1^2s_2 - 3s_1s_2^3 - 3s_1^3s_2 + 7s_1^2s_3 - 7s_1^3s_3 - 14s_2^2s_3 + 4s_1^4 \\ &+ 7s_2^3s_3 - 3s_1^3 + 4s_2^2 + 4s_2^3 - 3s_2^4 + 7s_1s_2^2s_3 + 7s_1^2s_2s_3 - 14s_1s_2s_3 - 3) \ , \end{aligned}$$

$$\begin{aligned} E_{12}^{[3]_3} =& -\frac{(s_1-1)}{840s_2(s_1-s_2)(s_2-s_3)}(4s_1 - 3s_2 + 7s_3 - 3s_1^2s_2^2 + 4s_1s_2 - 14s_1s_3 - 3s_2^2 \\ &+ 7s_2s_3 + 4s_1s_2^2 + 4s_1^2s_2 - 3s_1s_2^3 - 3s_1^3s_2 - 14s_1^2s_3 + 7s_1^3s_3 + 7s_2^2s_3 - 3s_2^3 \\ &- 7s_2^3s_3 + 4s_1^2 + 4s_1^3 - 3s_1^4 + 4s_2^4 + 7s_1s_2^2s_3 + 7s_1^2s_2s_3 - 14s_1s_2s_3 - 3) \ , \end{aligned}$$

$$\begin{aligned} E_{13}^{[3]_3} =& -\frac{(s_2-1)(s_1-1)}{(840s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(3s_1^4 - 4s_1^3s_2 - 4s_1^3 - 4s_1^2s_2^2 + 10s_1^2s_2 \\ &- 4s_1^2 - 4s_1s_2^3 + 10s_1s_2^2 + 10s_1s_2 - 4s_1 + 3s_2^4 - 4s_2^3 - 4s_2^2 - 4s_2 + 3) \ , \end{aligned}$$

$$E_{14}^{[3]_3} = -\frac{1}{(840s_3 - 840)}(4s_1^2s_2^2 - 3s_2 - 7s_3 - 3s_1 + 4s_1s_2 + 7s_1s_3 + 7s_2s_3 + 4s_1s_2^2 + 4s_1^2s_2 + 4s_1s_2^3 + 4s_1^3s_2 + 7s_1^2s_3 + 7s_1^3s_3 + 7s_2^2s_3 + 7s_2^3s_3 - 3s_1^2 - 3s_1^3 - 3s_1^4 - 3s_2^2 - 3s_2^3 - 3s_2^4 - 14s_1s_2^2s_3 - 14s_1^2s_2s_3 - 14s_1s_2s_3 + 4) ,$$

$$E_{21}^{[3]_3} = \frac{s_3(s_2 - s_3)}{(840s_1(s_1 - s_2)(s_1 - 1))}(4s_1^4 - 3s_1^3s_2 - 3s_1^3s_3 - 7s_1^3 - 3s_1^2s_2^2 + 4s_1^2s_2s_3 - 3s_3^4 + 7s_1^2s_2 - 3s_1^2s_3^2 + 7s_1^2s_3 - 3s_1s_2^3 + 4s_1s_2^2s_3 + 7s_1s_2^2 + 4s_1s_2s_3^2 - 14s_1s_2s_3 + 7s_3^3 - 3s_1s_3^3 + 7s_1s_3^2 - 3s_2^4 + 4s_2^3s_3 + 7s_2^3 + 4s_2^2s_3^2 - 14s_2^2s_3 + 4s_2s_3^3 - 14s_2s_3^2) ,$$

$$E_{22}^{[3]_3} = \frac{-s_3(s_1 - s_3)}{840s_2(s_1 - s_2)(s_2 - 1)}(-3s_1^4 - 3s_1^3s_2 + 4s_1^3s_3 + 7s_1^3 - 3s_1^2s_2^2 + 4s_1^2s_2s_3 + 4s_2^4 + 7s_1^2s_2 + 4s_1^2s_3^2 - 14s_1^2s_3 - 3s_1s_2^3 + 4s_1s_2^2s_3 + 7s_1s_2^2 + 4s_1s_2s_3^2 - 14s_1s_2s_3 - 3s_3^4 + 4s_1s_3^3 - 14s_1s_3^2 - 3s_2^3s_3 - 7s_2^3 - 3s_2^2s_3^2 + 7s_2^2s_3 - 3s_2s_3^3 + 7s_2s_3^2 + 7s_3^3) ,$$

$$E_{23}^{[3]_3} = \frac{1}{(840s_3 - 840)}(-3s_1^4 + 4s_1^3s_2 - 3s_1^3s_3 + 7s_1^3 + 4s_1^2s_2^2 + 4s_1^2s_2s_3 - 14s_1^2s_2 - 3s_1^2s_3^2 + 7s_1^2s_3 + 4s_1s_2^3 + 4s_1s_2^2s_3 - 14s_1s_2^2 + 4s_1s_2s_3^2 - 14s_1s_2s_3 - 3s_1s_3^3 + 7s_1s_3^2 - 3s_2^4 - 3s_2^3s_3 + 7s_2^3 - 3s_2^2s_3^2 + 7s_2^2s_3 - 3s_2s_3^3 + 7s_2s_3^2 + 4s_3^4 - 7s_3^3) ,$$

$$E_{24}^{[3]_3} = \frac{s_3(s_2 - s_3)(s_1 - s_3)}{840(s_1 - 1)(s_2 - 1)(s_3 - 1)}(3s_1^4 - 4s_1^3s_2 - 4s_1^3s_3 - 4s_1^2s_2^2 + 10s_1^2s_2s_3 + 3s_2^4 - 4s_1^2s_3^2 - 4s_1s_2^3 + 10s_1s_2^2s_3 + 10s_1s_2s_3^2 - 4s_1s_3^3 - 4s_2^3s_3 - 4s_2^2s_3^2 - 4s_2s_3^3 + 3s_3^4) ,$$

$$E_{31}^{[3]_3} = -\frac{s_2}{h(840s_1 - 840s_2)(s_1 - s_3)(s_1 - 1)}(3s_1^2s_2^2 - 7s_1s_2^2 - 7s_1^2s_2 + 3s_1s_2^3 + 3s_1^3s_2 - 14s_1^2s_3 + 7s_1^3s_3 + 21s_2^2s_3 - 7s_2^3s_3 + 7s_1^3 - 4s_1^4 - 7s_2^3 + 3s_2^4 - 7s_1s_2^2s_3 - 7s_1^2s_2s_3 + 21s_1s_2s_3) ,$$

$$E_{32}^{[3]_3} = \frac{s_1}{(840s_1 - 840s_2)(s_2 - s_3)(s_2 - 1)}(3s_1^2s_2^2 - 7s_1s_2^2 - 7s_1^2s_2 + 3s_1s_2^3 + 3s_1^3s_2 - 7s_1^3 + 21s_1^2s_3 - 7s_1^3s_3 - 14s_2^2s_3 + 7s_2^3s_3 + 3s_1^4 + 7s_2^3 - 4s_2^4 - 7s_1s_2^2s_3 - 7s_1^2s_2s_3 + 21s_1s_2s_3) ,$$

$$E_{33}^{[3]_3} = \frac{s_1s_2}{(840hs_3(s_1 - s_3)(s_2 - s_3)(s_3 - 1))}(-3s_1^4 + 4s_1^3s_2 + 7s_1^3 + 4s_1^2s_2^2 - 14s_1^2s_2 + 4s_1s_2^3 - 14s_1s_2^2 - 3s_2^4 + 7s_2^3) ,$$

$$E_{34}^{[3]_3} = -\frac{s_1s_2}{((840s_1 - 840)(s_2 - 1)(s_3 - 1))}(-3s_1^4 + 4s_1^3s_2 + 7s_3s_1^3 + 4s_1^2s_2^2 - 14s_3s_1^2s_2 + 4s_1s_2^3 - 14s_3s_1s_2^2 - 3s_2^4 + 7s_3s_2^3) ,$$

$$E_{41}^{[3]_3} = -\frac{1}{(420h^2s_1(s_1 - s_2)(s_1 - s_3)(s_1 - 1))}(7s_1^2s_2^2 - 3s_1^2s_2^3 - 3s_1^3s_2^2 + 7s_1s_2^3 - 3s_2^5$$

$$- 3s_1s_2^4 - 3s_1^4s_2 + 14s_1^3s_3 - 7s_1^4s_3 - 21s_2^3s_3 + 7s_2^4s_3 - 7s_1^4 + 4s_1^5 + 7s_2^4 + 7s_1^3s_2 - 21s_1s_2^2s_3 - 21s_1^2s_2s_3 + 7s_1s_2^3s_3 + 7s_1^3s_2s_3 + 7s_1^2s_2^2s_3) \ ,$$

$$E_{42}^{[3]_3} = \frac{1}{(420h^2s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(7s_1^2s_2^2 - 3s_1^2s_2^3 - 3s_1^3s_2^2 + 7s_1s_2^3 + 7s_1^3s_2 - 3s_1s_2^4 - 3s_1^4s_2 - 21s_1^3s_3 + 7s_1^4s_3 + 14s_2^3s_3 - 7s_2^4s_3 + 7s_1^4 - 3s_1^5 - 7s_2^4 + 4s_2^5 - 21s_1s_2^2s_3 - 21s_1^2s_2s_3 + 7s_1s_2^3s_3 + 7s_1^3s_2s_3 + 7s_1^2s_2^2s_3) \ ,$$

$$E_{43}^{[3]_3} = \frac{1}{(420h^2s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(3s_1^5 - 4s_1^4s_2 - 7s_1^4 - 4s_1^3s_2^2 + 14s_1^3s_2 - 4s_1^2s_2^3 + 14s_1^2s_2^2 - 4s_1s_2^4 + 14s_1s_2^3 + 3s_2^5 - 7s_2^4) \ ,$$

$$E_{44}^{[3]_3} = \frac{1}{420h^2((s_1-1)(s_2-1)(s_3-1))}(-3s_1^5 + 4s_1^4s_2 + 7s_3s_1^4 + 4s_1^3s_2^2 - 14s_3s_1^3s_2 + 4s_1^2s_2^3 - 14s_3s_1^2s_2^2 + 4s_1s_2^4 - 14s_3s_1s_2^3 - 3s_2^5 + 7s_3s_2^4) \ .$$

Multiplying Equation (2.5) by $(A^{[3]_3})^{-1}$ produces the following hybrid block method.

$$I^{[3]_3}Y_m^{[3]_3} = \bar{B}_1^{[3]_3}R_1^{[3]_3} + \bar{B}_2^{[3]_3}R_2^{[3]_3} + \bar{B}_3^{[2]_3}R_3^{[3]_3} + h^3\bar{D}^{[3]_3}R_4^{[3]_3} + h^3\bar{E}^{[3]_3}R_5^{[3]_3} \tag{2.6}$$

where

$$I^{[3]_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \bar{B}_1^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ \bar{B}_2^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & s_1h \\ 0 & 0 & 0 & s_2h \\ 0 & 0 & 0 & s_3h \\ 0 & 0 & 0 & h \end{pmatrix}$$

$$\bar{B}_3^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & \frac{h^2s_1^2}{2} \\ 0 & 0 & 0 & \frac{h^2s_2^2}{2} \\ 0 & 0 & 0 & \frac{h^2s_3^2}{2} \\ 0 & 0 & 0 & \frac{h^2}{2} \end{pmatrix}, \ \bar{D}^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & \bar{D}_{14}^{[3]_3} \\ 0 & 0 & 0 & \bar{D}_{24}^{[3]_3} \\ 0 & 0 & 0 & \bar{D}_{34}^{[3]_3} \\ 0 & 0 & 0 & \bar{D}_{44}^{[3]_3} \end{pmatrix}, \ \bar{E}^{[3]_3} = \begin{pmatrix} \bar{E}_{11}^{[3]_3} & E_{12}^{[3]_3} & E_{13}^{[3]_3} & E_{14}^{[3]_3} \\ E_{21}^{[3]_3} & E_{21}^{[3]_3} & E_{23}^{[3]_3} & E_{24}^{[3]_3} \\ E_{31}^{[3]_3} & E_{32}^{[3]_3} & E_{33}^{[3]_3} & E_{34}^{[3]_3} \\ E_{41}^{[3]_3} & E_{42}^{[3]_3} & E_{43}^{[3]_3} & E_{44}^{[3]_3} \end{pmatrix},$$

and the terms of $\bar{D}^{[3]_3}$ and $\bar{E}^{[3]_3}$ are given as

$$\bar{D}_{13}^{[3]_3} = -\frac{(s_1^3}{(840s_2s_3)}(21s_1s_2 + 21s_1s_3 - 105s_2s_3 - 7s_1^2s_2 - 7s_1^2s_3 - 7s_1^2 + 3s_1^3 + 21s_1s_2s_3) \ ,$$

$$\bar{D}_{23}^{[3]_3} = \frac{(s_2^3}{(840s_1s_3)}(105s_1s_3 - 21s_1s_2 - 21s_2s_3 + 7s_1s_2^2 + 7s_2^2s_3 + 7s_2^2 - 3s_2^3 - 21s_1s_2s_3) \ ,$$

$$\bar{D}_{33}^{[3]_3} = -\frac{s_3^3}{(840s_1s_2)}(21s_1s_3 - 105s_1s_2 + 21s_2s_3 - 7s_1s_3^2 - 7s_2s_3^2 - 7s_3^2 + 3s_3^3 + 21s_1s_2s_3) \ ,$$

$$\bar{D}_{33}^{[3]_3} = \frac{1}{(840s_1s_2s_3)}(7s_1 + 7s_2 + 7s_3 - 21s_1s_2 - 21s_1s_3 - 21s_2s_3$$

$$+ 105s_1s_2s_3 - 3)\ ,$$

$$\bar{E}_{11}^{[3]_3} = \frac{h^3s_1^3}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}(14s_1s_2 + 14s_1s_3 - 35s_2s_3 - 7s_1^2s_2 - 7s_1^2s_3 - 7s_1^2 + 4s_1^3 + 14s_1s_2s_3)\ ,$$

$$\bar{E}_{12}^{[3]_3} = \frac{(h^3s_1^5}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}(21s_3 - 7s_1 - 7s_1s_3 + 3s_1^2)\ ,$$

$$\bar{E}_{13}^{[3]_3} = -\frac{(h^3s_1^5}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(21s_2 - 7s_1 - 7s_1s_2 + 3s_1^2)\ ,$$

$$\bar{E}_{14}^{[3]_3} = \frac{(h^3s_1^5}{(840(s_3-1)(s_2-1)(s_1-1))}(21s_2s_3 - 7s_1s_3 - 7s_1s_2 + 3s_1^2)\ ,$$

$$\bar{E}_{21}^{[3]_3} = -\frac{h^3s_2^5}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}(21s_3 - 7s_2 - 7s_2s_3 + 3s_2^2)\ ,$$

$$\bar{E}_{22}^{[2]_3} = \frac{(h^3s_2^3}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}(35s_1s_3 - 14s_1s_2 - 14s_2s_3 + 7s_1s_2^2 + 7s_2^2s_3 + 7s_2^2 - 4s_2^3 - 14s_1s_2s_3)\ ,$$

$$\bar{E}_{23}^{[3]_3} = \frac{h^3s_2^5}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(7s_2 - 21s_1 + 7s_1s_2 - 3s_2^2)\ ,$$

$$\bar{E}_{24}^{[3]_3} = -\frac{h^3s_2^5}{(840(s_3-1)(s_1-1)(s_2-1))}(7s_1s_2 - 21s_1s_3 + 7s_2s_3 - 3s_2^2)\ ,$$

$$\bar{E}_{31}^{[3]_3} = \frac{h^3s_3^5}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}(7s_3 - 21s_2 + 7s_2s_3 - 3s_3^2)\ ,$$

$$\bar{E}_{32}^{[3]_3} = -\frac{h^3s_3^5}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}(7s_3 - 21s_1 + 7s_1s_3 - 3s_3^2)\ ,$$

$$\bar{E}_{33}^{[3]_3} = \frac{h^3s_3^3}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}(14s_1s_3 - 35s_1s_2 + 14s_2s_3 - 7s_1s_3^2 - 7s_2s_3^2 - 7s_3^2 + 4s_3^3 + 14s_1s_2s_3)\ ,$$

$$\bar{E}_{34}^{[3]_3} = \frac{h^3s_3^5(21s_1s_2 - 7s_1s_3 - 7s_2s_3 + 3s_3^2)}{(840(s_2-1)(s_1-1)(s_3-1))}\ ,$$

$$\bar{E}_{41}^{[3]_3} = -\frac{h^3(21s_2s_3 - 7s_3 - 7s_2 + 3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\ ,$$

$$\bar{E}_{42}^{[3]_3} = \frac{h^3(21s_1s_3 - 7s_3 - 7s_1 + 3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\ ,$$

$$\bar{E}_{43}^{[3]_3} = -\frac{h^3(21s_1s_2 - 7s_2 - 7s_1 + 3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\ ,$$

$$\bar{E}_{44}^{[3]_3} = \frac{h^3(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_3-1)(s_2-1)(s_1-1))} .$$

Substituting y_{n+s_1} and y_{n+s_2} into first derivative of discrete schemes to give the block of first derivative:

$$\begin{aligned}
y'_{n+s_1} =& y'_n + s_1 h y''_n \\
&- \frac{h^2 s_1^2(5s_1s_2+5s_1s_3-20s_2s_3-2s_1^2s_2-2s_1^2s_3-2s_1^2+s_1^3+5s_1s_2s_3)}{(60s_2s_3)} f_n \\
&+ \frac{(h^2 s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{(h^2 s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} - \frac{(h^2 s_1^4(5s_2-2s_1-2s_1s_2+s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{(h^2 s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} ,
\end{aligned}$$

$$\begin{aligned}
y'_{n+s_2} =& y'_n + s_2 h y''_n \\
&+ \frac{h^2 s_2^2(20s_1s_3-5s_1s_2-5s_2s_3+2s_1s_2^2+2s_2^2s_3+2s_2^2-s_2^3-5s_1s_2s_3)}{(60s_1s_3)} f_n \\
&- \frac{h^2 s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{h^2 s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{h^2 s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} - \frac{h^2 s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))} f_{n+1} ,
\end{aligned}$$

$$\begin{aligned}
y'_{n+s_3} =& y'_n + s_3 h y''_n \\
&- \frac{h^2 s_3^2(5s_1s_3-20s_1s_2+5s_2s_3-2s_1s_3^2-2s_2s_3^2-2s_3^2+s_3^3+5s_1s_2s_3)}{(60s_1s_2)} f_n \\
&+ \frac{h^2 s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} \\
&+ \frac{h^2 s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)} f_{n+s_1} - \frac{h^2 s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)} f_{n+s_2} \\
&+ \frac{h^2 s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} ,
\end{aligned}$$

$$\begin{aligned}
y'_{n+1} =& y'_n + h y''_n \\
&+ \frac{h^2(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1)}{(60s_1s_2s_3)} f_n \\
&- \frac{h^2(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} + \frac{h^2(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&- \frac{h^2(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} + \\
&\frac{h^2(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} .
\end{aligned}$$

Substituting y_{n+s_1} and y_{n+s_2} into second derivative of discrete schemes gives the block of second derivative:

$$
\begin{aligned}
y''_{n+s_1} =& y''_n \\
&- \frac{hs_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3)}{(60s_2s_3)} f_n \\
&+ \frac{hs_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{hs_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2)} f_{n+s_2} - \frac{hs_1^3(10s_2-5s_1-5s_1s_2+3s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{hs_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} ,
\end{aligned}
$$

$$
\begin{aligned}
y''_{n+s_2} =& y''_n \\
&+ \frac{hs_2(30s_1s_3-10s_1s_2-10s_2s_3+5s_1s_2^2+5s_2^2s_3+5s_2^2-3s_2^3-10s_1s_2s_3)}{(60s_1s_3)} f_n \\
&- \frac{hs_2^3(10s_3-5s_2-5s_2s_3+3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{hs_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2)} f_{n+s_2} \\
&+ \frac{hs_2^3(5s_2-10s_1+5s_1s_2-3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} - \frac{hs_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} ,
\end{aligned}
$$

$$
\begin{aligned}
y''_{n+s_3} =& y''_n \\
&- \frac{hs_3(10s_1s_3-30s_1s_2+10s_2s_3-5s_1s_3^2-5s_2s_3^2-5s_3^2+3s_3^3+10s_1s_2s_3)}{(60s_1s_2)} f_n \\
&+ \frac{hs_3^3(5s_3-10s_2+5s_2s_3-3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} - \frac{hs_3^3(5s_3-10s_1+5s_1s_3-3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{hs_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{hs_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} ,
\end{aligned}
$$

$$
\begin{aligned}
y''_{n+1} =& y''_n \\
&+ \frac{h(5s_1+5s_2+5s_3-10s_1s_2-10s_1s_3-10s_2s_3+30s_1s_2s_3-3)}{(60s_1s_2s_3)} f_n \\
&- \frac{s_1h(10s_2s_3-5s_3-5s_2+3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} + \frac{s_2h(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&- \frac{s_3h(10s_1s_2-5s_2-5s_1+3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3)} f_{n+s_3} \\
&+ \frac{h(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} .
\end{aligned}
$$

3. Analysis of the Method

3.1. Order of the Method

In order to find the order of the block, expanding y and f-function in Taylor series, that is

$$\left[\begin{array}{c}
\sum_{j=0}^{\infty}\frac{(s_1)^j h^j}{j!}y_n^{j}-y_n-s_1hy_n^{'}-\frac{h^2s_1^2}{2}y_n^{''}\\
+\frac{(s_1^3h^3(21s_1s_2+21s_1s_3-105s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+3s_1^3+21s_1s_2s_3)}{(840s_2s_3)}y_n^{'''}\\
-\frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+3}}{j!}y_n^{j+3}\\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{s_2^jh^{j+3}}{j!}y_n^{j+3}\\
+\frac{s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{s_2^jh^{j+3}}{j!}y_n^{j+3}\\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\\
\sum_{j=0}^{\infty}\frac{(s_2)^j h^j}{j!}y_n^{j}-y_n-s_2hy_n^{'}-\frac{h^2s_2^2}{2}y_n^{''}\\
-\frac{s_2^3h^3(105s_1s_3-21s_1s_2-21s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-3s_2^3-21s_1s_2s_3)}{(840s_1s_3)}y_n^{'''}\\
+\frac{s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+3}}{j!}y_n^{j+3}\\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3)}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+3}}{j!}y_n^{j+3}\\
-\frac{s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+3}}{j!}y_n^{j+3}\\
+\frac{s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2)}{(840(s_3-1)(s_1-1)(s_2-1))}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\\
\sum_{j=0}^{\infty}\frac{(s_3)^j h^j}{j!}y_n^{j}-y_n-s_3hy_n^{'}-\frac{h^2s_3^2}{2}y_n^{''}\\
+\frac{s_3^3h^3(21s_1s_3-105s_1s_2+21s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+3s_3^3+21s_1s_2s_3)}{(840s_1s_2)}y_n^{'''}\\
-\frac{s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+3}}{j!}y_n^{j+3}\\
+\frac{s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+3}}{j!}y_n^{j+3}\\
-\frac{s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3)}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+3}}{j!}y_n^{j+3}\\
-\frac{s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2)}{(840(s_2-1)(s_1-1)(s_3-1))}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\\
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{j}-y_n-hy_n^{'}-\frac{h^2}{2}y_n^{''}\\
-\frac{h^3(7s_1+7s_2+7s_3-21s_1s_2-21s_1s_3-21s_2s_3+105s_1s_2s_3-3)}{(840s_1s_2s_3)}y_n^{'''}\\
+\frac{(21s_2s_3-7s_3-7s_2+3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+3}}{j!}y_n^{j+3}\\
-\frac{(21s_1s_3-7s_3-7s_1+3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+3}}{j!}y_n^{j+3}\\
+\frac{(21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+3}}{j!}y_n^{j+3}\\
-\frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}
\end{array}\right]=\left[\begin{array}{c}0\\0\\0\\0\end{array}\right].$$

By comparing the coefficient of h, the order of the method is $[5,5,5,5]^T$ with the following error constants vector

$$\left[\begin{array}{c}
\frac{-(s_1^5(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{201600}\\
\frac{(s_2^5(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{201600}\\
\frac{-(s_3^5(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{201600}\\
\frac{(6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3)}{201600}
\end{array}\right],$$

for all $s_1,s_2,s_3\in(0,1)\setminus\{\{s_2=\dfrac{-14s_1s_3+6s_1^2s_3+6s_1^2-3s_1^3}{14s_1-42s_3-6s_1^2+14s_1s_3}\}\cup\{s_1=\dfrac{14s_2s_3-6s_2^2s_3-6s_2^2+3s_2^3}{42s_3-14s_2+6s_2^2-14s_2s_3}\}$

$\cup\{s_2=\dfrac{-14s_1s_3+6s_1s_3^2+6s_3^2-3s_3^3}{-42s_1+14s_3-6s_3^2+14s_1s_3}\}\cup\{s_1=\dfrac{-6s_2-6s_3+14s_2s_3+3}{6-14s_2-14s_3+42s_2s_3}\}\}.$

3.2. *Zero Stability of Method*

In finding the zero-stability of the block method (2.6), roots of the first characteristic function $\Pi(z) = |z\, I^{[3]_3} - \bar{B}_1^{[3]_3}|$ must be simple or less than one. Where $I^{[2]_3}$ and $\bar{B}_1^{[3]_3}$ are the coefficients of $y_{n+i}; i = s_1, s_2, s_3, 1$ and y_n. That is

$$\Pi(z) = |z\, I^{[3]_3} - \bar{B}_1^{[3]_3}| = \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = z^3(z-1) = 0 \ ,$$

which gives $z = 0, 0, 0, 1$. This implies that our method is zero stable. The new method is consistent Since the order of the method is greater than one. However, zero stability and consistency are sufficient conditions of the method to be convergent [6].

3.3. *Region of Absolute Stability*

The hybrid block method in (2.6) is said to be absolutely stable, if for a given h, all roots of the characteristic polynomial $\pi(z, h) = \rho(z) - \bar{h}\sigma(z)$, satisfies $|z_t| < 1$. In this article, the locus method was adopted to determine the region of absolute stability. The test equation $y''' = \lambda^3 y$ is substituted in the main method in (2.6) where $\bar{h} = \lambda^3 h^3$ and $\lambda = \frac{df}{dy}$. Substituting $r = \cos\theta - i \sin\theta$ and considering real part yields the equation of region stability

$$\bar{h}(\theta, h) = \frac{36288000(e^{i\theta} - 1)}{(s_1^2 s_2^2 s_3^2 (10s_1 + 10s_2 + 10s_3 - 6s_1s_2 - 6s_1s_3 - 6s_2s_3 + 3s_1s_2s_3 + s_1s_2s_3e^{i\theta} - 15))} \ . \tag{3.1}$$

4. Numerical Example

The following two third order (IVPs) were solved using 8 and 7 order block method by Kuboye (2015) and Omar(2015) respectively. The performance of our method is confirmed by solving the same problem. This is demonstrated in Table 1 and Table 2.

Problem 1: $y''' + y = 0$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 1$.
Exact solution: $y(x) = e^{-x}$ with $h = 0.1$.

Table 1: Comparison of the new method with Kuboye and Omar(2015) for solving Problem 1

x	Exact solution	Computed solution in our method with one off-step points $s_1 = \frac{1}{7}$, $s_2 = \frac{2}{5}$, $s_3 = \frac{9}{10}$	Error in our method, $P = 5$	Error in [8], $P = 8$
0.1	0.904837418035959520	0.904837418035952970	$6.550316E^{-15}$	$2.138401E^{-12}$
0.2	0.818730753077981820	0.818730753077960730	$2.109424E^{-14}$	$6.055156E^{-13}$
0.3	0.740818220681717770	0.740818220681712440	$5.329071E^{-15}$	$7.395751E^{-12}$
0.4	0.670320046035639330	0.670320046035714160	$7.482903E^{-14}$	$2.158163E^{-12}$
0.5	0.606530659712633420	0.606530659712884330	$2.509104E^{-13}$	$1.484579E^{-11}$
0.6	0.548811636094026500	0.548811636094576840	$5.503376E^{-13}$	$1.098521E^{-11}$
0.7	0.496585303791409530	0.496585303792408010	$9.984791E^{-13}$	$3.142886E^{-11}$
0.8	0.449328964117221620	0.449328964118839050	$1.617428E^{-12}$	$2.309530E^{-11}$
0.9	0.406569659740599170	0.406569659743025950	$2.426781E^{-12}$	$5.154149E^{-11}$
1.0	0.367879441171442330	0.367879441174885860	$3.443523E^{-12}$	$8.200535E^{-11}$

Problem 2: $y''' + e^x = 0$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 3$.
Exact solution: $y(x) = 2x^2 - e^x + 2$ with $h = 0.1$.

Table 2: Comparison of the new method with Omar and Kuboye (2015) for solving Problem 2

x	Exact solution	Computed solution in our method with one off-step points $s_1 = \frac{1}{7}$, $s_2 = \frac{2}{5}$, $s_3 = \frac{9}{10}$	Error in our method, $P = 5$	Error in [11], $P = 7$
0.1	0.914829081924352310	0.914829081924363410	$1.110223E^{-14}$	$2.886580E^{-13}$
0.2	0.858597241839830220	0.858597241839990980	$1.607603E^{-13}$	$1.836753E^{-12}$
0.3	0.830141192423996980	0.830141192424628030	$6.310508E^{-13}$	$4.572787E^{-12}$
0.4	0.828175302358729710	0.828175302360352860	$1.623146E^{-12}$	$8.563816E^{-12}$
0.5	0.851278729299871810	0.851278729303230900	$3.359091E^{-12}$	$1.374012E^{-11}$
0.6	0.897881199609491090	0.897881199615575220	$6.084133E^{-12}$	$2.017764E^{-11}$
0.7	0.966247292529523350	0.966247292539593290	$1.006994E^{-11}$	$2.736122E^{-11}$
0.8	1.054459071507532400	1.054459071523148300	$1.561595E^{-11}$	$3.675460E^{-11}$
0.9	1.160396888843050300	1.160396888866103800	$2.305356E^{-11}$	$4.822365E^{-11}$
1.0	1.281718171540954500	1.281718171573704000	$3.274958E^{-11}$	$6.189094E^{-11}$

5. Conclusion

An accurate hybrid block one step method for solving third order initial value problem directly has been developed in this work. The accuracy of the new method when was applied for solving some problems is demonstrated in Table 1 and Table 2. numerical properties of the method which includes, consistency, order, zero stability, error constant and convergence are also established.

References

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